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Solutions and nonnegative solutions for a weighted variable exponent impulsive integro-differential system with multi-point and integral mixed boundary value problems

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Abstract

This paper investigates the existence of solutions for a weighted $p(t)$-Laplacian impulsive integro-differential system with multi-point and integral mixed boundary value problems via Leray-Schauder's degree; sufficient conditions for the existence of solutions are given. Moreover, we get the existence of nonnegative solutions.

MSC: 34B37

Keywords: weighted $p(t)$-Laplacian; impulsive integro-differential system; Leray-Schauder's degree

1 Introduction

In this paper, we consider the existence of solutions and nonnegative solutions for the following weighted $p(t)$-Laplacian integro-differential system:

$$-\triangle_{p(t)}u + f\bigl(t, u, \bigl(w(t)\bigr)^{\frac{1}{p(t)-1}}u', S(u), T(u)\bigr) = 0, \quad t \in (0,1), t \neq t_i, \tag{1}$$

where $u : [0,1] \to \mathbb{R}^N$, $f(\cdot,\cdot,\cdot,\cdot,\cdot) : [0,1] \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$, $t_i \in (0,1)$, $i = 1,\dots,k$, with the following impulsive boundary value conditions:

$$\lim_{t\to t_i^+} u(t) - \lim_{t\to t_i^-} u(t) = A_i\Bigl(\lim_{t\to t_i^-} u(t), \lim_{t\to t_i^-} \bigl(w(t)\bigr)^{\frac{1}{p(t)-1}}u'(t)\Bigr), \quad i = 1,\dots,k, \tag{2}$$

$$\begin{aligned}&\lim_{t\to t_i^+} w(t)\bigl|u'\bigr|^{p(t)-2}u'(t) - \lim_{t\to t_i^-} w(t)\bigl|u'\bigr|^{p(t)-2}u'(t)\\ &\quad = B_i\Bigl(\lim_{t\to t_i^-} u(t), \lim_{t\to t_i^-} \bigl(w(t)\bigr)^{\frac{1}{p(t)-1}}u'(t)\Bigr), \quad i = 1,\dots,k,\end{aligned} \tag{3}$$

$$u(0) = \int_0^1 g(t)u(t)\,dt, \qquad u(1) = \sum_{\ell=1}^{m-2} \alpha_\ell u(\xi_\ell) - \int_0^1 h(t)u(t)\,dt, \tag{4}$$

where $p \in C([0,1],\mathbb{R})$ and $p(t) > 1$, $-\triangle_{p(t)}u := -(w(t)|u'|^{p(t)-2}u')'$ is called the weighted $p(t)$-Laplacian; $0 < t_1 < t_2 < \cdots < t_k < 1$, $0 < \xi_1 < \cdots < \xi_{m-2} < 1$; $\alpha_\ell \geq 0$ ($\ell = 1,\dots,m-2$); $g \in L^1[0,1]$ is nonnegative, $\int_0^1 g(t)\,dt = \sigma \in [0,1]$; $h \in L^1[0,1]$, $\int_0^1 h(t)\,dt = \delta$; $A_i, B_i \in C(\mathbb{R}^N \times$

$\mathbb{R}^N,\mathbb{R}^N)$; T and S are linear operators defined by $(Su)(t) = \int_0^1 h_*(t,s)u(s)\,ds$, $(Tu)(t) = \int_0^t k_*(t,s)u(s)\,ds$, $t \in [0,1]$, where $k_*, h_* \in C([0,1] \times [0,1], \mathbb{R})$.

If $\sigma < 1$ and $\sum_{\ell=1}^{m-2} \alpha_\ell - \delta \neq 1$, we say the problem is nonresonant, but if $\sigma = 1$ or $\sum_{\ell=1}^{m-2} \alpha_\ell - \delta = 1$, we say the problem is resonant.

Throughout the paper, $o(1)$ means functions which are uniformly convergent to 0 (as $n \to +\infty$); for any $v \in \mathbb{R}^N$, v^j will denote the jth component of v; the inner product in $\mathbb{R}^N$ will be denoted by $\langle \cdot,\cdot \rangle$, $|\cdot|$ will denote the absolute value and the Euclidean norm on $\mathbb{R}^N$. Denote $J = [0,1]$, $J' = (0,1)\backslash\{t_1,\dots,t_k\}$, $J_0 = [t_0,t_1]$, $J_i = (t_i,t_{i+1}]$, $i = 1,\dots,k$, where $t_0 = 0$, $t_{k+1} = 1$. Denote by J_i^o the interior of J_i, $i = 0,1,\dots,k$. Let

$$PC(J,\mathbb{R}^N) = \left\{ x : J \to \mathbb{R}^N \,\middle|\, \begin{array}{l} x \in C(J_i,\mathbb{R}^N), i = 0,1,\dots,k \\ \text{and } \lim_{t\to t_i^+} x(t) \text{ exists for } i = 1,\dots,k \end{array} \right\},$$

$w \in PC(J,\mathbb{R})$ satisfy $0 < w(t)$, $\forall t \in (0,1)\backslash\{t_1,\dots,t_k\}$, and $(w(t))^{\frac{-1}{p(t)-1}} \in L^1(0,1)$,

$$PC^1(J,\mathbb{R}^N) = \left\{ x \in PC(J,\mathbb{R}^N) \,\middle|\, \begin{array}{l} x' \in C(J_i^o,\mathbb{R}^N), \lim_{t\to t_i^+} (w(t))^{\frac{1}{p(t)-1}} x'(t) \\ \text{and } \lim_{t\to t_{i+1}^-} (w(t))^{\frac{1}{p(t)-1}} x'(t) \text{ exists for } i = 0,1,\dots,k \end{array} \right\}.$$

For any $x = (x^1,\dots,x^N) \in PC(J,\mathbb{R}^N)$, denote $|x^i|_0 = \sup\{|x^i(t)| \mid t \in J'\}$.

Obviously, $PC(J,\mathbb{R}^N)$ is a Banach space with the norm $\|x\|_0 = (\sum_{i=1}^N |x^i|_0^2)^{\frac{1}{2}}$, and $PC^1(J,\mathbb{R}^N)$ is a Banach space with the norm $\|x\|_1 = \|x\|_0 + \|(w(t))^{\frac{1}{p(t)-1}} x'\|_0$. Denote $L^1 = L^1(J,\mathbb{R}^N)$ with the norm

$$\|x\|_{L^1} = \left(\sum_{i=1}^N |x^i|_{L^1}^2 \right)^{\frac{1}{2}}, \quad \forall x \in L^1, \text{ where } |x^i|_{L^1} = \int_0^1 |x^i(t)|\,dt.$$

In the following, $PC(J,\mathbb{R}^N)$ and $PC^1(J,\mathbb{R}^N)$ will be simply denoted by PC and PC^1, respectively. We denote

$$u(t_i^+) = \lim_{t\to t_i^+} u(t), \qquad u(t_i^-) = \lim_{t\to t_i^-} u(t),$$

$$w(0)|u'|^{p(0)-2}u'(0) = \lim_{t\to 0^+} w(t)|u'|^{p(t)-2}u'(t),$$

$$w(1)|u'|^{p(1)-2}u'(1) = \lim_{t\to 1^-} w(t)|u'|^{p(t)-2}u'(t),$$

$$A_i = A_i\Big(\lim_{t\to t_i^-} u(t), \lim_{t\to t_i^-} (w(t))^{\frac{1}{p(t)-1}} u'(t) \Big), \quad i = 1,\dots,k,$$

$$B_i = B_i\Big(\lim_{t\to t_i^-} u(t), \lim_{t\to t_i^-} (w(t))^{\frac{1}{p(t)-1}} u'(t) \Big), \quad i = 1,\dots,k.$$

The study of differential equations and variational problems with nonstandard $p(t)$-growth conditions has attracted more and more interest in recent years (see [1–4]). The applied background of these kinds of problems includes nonlinear elasticity theory [4], electro-rheological fluids [1, 3], and image processing [2]. Many results have been obtained on these kinds of problems; see, for example, [5–15]. Recently, the applications of variable exponent analysis in image restoration have attracted more and more attention

[16–19]. If $p(t) \equiv p$ (a constant), (1)-(4) becomes the well-known p-Laplacian problem. If $p(t)$ is a general function, one can see easily $-\triangle_{p(t)} cu \neq c^{p(t)-1}(-\triangle_{p(t)} u)$ in general, but $-\triangle_p cu = c^{p-1}(-\triangle_p u)$, so $-\triangle_{p(t)}$ represents a non-homogeneity and possesses more nonlinearity, thus $-\triangle_{p(t)}$ is more complicated than $-\triangle_p$. For example:

(a) If $\Omega \subset \mathbb{R}^N$ is a bounded domain, the Rayleigh quotient

$$\lambda_{p(x)} = \inf_{u \in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\,dx}{\int_\Omega \frac{1}{p(x)} |u|^{p(x)}\,dx}$$

is zero in general, and only under some special conditions $\lambda_{p(x)} > 0$ (see [9]), when $\Omega \subset \mathbb{R}$ ($N = 1$) is an interval, the results show that $\lambda_{p(x)} > 0$ if and only if $p(x)$ is monotone. But the property of $\lambda_p > 0$ is very important in the study of p-Laplacian problems, for example, in [20], the authors use this property to deal with the existence of solutions.

(b) If $w(t) \equiv 1$ and $p(t) \equiv p$ (a constant) and $-\triangle_p u > 0$, then u is concave, this property is used extensively in the study of one-dimensional p-Laplacian problems (see [21]), but it is invalid for $-\triangle_{p(t)}$. It is another difference between $-\triangle_p$ and $-\triangle_{p(t)}$.

In recent years, many results have been devoted to the existence of solutions for the Laplacian impulsive differential equation boundary value problems; see, for example, [22–29]. There are some methods to deal with these problems, for example, sub-super-solution method, fixed point theorem, monotone iterative method, coincidence degree. Because of the nonlinear property of $-\triangle_p$, results on the existence of solutions for p-Laplacian impulsive differential equation boundary value problems are rare (see [30–33]). In [34], using the coincidence degree method, the present author investigates the existence of solutions for $p(r)$-Laplacian impulsive differential equation with multi-point boundary value conditions, when the problem is nonresonant. Integral boundary conditions for evolution problems have various applications in chemical engineering, thermo-elasticity, underground water flow and population dynamics. There are many papers on the differential equations with integral boundary value problems; see, for example, [35–38].

