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Existence of three solutions for a nonlocal elliptic system of (p,q)-Kirchhoff type

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Abstract

In this paper, we study the solutions of a nonlocal elliptic system of (p,q)-Kirchhoff type on a bounded domain based on the three critical points theorem introduced by Ricceri. Firstly, we establish the existence of three weak solutions under appropriate hypotheses; then, we prove the existence of at least three weak solutions for the nonlocal elliptic system of (p,q)-Kirchhoff type.

Keywords: (p,q)-Kirchhoff type system; multiple solutions; three critical points theory

1 Introduction and main results

We consider the boundary problem involving (p,q)-Kirchhoff

$$\begin{cases} -[M_1(\int_\Omega |\nabla u|^p)]^{p-1}\Delta_p u = \lambda F_u(x,u,\upsilon) + \mu G_u(x,u,\upsilon), & \text{in } \Omega, \\ -[M_2(\int_\Omega |\nabla \upsilon|^q)]^{q-1}\Delta_q \upsilon = \lambda F_\upsilon(x,u,\upsilon) + \mu G_v(x,u,\upsilon), & \text{in } \Omega, \\ u = \upsilon = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset R^N$ $(N \geq 1)$ is a bounded smooth domain, $\lambda, \mu \in [0, +\infty)$, $p > N$, $q > N$, Δ_p is the p-Laplacian operator $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. $F, G : \Omega \times \mathrm{R} \times \mathrm{R} \mapsto \mathrm{R}$ are functions such that $F(\cdot, s, t)$, $G(\cdot, s, t)$ are measurable in Ω for all $(s,t) \in \mathrm{R} \times \mathrm{R}$ and $F(x, \cdot, \cdot)$, $G(x, \cdot, \cdot)$ are continuously differentiable in $\mathrm{R} \times \mathrm{R}$ for a.e. $x \in \Omega$. F_i is the partial derivative of F with respect to i, $i = u, v$, so is G_i. $M_i : R^+ \to R$, $i = 1, 2$, are continuous functions which satisfy the following bounded conditions.

(M) There exist two positive constants m_0, m_1 such that

$$m_0 \leq M_i(t) \leq m_1, \quad \forall t \geq 0, i = 1, 2. \tag{1.2}$$

Here and in the sequel, X denotes the Cartesian product of two Sobolev spaces $W_0^{1,p}(\Omega)$ and $W_0^{1,q}(\Omega)$, *i.e.*, $X = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$. The reflexive real Banach space X is endowed with the norm

$$\|(u,\upsilon)\| = \|u\|_p + \|\upsilon\|_q, \qquad \|u\|_p = \left(\int_\Omega |\nabla u|^p\right)^{1/p}, \qquad \|\upsilon\|_q = \left(\int_\Omega |\nabla \upsilon|^q\right)^{1/q}.$$

Since $p > N$ and $q > N$, $W_0^{1,p}(\Omega)$ and $W_0^{1,q}(\Omega)$ are compactly embedded in $C^0(\bar{\Omega})$. Let

$$C = \max\left\{ \sup_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\max_{x\in\bar{\Omega}}\{|u(x)|^p\}}{\|u\|_p^p}, \sup_{v \in W_0^{1,q}(\Omega)\setminus\{0\}} \frac{\max_{x\in\bar{\Omega}}\{|\upsilon(x)|^q\}}{\|\upsilon\|_q^q} \right\}, \tag{1.3}$$

then one has $C < +\infty$. Furthermore, it is known from [1] that

$$\sup_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\max_{x\in\bar{\Omega}}\{|u(x)|^p\}}{\|u\|_p} \le \frac{N^{-1/p}}{\sqrt{\pi}} \left(\Gamma\left(1 + \frac{N}{2}\right)\right)^{1/N} \left(\frac{p-1}{p-N}\right)^{1-1/p} |\Omega|^{(1/N)-(1/p)},$$