In this paper, when $p(t)$ is a general function, we investigate the existence of solutions and nonnegative solutions for the weighted $p(t)$-Laplacian impulsive integro-differential system with integral and multi-point boundary value conditions. Results on these kinds of problems are rare. Our results contain both of the cases of resonance and nonresonance. Our method is based upon Leray-Schauder's degree. The homotopy transformation used in [34] is unsuitable for this paper. Moreover, this paper will consider the existence of (1) with (2), (4) and the following impulsive condition:

$$\begin{aligned} &\lim_{t \to t_i^+} \bigl(w(t)\bigr)^{\frac{1}{p(t)-1}} u'(t) - \lim_{t \to t_i^-} \bigl(w(t)\bigr)^{\frac{1}{p(t)-1}} u'(t) \\ &\quad = D_i\Bigl(\lim_{t \to t_i^-} u(t), \lim_{t \to t_i^-} \bigl(w(t)\bigr)^{\frac{1}{p(t)-1}} u'(t)\Bigr), \quad i = 1, \ldots, k, \end{aligned} \tag{5}$$

where $D_i \in C(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N)$, the impulsive condition (5) is called a linear impulsive condition (LI for short), and (3) is called a nonlinear impulsive condition (NLI for short). In general, p-Laplacian impulsive problems have two kinds of impulsive conditions, including LI and NLI; but Laplacian impulsive problems only have LI in general. It is another difference between p-Laplacian impulsive problems and Laplacian impulsive problems.

Moreover, since the Rayleigh quotient $\lambda_{p(x)} = 0$ in general and the $p(t)$-Laplacian is non-homogeneity, when we deal with the existence of solutions of variable exponent impulsive problems like (1)-(4), we usually need the nonlinear term that satisfies the sub-$(p^- - 1)$ growth condition, but for the p-Laplacian impulsive problems, the nonlinear term only needs to satisfy the sub-$(p - 1)$ growth condition.

Let $N \geq 1$, the function $f : J \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is assumed to be Caratheodory, by which we mean:

(i) For almost every $t \in J$, the function $f(t, \cdot, \cdot, \cdot, \cdot)$ is continuous;
(ii) For each $(x, y, s, z) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$, the function $f(\cdot, x, y, s, z)$ is measurable on J;
(iii) For each $R > 0$, there is a $\alpha_R \in L^1(J, \mathbb{R})$ such that, for almost every $t \in J$ and every $(x, y, s, z) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$ with $|x| \leq R$, $|y| \leq R$, $|s| \leq R$, $|z| \leq R$, one has

$$\left|f(t, x, y, s, z)\right| \leq \alpha_R(t).$$

We say a function $u : J \to \mathbb{R}^N$ is a solution of (1) if $u \in PC^1$ with $w(t)|u'|^{p(t)-2}u'$ absolutely continuous on J_i^o, $i = 0, 1, \ldots, k$, which satisfies (1) a.e. on J.

In this paper, we always use C_i to denote positive constants, if it cannot lead to confusion. Denote

$$z^- = \inf_{t \in J} z(t), \qquad z^+ = \sup_{t \in J} z(t) \quad \text{for any } z \in PC(J, \mathbb{R}).$$

We say f satisfies the sub-$(p^- - 1)$ growth condition if f satisfies

$$\lim_{|u|+|v|+|s|+|z| \to +\infty} \frac{f(t, u, v, s, z)}{(|u| + |v| + |s| + |z|)^{q(t)-1}} = 0 \quad \text{for } t \in J \text{ uniformly},$$

where $q(t) \in PC(J, \mathbb{R})$ and $1 < q^- \leq q^+ < p^-$.

We will discuss the existence of solutions for system (1)-(4) or (1) with (2), (4) and (5) in the following three cases:

Case (i): $\sigma < 1$, $\sum_{\ell=1}^{m-2} \alpha_\ell - \delta = 1$;
Case (ii): $\sigma = 1$, $\sum_{\ell=1}^{m-2} \alpha_\ell - \delta \neq 1$;
Case (iii): $\sigma < 1$, $\sum_{\ell=1}^{m-2} \alpha_\ell - \delta < 1$.

This paper is organized as five sections. In Section 2, we present some preliminaries and give the operator equation which has the same solutions of (1)-(4) in the three cases, respectively. In Section 3, we give the existence of solutions for system (1)-(4) or (1) with (2), (4) and (5) when $\sigma < 1$, $\sum_{\ell=1}^{m-2} \alpha_\ell - \delta = 1$. In Section 4, we give the existence of solutions for system (1)-(4) or (1) with (2), (4) and (5) when $\sigma = 1$, $\sum_{\ell=1}^{m-2} \alpha_\ell - \delta \neq 1$. Finally, in Section 5, we give the existence of solutions and nonnegative solutions for system (1)-(4) or (1) with (2), (4) and (5) when $\sigma < 1$, $\sum_{\ell=1}^{m-2} \alpha_\ell - \delta < 1$.

2 Preliminary

For any $(t, x) \in J \times \mathbb{R}^N$, denote $\varphi(t, x) = |x|^{p(t)-2}x$. Obviously, φ has the following properties.

Lemma 2.1 (see [34]) *φ is a continuous function and satisfies*:

(i) *For any* $t \in [0,1]$, $\varphi(t,\cdot)$ *is strictly monotone, i.e.*,

$$\langle \varphi(t,x_1) - \varphi(t,x_2), x_1 - x_2 \rangle > 0 \quad \text{for any } x_1, x_2 \in \mathbb{R}^N, x_1 \neq x_2.$$

(ii) *There exists a function* $\alpha : [0,+\infty) \to [0,+\infty)$, $\alpha(s) \to +\infty$ *as* $s \to +\infty$ *such that*

$$\langle \varphi(t,x), x \rangle \geq \alpha(|x|)|x| \quad \text{for all } x \in \mathbb{R}^N.$$

It is well known that $\varphi(t,\cdot)$ is a homeomorphism from $\mathbb{R}^N$ to $\mathbb{R}^N$ for any fixed $t \in J$. Denote

$$\varphi^{-1}(t,x) = |x|^{\frac{2-p(t)}{p(t)-1}} x \quad \text{for } x \in \mathbb{R}^N \setminus \{0\}, \varphi^{-1}(t,0) = 0, \forall t \in J.$$

It is clear that $\varphi^{-1}(t,\cdot)$ is continuous and sends bounded sets to bounded sets.

In this section, we will do some preparation and give the operator equation which has the same solutions of (1)-(4) in three cases, respectively. At first, let us now consider the following simple impulsive problem with boundary value condition (4):

$$\left.\begin{array}{l} (w(t)\varphi(t,u'(t)))' = f(t), \quad t \in (0,1), t \neq t_i, \\ \lim_{t\to t_i^+} u(t) - \lim_{t\to t_i^-} u(t) = a_i, \quad i = 1,\ldots,k, \\ \lim_{t\to t_i^+} w(t)|u'|^{p(t)-2}u'(t) - \lim_{t\to t_i^-} w(t)|u'|^{p(t)-2}u'(t) = b_i, \quad i = 1,\ldots,k, \end{array}\right\} \tag{6}$$

where $a_i, b_i \in \mathbb{R}^N$; $f \in L^1$.

Denote $a = (a_1,\ldots,a_k)$, $b = (b_1,\ldots,b_k)$. Obviously, $a, b \in \mathbb{R}^{kN}$.

We will discuss it in three cases, respectively.

2.1 Case (i)

Suppose that $\sigma < 1$ and $\sum_{\ell=1}^{m-2} \alpha_\ell - \delta = 1$. If u is a solution of (6) with (4), we have

$$w(t)\varphi(t,u'(t)) = w(0)\varphi(0,u'(0)) + \sum_{t_i<t} b_i + \int_0^t f(s)\,ds, \quad \forall t \in J'. \tag{7}$$

Denote $\rho_1 = w(0)\varphi(0,u'(0))$. It is easy to see that ρ_1 is dependent on a, b and $f(\cdot)$. Define the operator $F : L^1 \to PC$ as

$$F(f)(t) = \int_0^t f(s)\,ds, \quad \forall t \in J, \forall f \in L^1.$$

By solving for u' in (7) and integrating, we find

$$u(t) = u(0) + \sum_{t_i<t} a_i + F\left\{\varphi^{-1}\left[t, (w(t))^{-1}\left(\rho_1 + \sum_{t_i<t} b_i + F(f)(t)\right)\right]\right\}(t), \quad \forall t \in J,$$

which together with boundary value condition (4) implies

$$u(0) = \frac{1}{(1-\sigma)} \int_0^1 g(t)\left(F\left\{\varphi^{-1}\left[t, (w(t))^{-1}\left(\rho_1 + \sum_{t_i<t} b_i + F(f)(t)\right)\right]\right\}(t) + \sum_{t_i<t} a_i\right) dt,$$

and

$$\sum_{\ell=1}^{m-2}\alpha_\ell\left\{\sum_{t_i<\xi_\ell}a_i+\int_0^{\xi_\ell}\varphi^{-1}\left[t,\left(w(t)\right)^{-1}\left(\rho_1+\sum_{t_i<t}b_i+F(f)(t)\right)\right]dt\right\}$$
$$-\sum_{i=1}^{k}a_i-\int_0^1\varphi^{-1}\left[t,\left(w(t)\right)^{-1}\left(\rho_1+\sum_{t_i<t}b_i+F(f)(t)\right)\right]dt$$
$$-\int_0^1 h(t)\left(F\left\{\varphi^{-1}\left[t,\left(w(t)\right)^{-1}\left(\rho_1+\sum_{t_i<t}b_i+F(f)(t)\right)\right]\right\}(t)+\sum_{t_i<t}a_i\right)dt=0.$$

Denote $W=\mathbb{R}^{2kN}\times L^1$ with the norm

$$\|\omega\|=\sum_{i=1}^{k}|a_i|+\sum_{i=1}^{k}|b_i|+\|\vartheta\|_{L^1},\quad \forall\omega=(a,b,\vartheta)\in W,$$

then W is a Banach space.

For any $\omega\in W$, we denote

$$\Lambda_\omega(\rho_1)=\sum_{\ell=1}^{m-2}\alpha_\ell\left\{\sum_{t_i<\xi_\ell}a_i+\int_0^{\xi_\ell}\varphi^{-1}\left[t,\left(w(t)\right)^{-1}\left(\rho_1+\sum_{t_i<t}b_i+F(\vartheta)(t)\right)\right]dt\right\}$$
$$-\sum_{i=1}^{k}a_i-\int_0^1\varphi^{-1}\left[t,\left(w(t)\right)^{-1}\left(\rho_1+\sum_{t_i<t}b_i+F(\vartheta)(t)\right)\right]dt$$
$$-\int_0^1 h(t)\left(F\left\{\varphi^{-1}\left[t,\left(w(t)\right)^{-1}\left(\rho_1+\sum_{t_i<t}b_i+F(\vartheta)(t)\right)\right]\right\}(t)+\sum_{t_i<t}a_i\right)dt.$$

Denote $\xi_{m-1}=1$. Then

$$\Lambda_\omega(\rho_1)=-\sum_{\ell=1}^{m-2}\alpha_\ell\left\{\sum_{\xi_\ell\le t_i}a_i+\int_{\xi_\ell}^{1}\varphi^{-1}\left[t,\left(w(t)\right)^{-1}\left(\rho_1+\sum_{t_i<t}b_i+F(\vartheta)(t)\right)\right]dt\right\}$$
$$+\int_0^1 h(t)\left(\int_t^1\varphi^{-1}\left[t,\left(w(t)\right)^{-1}\left(\rho_1+\sum_{t_i<t}b_i+F(\vartheta)(t)\right)\right]dt+\sum_{t_i\ge t}a_i\right)dt$$
$$=-\sum_{\ell=1}^{m-2}\left(\alpha_\ell-\int_{\xi_\ell}^{\xi_{\ell+1}}h(t)\,dt\right)\int_{\xi_\ell}^{1}\varphi^{-1}\left[t,\left(w(t)\right)^{-1}\left(\rho_1+\sum_{t_i<t}b_i+F(\vartheta)(t)\right)\right]dt$$
$$-\sum_{\ell=1}^{m-2}\int_{\xi_\ell}^{\xi_{\ell+1}}h(t)\int_{\xi_\ell}^{t}\varphi^{-1}\left[s,\left(w(s)\right)^{-1}\left(\rho_1+\sum_{s_i<s}b_i+F(\vartheta)(s)\right)\right]ds\,dt$$
$$+\int_0^{\xi_1}h(t)\int_t^1\varphi^{-1}\left[s,\left(w(s)\right)^{-1}\left(\rho_1+\sum_{s_i<s}b_i+F(\vartheta)(s)\right)\right]ds\,dt$$
$$-\sum_{\ell=1}^{m-2}\alpha_\ell\sum_{\xi_\ell\le t_i}a_i+\int_0^1 h(t)\sum_{t_i\ge t}a_i\,dt.$$

Throughout the paper, we denote

$$\begin{aligned}E &= \int_0^{\xi_1} \left|h(t)\right| \int_t^1 \left(w(s)\right)^{\frac{-1}{p(s)-1}}\,ds\,dt + \sum_{\ell=1}^{m-2} \int_{\xi_\ell}^{\xi_{\ell+1}} h(t) \int_{\xi_\ell}^{t} \left(w(s)\right)^{\frac{-1}{p(s)-1}}\,ds\,dt \\ &\quad + \sum_{\ell=1}^{m-2}\left(\alpha_\ell - \int_{\xi_\ell}^{\xi_{\ell+1}} h(t)\,dt\right) \int_{\xi_\ell}^{1} \left(w(s)\right)^{\frac{-1}{p(s)-1}}\,ds, \\ \delta^* &= \sum_{\ell=1}^{m-2} \alpha_\ell + \int_0^1 \left|h(t)\right|\,dt.\end{aligned}$$

Lemma 2.2 *Suppose that* $h(t) \geq 0$ *on* $[\xi_1, 1]$, $\alpha_\ell \geq \int_{\xi_\ell}^{\xi_{\ell+1}} h(t)\,dt$ ($\ell = 1, \ldots, m-2$) *and* $h(t) \leq 0$ *on* $[0, \xi_1]$. *Then the function* $\Lambda_\omega(\cdot)$ *has the following properties*:

(i) *For any fixed* $\omega \in W$, *the equation*

$$\Lambda_\omega(\rho_1) = 0 \tag{8}$$

has a unique solution $\widetilde{\rho_1}(\omega) \in \mathbb{R}^N$.

(ii) *The function* $\widetilde{\rho_1} : W \to \mathbb{R}^N$, *defined in* (i), *is continuous and sends bounded sets to bounded sets. Moreover, for any* $\omega = (a, b, \vartheta) \in W$, *we have*

$$\left|\widetilde{\rho_1}(\omega)\right| \leq 3N\left[(2N)^{p^+}\left(\delta^* \frac{E+1}{E} \sum_{i=1}^{k} |a_i|\right)^{p^\#-1} + \sum_{i=1}^{k} |b_i| + \|\vartheta\|_{L^1}\right],$$

where the notation $M^{p^\#-1}$ *means*

$$M^{p^\#-1} = \begin{cases} M^{p^+-1}, & M > 1, \\ M^{p^--1}, & M \leq 1. \end{cases}$$

Proof (i) From Lemma 2.1, it is immediate that

$$\left\langle \Lambda_\omega(x_1) - \Lambda_\omega(x_2), x_1 - x_2 \right\rangle < 0 \quad \text{for } x_1 \neq x_2, \forall x_1, x_2 \in \mathbb{R}^N,$$

and hence, if (8) has a solution, then it is unique.

Set $R_0 = 3N[(2N)^{p^+}(\delta^* \frac{E+1}{E} \sum_{i=1}^{k} |a_i|)^{p^\#-1} + \sum_{i=1}^{k} |b_i| + \|\vartheta\|_{L^1}]$.

Suppose that $|\rho_1| > R_0$, it is easy to see that there exists some $j_0 \in \{1, \ldots, N\}$ such that the absolute value of the j_0th component $\rho_1^{j_0}$ of ρ_1 satisfies

$$\left|\rho_1^{j_0}\right| \geq \frac{|\rho_1|}{N} > 3\left[(2N)^{p^+}\left(\delta^* \frac{E+1}{E} \sum_{i=1}^{k} |a_i|\right)^{p^\#-1} + \sum_{i=1}^{k} |b_i| + \|\vartheta\|_{L^1}\right].$$

Thus the j_0th component of $\rho_1 + \sum_{t_i<t} b_i + F(\vartheta)(t)$ keeps sign on J, namely, for any $t \in J$, we have

$$\left|\left(\rho_1^{j_0} + \sum_{t_i<t} b_i^{j_0} + F(\vartheta)^{j_0}(t)\right)\right| \geq \frac{2|\rho_1|}{3N} > (2N)^{p^+}\left(\delta^* \frac{E+1}{E} \sum_{i=1}^{k} |a_i|\right)^{p^\#-1} + \sum_{i=1}^{k} |b_i| + \|\vartheta\|_{L^1}.$$

Obviously, we have

$$\left|\left(\rho_1 + \sum_{t_i<t} b_i + F(\vartheta)(t)\right)\right| \le \frac{4|\rho_1|}{3} \le 2N\left|\left(\rho_1^{j_0} + \sum_{t_i<t} b_i^{j_0} + F(\vartheta)^{j_0}(t)\right)\right|,$$

then it is easy to see that the j_0th component of $\Lambda_\omega(\rho_1)$ keeps the same sign of $\rho_1^{j_0}$. Thus,

$$\Lambda_\omega(\rho_1) \ne 0.$$

Let us consider the equation

$$\lambda\Lambda_\omega(\rho_1) + (1-\lambda)\rho_1 = 0, \quad \lambda \in [0,1]. \tag{9}$$

According to the preceding discussion, all the solutions of (9) belong to $b(R_0+1) = \{x \in \mathbb{R}^N \mid |x| < R_0 + 1\}$. Therefore

$$d_B\big[\Lambda_\omega(\rho_1), b(R_0+1), 0\big] = d_B\big[I, b(R_0+1), 0\big] \ne 0,$$

it means the existence of solutions of $\Lambda_\omega(\rho_1) = 0$.

In this way, we define a function $\widetilde{\rho}_1(\omega): W \to \mathbb{R}^N$, which satisfies $\Lambda_\omega(\widetilde{\rho}_1(\omega)) = 0$.

(ii) By the proof of (i), we also obtain $\widetilde{\rho}_1$ sends bounded sets to bounded sets, and

$$\big|\widetilde{\rho}_1(\omega)\big| \le 3N\left[(2N)^{p^+}\left(\delta^*\frac{E+1}{E}\sum_{i=1}^{k}|a_i|\right)^{p^\#-1} + \sum_{i=1}^{k}|b_i| + \|\vartheta\|_{L^1}\right].$$

It only remains to prove the continuity of $\widetilde{\rho}_1$. Let $\{\omega_n\}$ be a convergent sequence in W and $\omega_n \to \omega$, as $n \to +\infty$. Since $\{\widetilde{\rho}_1(\omega_n)\}$ is a bounded sequence, it contains a convergent subsequence $\{\widetilde{\rho}_1(\omega_{n_j})\}$. Suppose that $\widetilde{\rho}_1(\omega_{n_j}) \to \rho_0$ as $j \to +\infty$. Since $\Lambda_{\omega_{n_j}}(\widetilde{\rho}_1(\omega_{n_j})) = 0$, letting $j \to +\infty$, we have $\Lambda_\omega(\rho_0) = 0$, which together with (i) implies $\rho_0 = \widetilde{\rho}_1(\omega)$, it means $\widetilde{\rho}_1$ is continuous. This completes the proof. □

Now we denote by $N_f(u): [0,1] \times PC^1 \to L^1$ the Nemytskii operator associated to f defined by

$$N_f(u)(t) = f\big(t, u(t), \big(w(t)\big)^{\frac{1}{p(t)-1}} u'(t), S(u), T(u)\big) \quad \text{on } J. \tag{10}$$

We define $\rho_1: PC^1 \to \mathbb{R}^N$ as

$$\rho_1(u) = \widetilde{\rho}_1(A, B, N_f)(u), \tag{11}$$

where $A = (A_1, \ldots, A_k)$, $B = (B_1, \ldots, B_k)$.

It is clear that $\rho_1(\cdot)$ is continuous and sends bounded sets of PC^1 to bounded sets of $\mathbb{R}^N$, and hence it is compact continuous.