where Γ is the gamma function and $|\Omega|$ is the Lebesgue measure of Ω. As usual, by a weak solution of system (1.1), we mean any $(u, \upsilon) \in X$ such that

$$\begin{aligned} &\left[M_1\left(\int_\Omega |\nabla u|^p\right)\right]^{p-1} \int_\Omega |\nabla u|^{p-2}\nabla u \nabla \phi + \left[M_2\left(\int_\Omega |\nabla \upsilon|^q\right)\right]^{q-1} \int_\Omega |\nabla \upsilon|^{q-2}\nabla \upsilon \nabla \psi \\ &\quad - \lambda \int_\Omega (F_u\phi + F_v\psi)\,dx - \mu \int_\Omega (G_u\phi + G_v\psi)\,dx = 0 \end{aligned} \tag{1.4}$$

for all $(\phi, \psi) \in X$.

System (1.1) is related to the stationary version of a model established by Kirchhoff [2]. More precisely, Kirchhoff proposed the following model:

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.5}$$

which extends D'Alembert's wave equation with free vibrations of elastic strings, where ρ denotes the mass density, P_0 denotes the initial tension, h denotes the area of the cross-section, E denotes the Young modulus of the material, and L denotes the length of the string. Kirchhoff's model considers the changes in length of the string produced during the vibrations.

Later, (1.1) was developed into the following form:

$$u_{tt} - M\left(\int_\Omega |\nabla u|^2\right)\Delta u = f(x,u) \quad \text{in } \Omega, \tag{1.6}$$

where $M : R^+ \to R$ is a given function. After that, many authors studied the following problem:

$$-M\left(\int_\Omega |\nabla u|^2\right)\Delta u = f(x,u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.7}$$

which is the stationary counterpart of (1.6). By applying variational methods and other techniques, many results of (1.7) were obtained, the reader is referred to [3–13] and the references therein. In particular, Alves *et al.* [3, Theorem 4] supposed that M satisfies bounded condition (M) and $f(x,t)$ satisfies the condition

$$0 < \upsilon F(x,t) \le f(x,t)t, \quad \forall |t| \ge R, x \in \Omega \text{ for some } \nu > 2 \text{ and } R > 0, \tag{AR}$$

where $F(x,t) = \int_0^t f(x,s)\,ds$; one positive solution for (1.7) was given.

In [14], using Ekeland's variational principle, Corrêa and Nascimento proved the existence of a weak solution for the boundary problem associated with the nonlocal elliptic system of p-Kirchhoff type

$$\begin{cases} -[M_1(\int_\Omega |\nabla u|^p)]^{p-1}\Delta_p u = f(u,\upsilon) + \rho_1(x), & \text{in } \Omega, \\ -[M_2(\int_\Omega |\nabla \upsilon|^p)]^{p-1}\Delta_p \upsilon = g(,u,\upsilon) + \rho_2(x), & \text{in } \Omega, \\ \frac{\partial u}{\partial\eta} = \frac{\partial\upsilon}{\partial\eta} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

where η is the unit exterior vector on $\partial\Omega$, and M_i, ρ_i $(i=1,2)$, f, g satisfy suitable assumptions.

In [15], when $\mu = 0$ in (1.1), Bitao Cheng *et al.* studied the existence of two solutions and three solutions of the following nonlocal elliptic system:

$$\begin{cases} -[M_1(\int_\Omega |\nabla u|^p)]^{p-1}\Delta_p u = \lambda F_u(x,u,\upsilon), & \text{in } \Omega, \\ -[M_2(\int_\Omega |\nabla \upsilon|^q)]^{q-1}\Delta_q \upsilon = \lambda F_\upsilon(x,u,\upsilon), & \text{in } \Omega, \\ u = \upsilon = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.9}$$

In this paper, our objective is to prove the existence of three solutions of problem (1.1) by applying the three critical points theorem established by Ricceri [16]. Our result, under appropriate assumptions, ensures the existence of an open interval $\Lambda \subset [0,+\infty)$ and a positive real number ρ such that, for each $\lambda \in \Lambda$, problem (1.1) admits at least three weak solutions whose norms in X are less than ρ. The purpose of the present paper is to generalize the main result of [15].