If u is a solution of (6) with (4), we have

$$u(t) = u(0) + \sum_{t_i<t} a_i + F\left\{\varphi^{-1}\left[t, \big(w(t)\big)^{-1}\left(\widetilde{\rho}_1(\omega) + \sum_{t_i<t} b_i + F(f)(t)\right)\right]\right\}(t), \quad \forall t \in [0,1].$$

For fixed $a, b \in \mathbb{R}^{kN}$, we denote $K_{(a,b)} : L^1 \to PC^1$ as

$$K_{(a,b)}(\vartheta)(t) = F\left\{\varphi^{-1}\left[t, \left(w(t)\right)^{-1}\left(\widetilde{\rho_1}(a,b,\vartheta) + \sum_{t_i<t} b_i + F(\vartheta)(t)\right)\right]\right\}(t), \quad \forall t \in J.$$

Define $K_1 : PC^1 \to PC^1$ as

$$K_1(u)(t) = F\left\{\varphi^{-1}\left[t, \left(w(t)\right)^{-1}\left(\rho_1(u) + \sum_{t_i<t} B_i + F\left(N_f(u)\right)(t)\right)\right]\right\}(t), \quad \forall t \in J.$$

Lemma 2.3 (i) *The operator $K_{(a,b)}$ is continuous and sends equi-integrable sets in L^1 to relatively compact sets in PC^1.*

(ii) *The operator K_1 is continuous and sends bounded sets in PC^1 to relatively compact sets in PC^1.*

Proof (i) It is easy to check that $K_{(a,b)}(\vartheta)(\cdot) \in PC^1$, $\forall \vartheta \in L^1$, $\forall a, b \in \mathbb{R}^{kN}$. Since $(w(t))^{\frac{-1}{p(t)-1}} \in L^1$ and

$$K_{(a,b)}(\vartheta)'(t) = \varphi^{-1}\left[t, \left(w(t)\right)^{-1}\left(\widetilde{\rho_1}(a,b,\vartheta) + \sum_{t_i<t} b_i + F(\vartheta)\right)\right], \quad \forall t \in [0,1],$$

it is easy to check that $K_{(a,b)}(\cdot)$ is a continuous operator from L^1 to PC^1.

Let now U be an equi-integrable set in L^1, then there exists $\alpha \in L^1$ such that

$$\left|u(t)\right| \le \alpha(t) \quad \text{a.e. in } J \text{ for any } u \in L^1.$$

We want to show that $\overline{K_{(a,b)}(U)} \subset PC^1$ is a compact set.

Let $\{u_n\}$ be a sequence in $K_{(a,b)}(U)$, then there exists a sequence $\{\vartheta_n\} \in U$ such that $u_n = K_{(a,b)}(\vartheta_n)$. For any $t_1, t_2 \in J$, we have

$$\left|F(\vartheta_n)(t_1) - F(\vartheta_n)(t_2)\right| = \left|\int_0^{t_1} \vartheta_n(t)\,dt - \int_0^{t_2} \vartheta_n(t)\,dt\right| = \left|\int_{t_1}^{t_2} \vartheta_n(t)\,dt\right| \le \left|\int_{t_1}^{t_2} \alpha(t)\,dt\right|.$$

Hence the sequence $\{F(\vartheta_n)\}$ is uniformly bounded and equi-continuous. By the Ascoli-Arzela theorem, there exists a subsequence of $\{F(\vartheta_n)\}$ (which we rename the same) which is convergent in PC. According to the bounded continuity of the operator $\widetilde{\rho_1}$, we can choose a subsequence of $\{\widetilde{\rho_1}(a,b,\vartheta_n) + F(\vartheta_n)\}$ (which we still denote $\{\widetilde{\rho_1}(a,b,\vartheta_n) + F(\vartheta_n)\}$) which is convergent in PC, then $w(t)^{\frac{1}{p(t)-1}} K_{(a,b)}(\vartheta_n)'(t) = \varphi^{-1}(t, \widetilde{\rho_1}(a,b,\vartheta_n) + \sum_{t_i<t} b_i + F(\vartheta_n))$ is convergent in PC.

Since

$$K_{(a,b)}(\vartheta_n)(t) = F\left\{\varphi^{-1}\left[t, \left(w(t)\right)^{-1}\left(\widetilde{\rho_1}(a,b,\vartheta_n) + \sum_{t_i<t} b_i + F(\vartheta_n)\right)\right]\right\}(t), \quad \forall t \in [0,1],$$

it follows from the continuity of φ^{-1} and the integrability of $w(t)^{\frac{-1}{p(t)-1}}$ in L^1 that $K_{(a,b)}(\vartheta_n)$ is convergent in PC. Thus $\{u_n\}$ is convergent in PC^1.

(ii) It is easy to see from (i) and Lemma 2.2.

This completes the proof. □

Let us define $P_1 : PC^1 \to PC^1$ as

$$P_1(u) = \frac{\int_0^1 g(t)[K_1(u)(t) + \sum_{t_i<t} A_i]\,dt}{1-\sigma}.$$

It is easy to see that P_1 is compact continuous.

Lemma 2.4 *Suppose that* $\sigma < 1$, $\sum_{\ell=1}^{m-2} \alpha_\ell - \delta = 1$; $h(t) \ge 0$ *on* $[\xi_1, 1]$, $\alpha_\ell \ge \int_{\xi_\ell}^{\xi_{\ell+1}} h(t)\,dt$ $(\ell = 1, \dots, m-2)$ *and* $h(t) \le 0$ *on* $[0, \xi_1]$. *Then* u *is a solution of* (1)-(4) *if and only if* u *is a solution of the following abstract operator equation*:

$$u = P_1(u) + \sum_{t_i<t} A_i + K_1(u). \tag{12}$$

Proof Suppose that u is a solution of (1)-(4). By integrating (1) from 0 to t, we find that

$$w(t)\varphi\big(t, u'(t)\big) = \rho_1(u) + \sum_{t_i<t} B_i + F\big(N_f(u)\big)(t), \quad \forall t \in (0,1), t \ne t_1, \dots, t_k. \tag{13}$$

It follows from (13) and (4) that

$$\begin{aligned} u(t) &= u(0) + \sum_{t_i<t} A_i \\ &\quad + F\left\{\varphi^{-1}\left[t, \big(w(t)\big)^{-1}\left(\rho_1(u) + \sum_{t_i<t} B_i + F\big(N_f(u)\big)\right)\right]\right\}(t), \quad \forall t \in [0,1], \\ u(0) &= \frac{1}{(1-\sigma)} \\ &\quad \times \int_0^1 g(t)\left(F\left\{\varphi^{-1}\left[t, \big(w(t)\big)^{-1}\left(\rho_1(u) + \sum_{t_i<t} B_i + F\big(N_f(u)\big)\right)\right]\right\}(t) + \sum_{t_i<t} A_i\right) dt \\ &= \frac{\int_0^1 g(t)[K_1(u)(t) + \sum_{t_i<t} A_i]\,dt}{1-\sigma} = P_1(u). \end{aligned} \tag{14}$$

Combining the definition of ρ_1, we can see

$$u = P_1(u) + \sum_{t_i<t} A_i + K_1(u).$$

Conversely, if u is a solution of (12), then (2) is satisfied. It is easy to check that

$$\begin{aligned} u(0) &= P_1(u) = \frac{\int_0^1 g(t)[K_1(u)(t) + \sum_{t_i<t} A_i]\,dt}{1-\sigma}, \\ u(0) &= \sigma u(0) + \int_0^1 g(t)\left[K_1(u)(t) + \sum_{t_i<t} A_i\right] dt = \int_0^1 g(t)u(t)\,dt, \end{aligned} \tag{15}$$

and

$$u(1) = P_1(u) + \sum_{i=1}^{k} A_i + K_1(u)(1).$$

By the condition of the mapping ρ_1, we have

$$\begin{aligned}&\sum_{\ell=1}^{m-2}\alpha_\ell\left\{\sum_{t_i<\xi_\ell}A_i+\int_0^{\xi_\ell}\varphi^{-1}\left[t,\left(w(t)\right)^{-1}\left(\rho_1+\sum_{t_i<t}B_i+F\left(N_f(u)\right)(t)\right)\right]dt\right\}\\&\quad-\sum_{i=1}^{k}A_i-\int_0^1\varphi^{-1}\left[t,\left(w(t)\right)^{-1}\left(\rho_1+\sum_{t_i<t}B_i+F\left(N_f(u)\right)(t)\right)\right]dt\\&\quad-\int_0^1 h(t)\left(F\left\{\varphi^{-1}\left[t,\left(w(t)\right)^{-1}\left(\rho_1+\sum_{t_i<t}B_i+F\left(N_f(u)\right)(t)\right)\right]\right\}(t)+\sum_{t_i<t}A_i\right)dt=0.\end{aligned}$$

Thus

$$u(1)=\sum_{\ell=1}^{m-2}\alpha_\ell u(\xi_\ell)-\int_0^1 h(t)u(t)\,dt. \tag{16}$$

It follows from (15) and (16) that (4) is satisfied.

From (12), we have

$$w(t)\varphi\left(t,u'(t)\right)=\rho_1(u)+\sum_{t_i<t}B_i+F\left(N_f(u)\right)(t),\quad t\in(0,1),t\neq t_i, \tag{17}$$

$$\left(w(t)\varphi\left(t,u'\right)\right)'=N_f(u)(t),\quad t\in(0,1),t\neq t_i.$$

It follows from (17) that (3) is satisfied.

Hence u is a solution of (1)-(4). This completes the proof. □

2.2 Case (ii)

Suppose that $\sigma=1$ and $\sum_{\ell=1}^{m-2}\alpha_\ell-\delta\neq 1$. If u is a solution of (6) with (4), we have

$$w(t)\varphi\left(t,u'(t)\right)=w(0)\varphi\left(0,u'(0)\right)+\sum_{t_i<t}b_i+\int_0^t f(s)\,ds,\quad\forall t\in J'.$$

Denote $\rho_2=w(0)\varphi(0,u'(0))$. It is easy to see that ρ_2 is dependent on a, b and $f(\cdot)$. Boundary value condition (4) implies that

$$\int_0^1 g(t)\left(F\left\{\varphi^{-1}\left[t,\left(w(t)\right)^{-1}\left(\rho_2+\sum_{t_i<t}b_i+F(f)(t)\right)\right]\right\}(t)+\sum_{t_i<t}a_i\right)dt=0,$$

$$\begin{aligned}u(0)=&\frac{\sum_{\ell=1}^{m-2}\alpha_\ell\{\sum_{t_i<\xi_\ell}a_i+\int_0^{\xi_\ell}\varphi^{-1}[t,(w(t))^{-1}(\rho_2+\sum_{t_i<t}b_i+F(f)(t))]\,dt\}}{1-\sum_{i=1}^{m-2}\alpha_\ell+\delta}\\&-\frac{\sum_{i=1}^{k}a_i+\int_0^1\varphi^{-1}[t,(w(t))^{-1}(\rho_2+\sum_{t_i<t}b_i+F(f)(t))]\,dt}{1-\sum_{\ell=1}^{m-2}\alpha_\ell+\delta}\\&-\frac{\int_0^1 h(t)(F\{\varphi^{-1}[t,(w(t))^{-1}(\rho_2+\sum_{t_i<t}b_i+F(f)(t))]\}(t)+\sum_{t_i<t}a_i)\,dt}{1-\sum_{\ell=1}^{m-2}\alpha_\ell+\delta}.\end{aligned}$$

For any $\omega\in W$, we denote

$$\Gamma_\omega(\rho_2)=\int_0^1 g(t)\left(F\left\{\varphi^{-1}\left[t,\left(w(t)\right)^{-1}\left(\rho_2+\sum_{t_i<t}b_i+F(\vartheta)(t)\right)\right]\right\}(t)+\sum_{t_i<t}a_i\right)dt.$$

Throughout the paper, we denote $E_1 = \int_0^1 (w(t))^{\frac{-1}{p(t)-1}}\,dt$.

Lemma 2.5 *The function $\Gamma_\omega(\cdot)$ has the following properties:*

(i) *For any fixed $\omega \in W$, the equation $\Gamma_\omega(\rho_2) = 0$ has a unique solution $\widetilde{\rho}_2(\omega) \in \mathbb{R}^N$.*

(ii) *The function $\widetilde{\rho}_2 : W \to \mathbb{R}^N$, defined in (i), is continuous and sends bounded sets to bounded sets. Moreover, for any $\omega = (a, b, \vartheta) \in W$, we have*

$$\left|\widetilde{\rho}_2(\omega)\right| \le 3N\left[(2N)^{p^+}\left(\frac{E_1+1}{E_1}\sum_{i=1}^{k}|a_i|\right)^{p^\#-1} + \sum_{i=1}^{k}|b_i| + \|\vartheta\|_{L^1}\right],$$

where the notation $M^{p^\#-1}$ means

$$M^{p^\#-1} = \begin{cases} M^{p^+-1}, & M > 1, \\ M^{p^--1}, & M \le 1. \end{cases}$$

Proof Similar to the proof of Lemma 2.2, we omit it here. □

We define $\rho_2 : PC^1 \to \mathbb{R}^N$ as $\rho_2(u) = \widetilde{\rho}_2(A, B, N_f)(u)$, where $A = (A_1, \dots, A_k)$, $B = (B_1, \dots, B_k)$.

It is clear that $\rho_2(\cdot)$ is continuous and sends bounded sets of PC^1 to bounded sets of $\mathbb{R}^N$, and hence it is compact continuous.

For fixed $a, b \in \mathbb{R}^{kN}$, we denote $K^*_{(a,b)} : L^1 \to PC^1$ as

$$K^*_{(a,b)}(\vartheta)(t) = F\left\{\varphi^{-1}\left[t, \left(w(t)\right)^{-1}\left(\widetilde{\rho}_2(a, b, \vartheta) + \sum_{t_i<t} b_i + F(\vartheta)(t)\right)\right]\right\}(t), \quad \forall t \in J.$$

Define $K_2 : PC^1 \to PC^1$ as

$$K_2(u)(t) = F\left\{\varphi^{-1}\left[t, \left(w(t)\right)^{-1}\left(\rho_2(u) + \sum_{t_i<t} B_i + F\left(N_f(u)\right)(t)\right)\right]\right\}(t), \quad \forall t \in J.$$

Similar to the proof of Lemma 2.3, we have the following.

Lemma 2.6 (i) *The operator $K^*_{(a,b)}$ is continuous and sends equi-integrable sets in L^1 to relatively compact sets in PC^1.*

(ii) *The operator K_2 is continuous and sends bounded sets in PC^1 to relatively compact sets in PC^1.*

Let us define $P_2 : PC^1 \to PC^1$ as

$$P_2(u) = \frac{\sum_{\ell=1}^{m-2}\alpha_\ell[\sum_{t_i<\xi_\ell} A_i + K_2(u)(\xi_\ell)] - \sum_{i=1}^{k} A_i}{1 - \sum_{\ell=1}^{m-2}\alpha_\ell + \delta} - \frac{K_2(u)(1) + \int_0^1 h(t)[K_2(u)(t) + \sum_{t_i<t} A_i]\,dt}{1 - \sum_{\ell=1}^{m-2}\alpha_\ell + \delta}.$$

It is easy to see that P_2 is compact continuous.

Lemma 2.7 *Suppose that* $\sigma = 1$, $\sum_{\ell=1}^{m-2} \alpha_\ell - \delta \neq 1$, *then* u *is a solution of* (1)-(4) *if and only if* u *is a solution of the following abstract operator equation*:

$$u = P_2(u) + \sum_{t_i<t} A_i + K_2(u).$$

Proof Similar to the proof of Lemma 2.4, we omit it here. □

2.3 Case (iii)

Suppose that $\sigma < 1$ and $\sum_{\ell=1}^{m-2} \alpha_\ell - \delta < 1$. If u is a solution of (6) with (4), we have

$$w(t)\varphi\bigl(t, u'(t)\bigr) = w(0)\varphi\bigl(0, u'(0)\bigr) + \sum_{t_i<t} b_i + \int_0^t f(s)\,ds, \quad \forall t \in J'.$$

Denote $\rho_3 = w(0)\varphi(0, u'(0))$. It is easy to see that ρ_3 is dependent on a, b and $f(\cdot)$.
From $u(0) = \int_0^1 g(t)u(t)\,dt$, we have

$$\begin{aligned} u(0) = {} & \frac{1}{(1-\sigma)} \\ & \times \int_0^1 g(t)\biggl(F\biggl\{\varphi^{-1}\biggl[t, \bigl(w(t)\bigr)^{-1}\biggl(\rho_3 + \sum_{t_i<t} b_i + F(f)(t)\biggr)\biggr]\biggr\}(t) + \sum_{t_i<t} a_i\biggr)\,dt. \end{aligned} \tag{18}$$

From $u(1) = \sum_{\ell=1}^{m-2} \alpha_\ell u(\xi_\ell) - \int_0^1 h(t)u(t)\,dt$, we obtain

$$\begin{aligned} u(0) = {} & \frac{\sum_{\ell=1}^{m-2} \alpha_\ell \{\sum_{t_i<\xi_\ell} a_i + \int_0^{\xi_\ell} \varphi^{-1}[t, (w(t))^{-1}(\rho_3 + \sum_{t_i<t} b_i + F(f)(t))]\,dt\}}{1 - \sum_{\ell=1}^{m-2} \alpha_\ell + \delta} \\ & - \frac{\sum_{i=1}^{k} a_i + \int_0^1 \varphi^{-1}[t, (w(t))^{-1}(\rho_3 + \sum_{t_i<t} b_i + F(f)(t))]\,dt}{1 - \sum_{\ell=1}^{m-2} \alpha_\ell + \delta} \\ & - \frac{\int_0^1 h(t)(F\{\varphi^{-1}[t, (w(t))^{-1}(\rho_3 + \sum_{t_i<t} b_i + F(f)(t))]\}(t) + \sum_{t_i<t} a_i)\,dt}{1 - \sum_{\ell=1}^{m-2} \alpha_\ell + \delta}. \end{aligned} \tag{19}$$

For fixed $\omega \in W$, we denote

$$\begin{aligned} \Upsilon_\omega(\rho_3) = {} & \frac{1}{(1-\sigma)} \int_0^1 g(t)\biggl(F\biggl\{\varphi^{-1}\biggl[t, \bigl(w(t)\bigr)^{-1}\biggl(\rho_3 + \sum_{t_i<t} b_i + F(\vartheta)(t)\biggr)\biggr]\biggr\}(t) + \sum_{t_i<t} a_i\biggr)\,dt \\ & - \frac{\sum_{\ell=1}^{m-2} \alpha_\ell \{\sum_{t_i<\xi_\ell} a_i + \int_0^{\xi_\ell} \varphi^{-1}[t, (w(t))^{-1}(\rho_3 + \sum_{t_i<t} b_i + F(\vartheta)(t))]\,dt\}}{1 - \sum_{\ell=1}^{m-2} \alpha_\ell + \delta} \\ & + \frac{\sum_{i=1}^{k} a_i + \int_0^1 \varphi^{-1}[t, (w(t))^{-1}(\rho_3 + \sum_{t_i<t} b_i + F(\vartheta)(t))]\,dt}{1 - \sum_{\ell=1}^{m-2} \alpha_\ell + \delta} \\ & + \frac{\int_0^1 h(t)(F\{\varphi^{-1}[t, (w(t))^{-1}(\rho_3 + \sum_{t_i<t} b_i + F(\vartheta)(t))]\}(t) + \sum_{t_i<t} a_i)\,dt}{1 - \sum_{\ell=1}^{m-2} \alpha_\ell + \delta}, \\ & \forall \rho_3 \in \mathbb{R}^N. \end{aligned}$$

From (18) and (19), we have $\Upsilon_\omega(\rho_3) = 0$.