Now, for every $x_0 \in \Omega$ and choosing R_1, R_2 with $R_2 > R_1 > 0$, such that $B(x_0,R_2) \subseteq \Omega$, where $B(x,R) = \{y \in R^N : |y-x| < R\}$, put

$$\alpha_1 = \alpha_1(N,p,R_1,R_2) = \frac{C^{1/p}(R_2^N - R_1^N)^{1/p}}{R_2 - R_1}\left(\frac{\pi^{N/2}}{\Gamma(1+N/2)}\right)^{1/p}, \tag{1.10}$$

$$\alpha_2 = \alpha_2(N,q,R_1,R_2) = \frac{C^{1/q}(R_2^N - R_1^N)^{1/q}}{R_2 - R_1}\left(\frac{\pi^{N/2}}{\Gamma(1+N/2)}\right)^{1/q}. \tag{1.11}$$

Moreover, let a, c be positive constants and define

$$y(x) = \frac{a}{R_2 - R_1}\left(R_2 - \left\{\sum_{i=1}^N \left(x^i - x_0^i\right)^2\right\}^{1/2}\right), \quad \forall x \in B(x_0,R_2)\backslash B(x_0,R_1),$$

$$A(c) = \left\{(s,t) \in R \times R : |s|^p + |t|^q \le c\right\},$$

$$M^+ = \max\left\{\frac{m_1^{p-1}}{p}, \frac{m_1^{q-1}}{q}\right\}, \qquad M_- = \min\left\{\frac{m_0^{p-1}}{p}, \frac{m_0^{q-1}}{q}\right\}.$$

Our main result is stated as follows.

Theorem 1.1 *Assume that $R_2 > R_1 > 0$ such that $B(x_0,R_2) \subseteq \Omega$, and suppose that there exist four positive constants a, b, γ and β with $\gamma < p$, $\beta < q$, $(a\alpha_1)^p + (a\alpha_2)^q > bM^+/M_-$, and a function $\alpha(x) \in L^\infty(\Omega)$ such that*

(j1) *$F(x,s,t) \ge 0$ for a.e. $x \in \Omega\backslash B(x_0,R_1)$ and all $(s,t) \in [0,a] \times [0,a]$;*

(j2) $[(a\alpha_1)^p + (a\alpha_2)^q]|\Omega| \sup_{(x,s,t)\in\Omega\times A(bM^+/M_-)} F(x,s,t) < b \int_{B(x_0,R_1)} F(x,a,a)\,dx$;
(j3) $F(x,s,t) \le \alpha(x)(1 + |s|^\gamma + |t|^\beta)$ *for a.e.* $x \in \Omega$ *and all* $(s,t) \in R \times R$;
(j4) $F(x,0,0) = 0$ *for a.e.* $x \in \Omega$.

Then there exist an open interval $\Lambda \subseteq [0,\infty)$ *and a positive real number* ρ *with the following property: for each* $\lambda \in \Lambda$ *and for two Carathéodory functions* $G_u, G_v : \Omega \times R \times R \mapsto R$ *satisfying*

(j5) $\sup_{\{|s|\le\xi,|t|\le\xi\}}(|G_u(\cdot,s,t)| + |G_v(\cdot,s,t)|) \in L^1(\Omega)$ *for all* $\xi > 0$,

there exists $\delta > 0$ *such that, for each* $\mu \in [0,\delta]$, *problem* (1.1) *has at least three weak solutions* $w_i = (u_i, \upsilon_i) \in X$ $(i = 1,2,3)$ *whose norms* $\|w_i\|$ *are less than* ρ.

2 Proof of the main result

First we recall the modified form of Ricceri's three critical points theorem (Theorem 1 in [16]) and Proposition 3.1 of [17], which is our primary tool in proving our main result.