Obviously, $\Upsilon_\omega(\rho_3)$ can be rewritten as

$$\begin{aligned}\Upsilon_\omega(\rho_3) = {}& \frac{1}{(1-\sigma)}\int_0^1 g(t)\left(F\left\{\varphi^{-1}\left[t,\left(w(t)\right)^{-1}\left(\rho_3+\sum_{t_i<t}b_i+F(\vartheta)(t)\right)\right]\right\}(t)+\sum_{t_i<t}a_i\right)dt \\ & + \frac{\sum_{\ell=1}^{m-2}\alpha_\ell\{\sum_{\xi_\ell\le t_i}a_i+\int_{\xi_\ell}^1\varphi^{-1}[t,(w(t))^{-1}(\rho_3+\sum_{t_i<t}b_i+F(\vartheta)(t))]\,dt\}}{1-\sum_{\ell=1}^{m-2}\alpha_\ell+\delta} \\ & + \frac{(1-\sum_{\ell=1}^{m-2}\alpha_\ell)\int_0^1\varphi^{-1}[t,(w(t))^{-1}(\rho_3+\sum_{t_i<t}b_i+F(\vartheta)(t))]\,dt}{1-\sum_{\ell=1}^{m-2}\alpha_\ell+\delta} \\ & + \frac{\sum_{i=1}^k a_i(1-\sum_{\ell=1}^{m-2}\alpha_\ell)}{1-\sum_{\ell=1}^{m-2}\alpha_\ell+\delta} \\ & + \frac{\int_0^1 h(t)(F\{\varphi^{-1}[t,(w(t))^{-1}(\rho_3+\sum_{t_i<t}b_i+F(\vartheta)(t))]\}(t)+\sum_{t_i<t}a_i)\,dt}{1-\sum_{\ell=1}^{m-2}\alpha_\ell+\delta}.\end{aligned}$$

Denote $\xi_{m-1}=1$. Moreover, we also have

$$\begin{aligned}&\Upsilon_\omega(\rho_3) \\ &\quad = \frac{1}{(1-\sigma)}\int_0^1 g(t)\left(F\left\{\varphi^{-1}\left[t,\left(w(t)\right)^{-1}\left(\rho_3+\sum_{t_i<t}b_i+F(\vartheta)(t)\right)\right]\right\}(t)+\sum_{t_i<t}a_i\right)dt \\ &\qquad + \frac{\sum_{\ell=1}^{m-2}\alpha_\ell\sum_{\xi_\ell\le t_i}a_i}{1-\sum_{\ell=1}^{m-2}\alpha_\ell+\delta} \\ &\qquad + \frac{\sum_{\ell=1}^{m-2}(\alpha_\ell-\int_{\xi_\ell}^{\xi_{\ell+1}}h(t)\,dt)\int_{\xi_\ell}^1\varphi^{-1}[t,(w(t))^{-1}(\rho_3+\sum_{t_i<t}b_i+F(\vartheta)(t))]\,dt}{1-\sum_{\ell=1}^{m-2}\alpha_\ell+\delta} \\ &\qquad + \frac{\sum_{\ell=1}^{m-2}\int_{\xi_\ell}^{\xi_{\ell+1}}h(t)\int_{\xi_\ell}^t\varphi^{-1}[s,(w(s))^{-1}(\rho_3+\sum_{s_i<s}b_i+F(\vartheta)(s))]\,ds\,dt}{1-\sum_{\ell=1}^{m-2}\alpha_\ell+\delta} \\ &\qquad - \frac{\int_0^{\xi_1}h(t)\int_t^1\varphi^{-1}[s,(w(s))^{-1}(\rho_3+\sum_{s_i<s}b_i+F(\vartheta)(s))]\,ds\,dt+\int_0^1 h(t)\sum_{t_i\ge t}a_i\,dt}{1-\sum_{\ell=1}^{m-2}\alpha_\ell+\delta} \\ &\qquad + \int_0^1\varphi^{-1}\left[t,\left(w(t)\right)^{-1}\left(\rho_3+\sum_{t_i<t}b_i+F(\vartheta)(t)\right)\right]dt+\sum_{i=1}^k a_i.\end{aligned}$$

Lemma 2.8 *Suppose that* α_ℓ, g, h *satisfy one of the following*:

(1^0) $\sum_{\ell=1}^{m-2}\alpha_\ell\le 1$, $g(t)(1-\sum_{\ell=1}^{m-2}\alpha_\ell+\delta)+h(t)(1-\sigma)\ge 0$;
(2^0) $h(t)\ge 0$ *on* $[\xi_1,1]$, $\alpha_\ell\ge\int_{\xi_\ell}^{\xi_{\ell+1}}h(t)\,dt$ ($\ell=1,\dots,m-2$) *and* $h(t)\le 0$ *on* $[0,\xi_1]$.

Then the function $\Upsilon_\omega(\cdot)$ *has the following properties*:

(i) *For any fixed* $\omega\in W$, *the equation* $\Upsilon_\omega(\rho_3)=0$ *has a unique solution* $\widetilde{\rho}_3(\omega)\in\mathbb{R}^N$.
(ii) *The function* $\widetilde{\rho}_3: W\to\mathbb{R}^N$, *defined in* (i), *is continuous and sends bounded sets to bounded sets. Moreover, for any* $\omega=(a,b,\vartheta)\in W$, *we have*

$$\begin{aligned}\left|\widetilde{\rho}_3(\omega)\right| \le 3N\Biggl\{&(2N)^{p^+}\left[\left(\frac{E_1+1}{(1-\sigma)E_1}+\left(\delta^*+1\right)\frac{E+1}{(1-\sum_{\ell=1}^{m-2}\alpha_\ell+\delta)E}\right)\sum_{i=1}^k|a_i|\right]^{p^\#-1} \\ &+\sum_{i=1}^k|b_i|+\|\vartheta\|_{L^1}\Biggr\},\end{aligned}$$

where the notation $M^{p^{\#}-1}$ *means*

$$M^{p^{\#}-1} = \begin{cases} M^{p^{+}-1}, & M > 1, \\ M^{p^{-}-1}, & M \le 1. \end{cases}$$

Proof Similar to the proof of Lemma 2.2, we omit it here. □

We define $\rho_3 : PC^1 \to \mathbb{R}^N$ as $\rho_3(u) = \widetilde{\rho_3}(A, B, N_f)(u)$, where $A = (A_1, \ldots, A_k)$, $B = (B_1, \ldots, B_k)$.

It is clear that $\rho_3(\cdot)$ is continuous and sends bounded sets of PC^1 to bounded sets of $\mathbb{R}^N$, and hence it is compact continuous.

For fixed $a, b \in \mathbb{R}^{kN}$, we denote $K^{**}_{(a,b)} : L^1 \to PC^1$ as

$$K^{**}_{(a,b)}(\vartheta)(t) = F\left\{\varphi^{-1}\left[t, \left(w(t)\right)^{-1}\left(\widetilde{\rho_3}(a, b, \vartheta) + \sum_{t_i<t} b_i + F(\vartheta)(t)\right)\right]\right\}(t), \quad \forall t \in J.$$

Define $K_3 : PC^1 \to PC^1$ as

$$K_3(u)(t) = F\left\{\varphi^{-1}\left[t, \left(w(t)\right)^{-1}\left(\rho_3(u) + \sum_{t_i<t} B_i + F\left(N_f(u)\right)(t)\right)\right]\right\}(t), \quad \forall t \in J.$$

Similar to the proof of Lemma 2.3, we have

Lemma 2.9 (i) *The operator* $K^{**}_{(a,b)}$ *is continuous and sends equi-integrable sets in* L^1 *to relatively compact sets in* PC^1.

(ii) *The operator* K_3 *is continuous and sends bounded sets in* PC^1 *to relatively compact sets in* PC^1.

Let us define $P_3 : PC^1 \to PC^1$ *as*

$$P_3(u) = \frac{\int_0^1 g(t)[K_3(u)(t) + \sum_{t_i<t} A_i]\,dt}{1-\sigma}.$$

It is easy to see that P_3 *is compact continuous.*

Lemma 2.10 *Suppose that* $\sigma < 1$, $\sum_{\ell=1}^{m-2} \alpha_\ell - \delta < 1$ *and* α_ℓ, g, h *satisfy one of the following*:

(1^0) $\sum_{\ell=1}^{m-2} \alpha_\ell \le 1$, $g(t)(1 - \sum_{\ell=1}^{m-2} \alpha_\ell + \delta) + h(t)(1-\sigma) \ge 0$;
(2^0) $h(t) \ge 0$ *on* $[\xi_1, 1]$, $\alpha_\ell \ge \int_{\xi_\ell}^{\xi_{\ell+1}} h(t)\,dt$ ($\ell = 1, \ldots, m-2$) *and* $h(t) \le 0$ *on* $[0, \xi_1]$.

Then u is a solution of (1)-(4) *if and only if u is a solution of the following abstract operator equation*:

$$u = P_3(u) + \sum_{t_i<t} A_i + K_3(u).$$

Proof Similar to the proof of Lemma 2.4, we omit it here. □

3 Existence of solutions in Case (i)

In this section, we apply Leray-Schauder's degree to deal with the existence of solutions for system (1)-(4) or (1) with (2), (4) and (5) when $\sigma < 1$, $\sum_{\ell=1}^{m-2}\alpha_\ell - \delta = 1$.

When f satisfies the sub-$(p^- - 1)$ growth condition, we have the following theorem.

Theorem 3.1 *Suppose that* $\sigma < 1$, $\sum_{\ell=1}^{m-2}\alpha_\ell - \delta = 1$; $h(t) \geq 0$ *on* $[\xi_1, 1]$, $\alpha_\ell \geq \int_{\xi_\ell}^{\xi_{\ell+1}} h(t)\,dt$ $(\ell = 1, \ldots, m-2)$ *and* $h(t) \leq 0$ *on* $[0, \xi_1]$; f *satisfies the sub-*$(p^- - 1)$ *growth condition*; *and operators A and B satisfy the following conditions*:

$$\begin{aligned}
&\sum_{i=1}^{k}\bigl|A_i(u,v)\bigr| \leq C_1\bigl(1 + |u| + |v|\bigr)^{\frac{q^+-1}{p^+-1}},\\
&\sum_{i=1}^{k}\bigl|B_i(u,v)\bigr| \leq C_2\bigl(1 + |u| + |v|\bigr)^{q^+-1}, \quad \forall (u,v) \in \mathbb{R}^N \times \mathbb{R}^N,
\end{aligned} \tag{20}$$

then problem (1)-(4) *has at least a solution*.

Proof First we consider the following problem:

$$(S_1)\quad \begin{cases}
-\triangle_{p(t)} u = \lambda N_f(u)(t), \quad t \in (0,1), t \neq t_i,\\
\lim_{t\to t_i^+} u(t) - \lim_{t\to t_i^-} u(t)\\
\quad = \lambda A_i(\lim_{t\to t_i^-} u(t), \lim_{t\to t_i^-} (w(t))^{\frac{1}{p(t)-1}} u'(t)), \quad i = 1, \ldots, k,\\
\lim_{t\to t_i^+} w(t)|u'|^{p(t)-2}u'(t) - \lim_{t\to t_i^-} w(t)|u'|^{p(t)-2}u'(t)\\
\quad = \lambda B_i(\lim_{t\to t_i^-} u(t), \lim_{t\to t_i^-} (w(t))^{\frac{1}{p(t)-1}} u'(t)), \quad i = 1, \ldots, k,\\
u(0) = \int_0^1 g(t)u(t)\,dt, \qquad u(1) = \sum_{\ell=1}^{m-2}\alpha_\ell u(\xi_\ell) - \int_0^1 h(t)u(t)\,dt.
\end{cases}$$

Denote

$$\begin{aligned}
&\rho_{1,\lambda}(u) = \widetilde{\rho}_1(\lambda A, \lambda B, \lambda N_f)(u),\\
&K_{1,\lambda}(u) = F\left\{\varphi^{-1}\left[t, \bigl(w(t)\bigr)^{-1}\left(\rho_{1,\lambda}(u) + \lambda\sum_{t_i<t} B_i + F\bigl(\lambda N_f(u)\bigr)(t)\right)\right]\right\},\\
&P_{1,\lambda}(u) = \frac{\int_0^1 g(t)[K_{1,\lambda}(u)(t) + \sum_{t_i<t}\lambda A_i]\,dt}{1-\sigma},\\
&\Psi_f(u,\lambda) = P_{1,\lambda}(u) + \lambda\sum_{t_i<t} A_i + K_{1,\lambda}(u),
\end{aligned}$$

where $N_f(u)$ is defined in (10).

Obviously, (S_1) has the same solution as the following operator equation when $\lambda = 1$:

$$u = \Psi_f(u,\lambda). \tag{21}$$

It is easy to see that the operator $\rho_{1,\lambda}$ is compact continuous for any $\lambda \in [0,1]$. It follows from Lemma 2.2 and Lemma 2.3 that $\Psi_f(\cdot,\lambda)$ is compact continuous from PC^1 to PC^1 for any $\lambda \in [0,1]$.

We claim that all the solutions of (21) are uniformly bounded for $\lambda \in [0,1]$. In fact, if it is false, we can find a sequence of solutions $\{(u_n, \lambda_n)\}$ for (21) such that $\|u_n\|_1 \to +\infty$ as $n \to +\infty$ and $\|u_n\|_1 > 1$ for any $n = 1, 2, \ldots$.

From Lemma 2.2, we have

$$\left|\rho_{1,\lambda}(u)\right| \le C_3\left[\left(\sum_{i=1}^{k}|A_i|\right)^{p^{\#}-1} + \sum_{i=1}^{k}|B_i| + \left\|N_f(u)\right\|_{L^1}\right] \le C_4\left(1+\|u\|_1^{q^+-1}\right).$$

Thus

$$\left|\rho_{1,\lambda}(u) + \sum_{t_i<t}\lambda B_i + F(\lambda N_f)\right| \le \left|\rho_{1,\lambda}(u)\right| + \left|\sum_{t_i<t}B_i\right| + \left|F(N_f)\right| \le C_5\left(1+\|u\|_1^{q^+-1}\right). \tag{22}$$

From (S_1), we have

$$w(t)\left|u_n'(t)\right|^{p(t)-2}u_n'(t) = \rho_{1,\lambda}(u_n) + \sum_{t_i<t}\lambda B_i + \int_0^t \lambda N_f(u_n)(s)\,ds, \quad \forall t \in J'.$$

It follows from (11) and Lemma 2.2 that

$$w(t)\left|u_n'(t)\right|^{p(t)-1} \le \left|\rho_{1,\lambda}(u_n)\right| + \sum_{i=1}^{k}|B_i| + \int_0^1 \left|N_f(u_n)(s)\right|ds \le C_6 + C_7\|u_n\|_1^{q^+-1}, \quad \forall t \in J'.$$

Denote $\alpha = \frac{q^+-1}{p^--1}$. If the above inequality holds then

$$\left\|\left(w(t)\right)^{\frac{1}{p(t)-1}}u_n'(t)\right\|_0 \le C_8\|u_n\|_1^{\alpha}, \quad n = 1,2,\ldots. \tag{23}$$

It follows from (14), (20) and (22) that

$$\left|u_n(0)\right| \le C_9\|u_n\|_1^{\alpha}, \quad \text{where } \alpha = \frac{q^+-1}{p^--1}.$$

For any $j = 1,\ldots,N$, we have

$$\begin{aligned}
\left|u_n^j(t)\right| &= \left|u_n^j(0) + \sum_{t_i<t}A_i + \int_0^t \left(u_n^j\right)'(s)\,ds\right| \\
&\le \left|u_n^j(0)\right| + \left|\sum_{t_i<t}A_i\right| + \left|\int_0^t \left(w(s)\right)^{\frac{-1}{p(s)-1}} \sup_{t\in(0,1)}\left|\left(w(t)\right)^{\frac{1}{p(t)-1}}\left(u_n^j\right)'(t)\right|ds\right| \\
&\le \|u_n\|_1^{\alpha}[C_{10} + C_8E] + \left|\sum_{t_i<t}A_i\right| \le C_{11}\|u_n\|_1^{\alpha}, \quad \forall t \in J, n = 1,2,\ldots,
\end{aligned}$$

which implies that

$$\left|u_n^j\right|_0 \le C_{12}\|u_n\|_1^{\alpha}, \quad j = 1,\ldots,N; n = 1,2,\ldots.$$

Thus

$$\|u_n\|_0 \le NC_{12}\|u_n\|_1^{\alpha}, \quad n = 1,2,\ldots. \tag{24}$$

It follows from (23) and (24) that $\{\|u_n\|_1\}$ is uniformly bounded.

Thus, we can choose a large enough $R_0 > 0$ such that all the solutions of (21) belong to $B(R_0) = \{u \in PC^1 \mid \|u\|_1 < R_0\}$. Therefore the Leray-Schauder degree $d_{LS}[I - \Psi_f(\cdot, \lambda), B(R_0), 0]$ is well defined for $\lambda \in [0,1]$, and

$$d_{LS}\big[I - \Psi_f(\cdot,1), B(R_0), 0\big] = d_{LS}\big[I - \Psi_f(\cdot,0), B(R_0), 0\big].$$

It is easy to see that u is a solution of $u = \Psi_f(u,0)$ if and only if u is a solution of the following usual differential equation:

$$(S_2) \quad \begin{cases} -\triangle_{p(t)} u = 0, \quad t \in (0,1), \\ u(0) = \int_0^1 g(t)u(t)\,dt, \qquad u(1) = \sum_{\ell=1}^{m-2} \alpha_\ell u(\xi_\ell) - \int_0^1 h(t)u(t)\,dt. \end{cases}$$

Obviously, system (S_2) possesses a unique solution u_0. Since $u_0 \in B(R_0)$, we have

$$d_{LS}\big[I - \Psi_f(\cdot,1), B(R_0), 0\big] = d_{LS}\big[I - \Psi_f(\cdot,0), B(R_0), 0\big] \neq 0,$$

which implies that (1)-(4) has at least one solution. This completes the proof. □

Theorem 3.2 *Suppose that* $\sigma < 1$, $\sum_{\ell=1}^{m-2} \alpha_\ell - \delta = 1$; $h(t) \geq 0$ *on* $[\xi_1, 1]$, $\alpha_\ell \geq \int_{\xi_\ell}^{\xi_{\ell+1}} h(t)\,dt$ $(\ell = 1, \dots, m-2)$ *and* $h(t) \leq 0$ *on* $[0, \xi_1]$; f *satisfies the sub-*$(p^- - 1)$ *growth condition*; *and operators* A *and* $D = (D_1, \dots, D_k)$ *satisfy the following conditions*:

$$\sum_{i=1}^{k} \big|A_i(u,v)\big| \leq C_1\big(1 + |u| + |v|\big)^{\frac{q^+-1}{p^+-1}},$$

$$\sum_{i=1}^{k} \big|D_i(u,v)\big| \leq C_2\big(1 + |u| + |v|\big)^{\alpha_i^+}, \quad \forall (u,v) \in \mathbb{R}^N \times \mathbb{R}^N,$$

where $\alpha_i \leq \frac{q^+-1}{p(t_i)-1}$, *and* $p(t_i) - 1 \leq q^+ - \alpha_i$, $i = 1, \dots, k$.

Then problem (1) *with* (2), (4) *and* (5) *has at least a solution*.

Proof Obviously, $B_i(u,v) = \varphi(t_i, v + D_i(u,v)) - \varphi(t_i, v)$.

From Theorem 3.1, it suffices to show that

$$\sum_{i=1}^{k} \big|B_i(u,v)\big| \leq C_2\big(1 + |u| + |v|\big)^{q^+-1}, \quad \forall (u,v) \in \mathbb{R}^N \times \mathbb{R}^N. \tag{25}$$

(a) Suppose that $|v| \leq M^*|D_i(u,v)|$, where M^* is a large enough positive constant. From the definition of D, we have

$$\big|B_i(u,v)\big| \leq C_1\big|D_i(u,v)\big|^{p(t_i)-1} \leq C_2\big(1 + |u| + |v|\big)^{\alpha_i(p(t_i)-1)}.$$

Since $\alpha_i < \frac{q^+-1}{p(t_i)-1}$, we have $\alpha_i(p(t_i) - 1) \leq q^+ - 1$. Thus (25) is valid.

(b) Suppose that $|v| > M^*|D_i(u,v)|$, we can see that

$$\big|B_i(u,v)\big| \leq C_3|v|^{p(t_i)-1}\frac{|D_i(u,v)|}{|v|} = C_4|v|^{p(t_i)-2}\big|D_i(u,v)\big|.$$

There are two cases: Case (i): $p(t_i) - 1 \geq 1$; Case (ii): $p(t_i) - 1 < 1$.

Case (i): Since $p(t_i) - 1 \leq q^+ - \alpha_i$, we have $p(t_i) - 2 + \alpha_i \leq q^+ - 1$, and

$$\left|B_i(u,v)\right| \leq C_5|v|^{p(t_i)-2}\left|D_i(u,v)\right| \leq C_6\left(1+|u|+|v|\right)^{p(t_i)-2+\alpha_i} \leq C_6\left(1+|u|+|v|\right)^{q^+-1}.$$

Thus (25) is valid.

Case (ii): Since $\alpha_i < \frac{q^+-1}{p(t_i)-1}$, we have $\alpha_i(p(t_i)-1) \leq q^+ - 1$, and

$$\left|B_i(u,v)\right| \leq C_7|v|^{p(t_i)-2}\left|D_i(u,v)\right| \leq C_8\left|D_i(u,v)\right|^{p(t_i)-1} \leq C_9\left(1+|u|+|v|\right)^{\alpha_i(p(t_i)-1)}.$$

Thus (25) is valid.