Theorem 2.1 ([16], Theorem 1) *Suppose that* X *is a reflexive real Banach space and that* $\Phi : X \mapsto R$ *is a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on* X^*, *and that* Φ *is bounded on each bounded subset of* X; $\Psi : X \mapsto R$ *is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact*; $I \subseteq R$ *is an interval. Suppose that*

$$\lim_{\|x\|\to+\infty} \big(\Phi(x) + \lambda\Psi(x)\big) = +\infty$$

for all $\lambda \in I$, *and that there exists* $h \in R$ *such that*

$$\sup_{\lambda\in I} \inf_{x\in X} \big(\Phi(x) + \lambda\big(\Psi(x) + h\big)\big) < \inf_{x\in X} \sup_{\lambda\in I} \big(\Phi(x) + \lambda\big(\Psi(x) + h\big)\big). \tag{2.1}$$

Then there exist an open interval $\Lambda \subseteq I$ *and a positive real number* ρ *with the following property: for every* $\lambda \in \Lambda$ *and every* C^1 *functional* $J : X \mapsto R$ *with compact derivative, there exists* $\delta > 0$ *such that, for each* $\mu \in [0,\delta]$, *the equation*

$$\Phi'(x) + \lambda\Psi'(x) + \mu J'(x) = 0$$

has at least three solutions in X *whose norms are less than* ρ.

Proposition 2.1 ([17], Proposition 3.1) *Assume that* X *is a nonempty set and* Φ, Ψ *are two real functions on* X. *Suppose that there are* $r > 0$ *and* $x_0, x_1 \in X$ *such that*

$$\Phi(x_0) = -\Psi(x_0) = 0, \qquad \Phi(x_1) > 1, \qquad \sup_{x\in\Phi^{-1}([-\infty,r])} -\Psi(x) < r\frac{-\Psi(x_1)}{\Phi(x_1)}.$$

Then, for each h *satisfying*

$$\sup_{x\in\Phi^{-1}([-\infty,r])} -\Psi(x) < h < r\frac{-\Psi(x_1)}{\Phi(x_1)},$$

one has

$$\sup_{\lambda\geq 0}\inf_{x\in X}\big(\Phi(x)+\lambda\big(\Psi(x)+h\big)\big)<\inf_{x\in X}\sup_{\lambda\geq 0}\big(\Phi(x)+\lambda\big(\Psi(x)+h\big)\big).$$

Before proving Theorem 1.1, we define a functional and give a lemma.

The functional $H: X\to R$ is defined by

$$\begin{aligned} H(u,v)&=\Phi(u,v)+\lambda J(u,v)+\mu\psi(u,v)\\ &=\frac{1}{p}\widehat{M}_1\biggl(\int_\Omega|\nabla u|^p\biggr)+\frac{1}{q}\widehat{M}_2\biggl(\int_\Omega|\nabla\upsilon|^q\biggr)\\ &\quad-\lambda\int_\Omega F(x,u,v)\,dx-\mu\int_\Omega G(x,u,v)\,dx \end{aligned}\tag{2.2}$$

for all $(u,\upsilon)\in X$, where

$$\widehat{M}_1=\int_0^t\big[M_1(s)\big]^{p-1}\,ds,\qquad \widehat{M}_2=\int_0^t\big[M_2(s)\big]^{q-1}\,ds.\tag{2.3}$$

By conditions (M) and (j3), it is clear that $H\in C^1(X,R)$ and a critical point of H corresponds to a weak solution of system (1.1).

Lemma 2.2 *Assume that there exist two positive constants a, b with $(a\alpha_1)^p+(a\alpha_2)^q>bM^+/M_-$ such that*

(j1) $F(x,s,t)\geq 0$, *for a.e. $x\in\Omega\backslash B(x_0,R_1)$ and all $(s,t)\in[0,a]\times[0,a]$;*

(j2) $[(a\alpha_1)^p+(a\alpha_2)^q]|\Omega|\sup_{(x,s,t)\in\Omega\times A(bM^+/M_-)}F(x,s,t)<b\int_{B(x_0,R_1)}F(x,a,a)\,dx.$

Then there exist $r>0$ and $u_0\in W_0^{1,p}(\Omega)$, $\upsilon_0\in W_0^{1,q}(\Omega)$ such that

$$\Phi(u_0,v_0)>r$$

and

$$|\Omega|\sup_{(x,s,t)\in\Omega\times A(bM^+/M_-)}F(x,s,t)\leq\frac{bM^+}{C}\frac{\int_\Omega F(x,u_0,\upsilon_0)\,dx}{\Phi(u_0,\upsilon_0)}.$$

Proof We put

$$w_0(x)=\begin{cases}0, & x\in\bar{\Omega}\backslash B(x_0,R_2),\\ \frac{a}{R_2-R_1}(R_2-\{\sum_{i=1}^N(x^i-x_0^i)\}^{1/2}), & x\in B(x_0,R_2)\backslash B(x_0,R_1),\\ a, & x\in B(x_0,R_1),\end{cases}$$

and $u_0(x)=\upsilon_0(x)=w_0(x)$. Then we can verify easily $(u_0,\upsilon_0)\in X$ and, in particular, we have

$$\|u_0\|_p^p=\big(R_2^N-R_1^N\big)\frac{\pi^{N/2}}{\Gamma(1+N/2)}\biggl(\frac{a}{R_2-R_1}\biggr)^p,\tag{2.4}$$

and

$$\|\upsilon_0\|_q^q=\big(R_2^N-R_1^N\big)\frac{\pi^{N/2}}{\Gamma(1+N/2)}\biggl(\frac{a}{R_2-R_1}\biggr)^q.\tag{2.5}$$

Hence, we obtain from (1.10), (1.11), (2.4) and (2.5) that

$$\|u_0\|_p^p = \|w_0\|_p^p = \frac{(a\alpha_1)^p}{C}, \qquad \|\upsilon_0\|_q^q = \|w_0\|_q^q = \frac{(a\alpha_2)^q}{C}. \tag{2.6}$$

Under condition (M), by a simple computation, we have

$$M_-\bigl(\|u\|_p^p + \|\upsilon\|_q^q\bigr) \le \Phi(u,\upsilon) \le M^+\bigl(\|u\|_p^p + \|\upsilon\|_q^q\bigr). \tag{2.7}$$

Setting $r = \frac{bM^+}{C}$ and applying the assumption of Lemma 2.2

$$(a\alpha_1)^p + (a\alpha_2)^q > bM^+/M_-,$$

from (2.6) and (2.7), we obtain

$$\Phi(u_0,v_0) \ge M_-\bigl(\|u_0\|_p^p + \|v_0\|_q^q\bigr) = \frac{M_-}{C}\bigl[(a\alpha_1)^p + (a\alpha_2)^q\bigr] > \frac{M_-}{C}\frac{bM^+}{M_-} = r.$$

Since, $0 \le u_0 \le a$, $0 \le v_0 \le a$ for each $x \in \Omega$, from condition (j1) of Lemma 2.2, we have

$$\int_{\Omega\setminus B(x_0,R_2)} F(x,u_0,\upsilon_0)\,dx + \int_{B(x_0,R_2)\setminus B(x_0,R_1)} F(x,u_0,\upsilon_0)\,dx \ge 0.$$

Hence, based on condition (j2), we get

$$\begin{aligned}
|\Omega| \sup_{(x,s,t)\in\Omega\times A(bM^+/M_-)} F(x,s,t) &< \frac{b}{(a\alpha_1)^p + (a\alpha_2)^q}\int_{B(x_0,R_1)} F(x,a,a)\,dx \\
&= \frac{bM^+}{C}\frac{\int_{B(x_0,R_1)} F(x,a,a)\,dx}{M^+((a\alpha_1)^p + (a\alpha_2)^q)/C} \\
&\le \frac{bM^+}{C}\frac{\int_{\Omega\setminus B(x_0,R_1)} F(x,u_0,\upsilon_0)\,dx + \int_{B(x_0,R_1)} F(x,u_0,\upsilon_0)\,dx}{M^+(\|u_0\|_p^p + \|\upsilon_0\|_q^q)} \\
&\le \frac{bM^+}{C}\frac{\int_\Omega F(x,u_0,\upsilon_0)\,dx}{\Psi(u_0,\upsilon_0)}.
\end{aligned}$$

□

Now, we can prove our main result.

Proof of Theorem 1.1 For each $(u,v) \in X$, let

$$\begin{aligned}
\Phi(u,v) &= \frac{\widehat{M}_1(\|u\|_p^p)}{p} + \frac{\widehat{M}_2(\|v\|_q^q)}{q}, \\
\Psi(u,\upsilon) &= -\int_\Omega F(x,u,\upsilon)\,dx, \qquad J(u,v) = -\int_\Omega G(x,u,v)\,dx.
\end{aligned}$$

From the assumption of Theorem 1.1, we know that Φ is a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional. Additionally, the Gâteaux derivative of Φ has a continuous inverse on X^*. Since $p > N$, $q > N$, Ψ and J are continuously Gâteaux differential functionals whose Gâteaux derivatives are compact. Obviously,

Φ is bounded on each bounded subset of X. In particular, for each $(u,v),(\xi,\eta)\in X$,

$$\begin{aligned}\langle\Phi'(u,v),(\xi,\eta)\rangle &= \left[M_1\left(\int_\Omega |\nabla u|^p\right)\right]^{p-1}\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\xi \\ &\quad + \left[M_2\left(\int_\Omega |\nabla \upsilon|^q\right)\right]^{q-1}\int_\Omega |\nabla \upsilon|^{q-2}\nabla \upsilon\nabla\eta, \\ \langle\Psi'(u,v),(\xi,\eta)\rangle &= -\int_\Omega F_u(x,u,v)\xi\,dx - \int_\Omega F_v(x,u,v)\eta\,dx, \\ \langle J'(u,v),(\xi,\eta)\rangle &= -\int_\Omega G_u(x,u,v)\xi\,dx - \int_\Omega G_v(x,u,v)\eta\,dx.\end{aligned}$$

Hence, the weak solutions of problem (1.1) are exactly the solutions of the following equation:

$$\Phi'(u,v)+\lambda\Psi'(u,v)+\mu J'(u,v)=0.$$

From (j3), for each $\lambda>0$, one has

$$\lim_{\|(u,v)\|\to+\infty}\big(\lambda\Phi(u,v)+\mu\Psi(u,v)\big)=+\infty, \tag{2.8}$$

and so the first condition of Theorem 2.1 is satisfied. By Lemma 2.2, there exists $(u_0,\upsilon_0)\in X$ such that

$$\begin{aligned}\Phi(u_0,v_0) &= \frac{\widehat{M}_1(\|u_0\|_p^p)}{p}+\frac{\widehat{M}_2(\|v_0\|_q^q)}{q} \\ &\geq M_-\big(\|u_0\|_p^p+\|v_0\|_q^q\big)=\frac{M_-}{C}\big[(a\alpha_1)^p+(a\alpha_2)^q\big] \\ &> \frac{M_-}{C}\frac{bM^+}{M_-}=\frac{bM^+}{C}>0=\Phi(0,0),\end{aligned} \tag{2.9}$$

and

$$|\Omega|\sup_{(x,s,t)\in\Omega\times A(bM^+/M_-)}F(x,s,t)\leq\frac{bM^+}{C}\frac{\int_\Omega F(x,u_0,\upsilon_0)\,dx}{\Phi(u_0,\upsilon_0)}. \tag{2.10}$$

From (1.3), we have

$$\max_{x\in\bar\Omega}\big\{|u(x)|^p\big\}\leq C\|u\|_p^p,\qquad \max_{x\in\bar\Omega}\big\{|\upsilon(x)|^q\big\}\leq C\|\upsilon\|_q^q$$

for each $(u,\upsilon)\in X$. We obtain

$$\max_{x\in\bar\Omega}\left\{\frac{|u(x)|^p}{p}+\frac{|v(x)|^q}{q}\right\}\leq C\left\{\frac{\|u\|_p^p}{p}+\frac{\|v\|_q^q}{q}\right\} \tag{2.11}$$

for each $(u,\upsilon)\in X$. Let $r=\frac{bM^+}{C}$ for each $(u,\upsilon)\in X$ such that

$$\Phi(u,\upsilon)=\frac{\widehat{M}_1(\|u\|_p^p)}{p}+\frac{\widehat{M}_2(\|v\|_q^q)}{q}\leq r.$$

From (2.11), we get

$$\left|u(x)\right|^p + \left|\upsilon(x)\right|^q \le C\left(\|u\|_p^p + \|\upsilon\|_q^q\right) \le \frac{Cr}{M_-} = \frac{C}{M_-}\frac{bM^+}{C} = \frac{bM^+}{M_-}. \tag{2.12}$$

Then, from (2.10) and (2.12), we find

$$\begin{aligned}\sup_{(u,\upsilon)\in\Phi^{-1}(-\infty,r)}\bigl(-\Psi(u,\upsilon)\bigr) &= \sup_{\{(u,\upsilon)|\in\Phi(u,\upsilon)\le r\}}\int_\Omega F(x,u,\upsilon)\,dx \\ &\le \sup_{\{(u,\upsilon)||u(x)|^p+|\upsilon(x)|^q\le bM^+/M_-\}}\int_\Omega F(x,u,\upsilon)\,dx \\ &\le \int_\Omega \sup_{(s,t)\in A(bM^+/M_-)} F(x,s,t)\,dx \\ &\le |\Omega| \sup_{(x,s,t)\in\Omega\times A(bM^+/M_-)} F(x,s,t) \\ &\le \frac{bM^+}{C}\frac{\int_\Omega F(x,u_0,\upsilon_0)\,dx}{\Phi(u_0,\upsilon_0)} \\ &= r\frac{-\Psi(u_0,\upsilon_0)}{\Phi(u_0,\upsilon_0)}.\end{aligned}$$

Hence, we have

$$\sup_{\{(u,v)|\Phi(u,v\le r\}}\bigl(-\Psi(u,v)\bigr) < r\frac{-\Psi(u_0,\upsilon_0)}{\Phi(u_0,\upsilon_0)}. \tag{2.13}$$

Fix h such that

$$\sup_{\{(u,v)|\Phi(u,v\le r\}}\bigl(-\Psi(u,v)\bigr) < h < r\frac{-\Psi(u_0,\upsilon_0)}{\Phi(u_0,\upsilon_0)},$$

by (2.9), (2.13) and Proposition 2.1, with $(u_1,v_1) = (0,0)$ and $(u^*,v^*) = (u_0,v_0)$, we obtain

$$\sup_{\lambda\ge 0}\inf_{x\in X}\bigl(\Phi(x) + \lambda\bigl(h + \Psi(x)\bigr)\bigr) < \inf_{x\in X}\sup_{\lambda\ge 0}\bigl(\Phi(x) + \lambda\bigl(h + \Psi(x)\bigr)\bigr), \tag{2.14}$$

and so assumption (2.1) of Theorem 2.1 is satisfied.

Now, with $I = [0,\infty)$, from (2.8) and (2.14), all the assumptions of Theorem 2.1 hold. Hence, our conclusion follows from Theorem 2.1. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
This paper is the result of joint work of all authors who contributed equally to the final version of this paper. All authors read and approved the final manuscript.

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