Thus problem (1) with (2), (4) and (5) has at least a solution. This completes the proof. □

Let us consider

$$-\left(w(t)\left|u'\right|^{p(t)-2}u'\right)' + \phi\left(t,u,\left(w(t)\right)^{\frac{1}{p(t)-1}}u',S(u),T(u),\varepsilon\right) = 0, \quad t \in (0,1), t \neq t_i, \tag{26}$$

where ε is a parameter, and

$$\begin{aligned}&\phi\left(t,u,\left(w(t)\right)^{\frac{1}{p(t)-1}}u',S(u),T(u),\varepsilon\right)\\&\quad = f\left(t,u,\left(w(t)\right)^{\frac{1}{p(t)-1}}u',S(u),T(u)\right) + \varepsilon h\left(t,u,\left(w(t)\right)^{\frac{1}{p(t)-1}}u',S(u),T(u)\right),\end{aligned}$$

where $h,f : J \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ are Caratheodory. We have the following theorem.

Theorem 3.3 *Suppose that* $\sigma < 1$, $\sum_{\ell=1}^{m-2}\alpha_\ell - \delta = 1$; $h(t) \geq 0$ *on* $[\xi_1,1]$, $\alpha_\ell \geq \int_{\xi_\ell}^{\xi_{\ell+1}} h(t)\,dt$ $(\ell = 1,\ldots,m-2)$ *and* $h(t) \leq 0$ *on* $[0,\xi_1]$; f *satisfies the sub-*$(p^- - 1)$ *growth condition*; *and we assume that*

$$\begin{aligned}&\sum_{i=1}^{k}\left|A_i(u,v)\right| \leq C_1\left(1+|u|+|v|\right)^{\frac{q^+-1}{p^+-1}},\\&\sum_{i=1}^{k}\left|B_i(u,v)\right| \leq C_2\left(1+|u|+|v|\right)^{q^+-1}, \quad \forall(u,v) \in \mathbb{R}^N \times \mathbb{R}^N,\end{aligned}$$

then problem (26) *with* (2)-(4) *has at least one solution when parameter* ε *is small enough*.

Proof Denote

$$\begin{aligned}&\phi_\lambda\left(t,u,\left(w(t)\right)^{\frac{1}{p(t)-1}}u',S(u),T(u),\varepsilon\right)\\&\quad = f\left(t,u,\left(w(t)\right)^{\frac{1}{p(t)-1}}u',S(u),T(u)\right) + \lambda\varepsilon h\left(t,u,\left(w(t)\right)^{\frac{1}{p(t)-1}}u',S(u),T(u)\right).\end{aligned}$$

We consider the existence of solutions of the following equation with (2)-(4)

$$-\left(w(t)\left|u'\right|^{p(t)-2}u'\right)' + \phi_\lambda\left(t,u,\left(w(t)\right)^{\frac{1}{p(t)-1}}u',S(u),T(u),\varepsilon\right) = 0, \quad t \in (0,1), t \neq t_i. \tag{27}$$

Denote

$$\rho^{\#}_{1,\lambda}(u,\varepsilon)=\widetilde{\rho}_1(A,B,N_{\phi_\lambda})(u),$$
$$K^{\#}_{1,\lambda}(u,\varepsilon)=F\left\{\varphi^{-1}\left[t,\left(w(t)\right)^{-1}\left(\rho^{\#}_{1,\lambda}(u,\varepsilon)+\sum_{t_i<t}B_i+F\left(N_{\phi_\lambda}(u)\right)(t)\right)\right]\right\},$$
$$P^{\#}_{1,\lambda}(u,\varepsilon)=\frac{\int_0^1 g(t)[K^{\#}_{1,\lambda}(u,\varepsilon)(t)+\sum_{t_i<t}A_i]\,dt}{(1-\sigma)},$$
$$\Phi_\varepsilon(u,\lambda)=P^{\#}_{1,\lambda}(u,\varepsilon)+\sum_{t_i<t}A_i+K^{\#}_{1,\lambda}(u,\varepsilon),$$

where $N_{\phi_\lambda}(u)$ is defined in (10).

We know that (27) with (2)-(4) has the same solution of $u=\Phi_\varepsilon(u,\lambda)$.

Obviously, $\phi_0=f$. So $\Phi_\varepsilon(u,0)=\Psi_f(u,1)$. As in the proof of Theorem 3.1, we know that all the solutions of $u=\Phi_\varepsilon(u,0)$ are uniformly bounded, then there exists a large enough $R_0>0$ such that all the solutions of $u=\Phi_\varepsilon(u,0)$ belong to $B(R_0)=\{u\in PC^1 \mid \|u\|_1<R_0\}$. Since $\Phi_\varepsilon(\cdot,0)$ is compact continuous from PC^1 to PC^1, we have

$$\inf_{u\in\partial B(R_0)}\left\|u-\Phi_\varepsilon(u,0)\right\|_1>0. \tag{28}$$

Since f and h are Caratheodory, we have

$$\left\|F\left(N_{\phi_\lambda}(u)\right)-F\left(N_{\phi_0}(u)\right)\right\|_0\to 0\quad\text{for } (u,\lambda)\in\overline{B(R_0)}\times[0,1] \text{ uniformly, as } \varepsilon\to 0,$$
$$\left|\rho^{\#}_{1,\lambda}(u,\varepsilon)-\rho^{\#}_{1,0}(u,\varepsilon)\right|\to 0\quad\text{for } (u,\lambda)\in\overline{B(R_0)}\times[0,1] \text{ uniformly, as } \varepsilon\to 0,$$
$$\left\|K^{\#}_{1,\lambda}(u,\varepsilon)-K^{\#}_{1,0}(u,\varepsilon)\right\|_1\to 0\quad\text{for } (u,\lambda)\in\overline{B(R_0)}\times[0,1] \text{ uniformly, as } \varepsilon\to 0,$$
$$\left|P^{\#}_{1,\lambda}(u,\varepsilon)-P^{\#}_{1,0}(u,\varepsilon)\right|\to 0\quad\text{for } (u,\lambda)\in\overline{B(R_0)}\times[0,1] \text{ uniformly, as } \varepsilon\to 0.$$

Thus

$$\left\|\Phi_\varepsilon(u,\lambda)-\Phi_0(u,\lambda)\right\|_1\to 0\quad\text{for } (u,\lambda)\in\overline{B(R_0)}\times[0,1] \text{ uniformly, as } \varepsilon\to 0.$$

Obviously, $\Phi_0(u,\lambda)=\Phi_\varepsilon(u,0)=\Phi_0(u,0)$. We obtain

$$\left\|\Phi_\varepsilon(u,\lambda)-\Phi_\varepsilon(u,0)\right\|_1\to 0\quad\text{for } (u,\lambda)\in\overline{B(R_0)}\times[0,1] \text{ uniformly, as } \varepsilon\to 0.$$

Thus, when ε is small enough, from (28), we can conclude that

$$\begin{aligned}&\inf_{(u,\lambda)\in\partial B(R_0)\times[0,1]}\left\|u-\Phi_\varepsilon(u,\lambda)\right\|_1\\&\quad\geq\inf_{u\in\partial B(R_0)}\left\|u-\Phi_\varepsilon(u,0)\right\|_1-\sup_{(u,\lambda)\in\overline{B(R_0)}\times[0,1]}\left\|\Phi_\varepsilon(u,0)-\Phi_\varepsilon(u,\lambda)\right\|_1>0.\end{aligned}$$

Thus $u=\Phi_\varepsilon(u,\lambda)$ has no solution on $\partial B(R_0)$ for any $\lambda\in[0,1]$, when ε is small enough. It means that the Leray-Schauder degree $d_{LS}[I-\Phi_\varepsilon(\cdot,\lambda),B(R_0),0]$ is well defined for any $\lambda\in[0,1]$, and

$$d_{LS}\left[I-\Phi_\varepsilon(u,\lambda),B(R_0),0\right]=d_{LS}\left[I-\Phi_\varepsilon(u,0),B(R_0),0\right].$$

Since $\Phi_\varepsilon(u,0) = \Psi_f(u,1)$, from the proof of Theorem 3.1, we can see that the right-hand side is nonzero. Thus (26) with (2)-(4) has at least one solution when ε is small enough. This completes the proof. □

Theorem 3.4 *Suppose that* $\sigma < 1$, $\sum_{\ell=1}^{m-2}\alpha_\ell - \delta = 1$; $h(t) \geq 0$ *on* $[\xi_1, 1]$, $\alpha_\ell \geq \int_{\xi_\ell}^{\xi_{\ell+1}} h(t)\,dt$ $(\ell = 1,\dots,m-2)$ *and* $h(t) \leq 0$ *on* $[0,\xi_1]$; *f satisfies the sub-*$(p^- - 1)$ *growth condition*; *and we assume that*

$$\sum_{i=1}^{k}\bigl|A_i(u,v)\bigr| \leq C_1\bigl(1+|u|+|v|\bigr)^{\frac{q^+-1}{p^+-1}},$$

$$\sum_{i=1}^{k}\bigl|D_i(u,v)\bigr| \leq C_2\bigl(1+|u|+|v|\bigr)^{\alpha_i^+}, \quad \forall (u,v) \in \mathbb{R}^N \times \mathbb{R}^N,$$

where $\alpha_i \leq \frac{q^+-1}{p(t_i)-1}$, *and* $p(t_i) - 1 \leq q^+ - \alpha_i$, $i = 1,\dots,k$, *then problem* (26) *with* (2), (4) *and* (5) *has at least one solution when parameter* ε *is small enough*.

Proof Similar to the proof of Theorem 3.2 and Theorem 3.3, we omit it here. □

4 Existence of solutions in Case (ii)

In this section, we apply Leray-Schauder's degree to deal with the existence of solutions for system (1)-(4) or (1) with (2), (4) and (5) when $\sigma = 1$, $\sum_{\ell=1}^{m-2}\alpha_\ell - \delta \neq 1$.

When f satisfies the sub-$(p^- - 1)$ growth condition, we have the following.

Theorem 4.1 *Suppose that* $\sigma = 1$, $\sum_{\ell=1}^{m-2}\alpha_\ell - \delta \neq 1$; *f satisfies the sub-*$(p^- - 1)$ *growth condition*; *and operators A and B satisfy the following conditions*:

$$\sum_{i=1}^{k}\bigl|A_i(u,v)\bigr| \leq C_1\bigl(1+|u|+|v|\bigr)^{\frac{q^+-1}{p^+-1}},$$

$$\sum_{i=1}^{k}\bigl|B_i(u,v)\bigr| \leq C_2\bigl(1+|u|+|v|\bigr)^{q^+-1}, \quad \forall (u,v) \in \mathbb{R}^N \times \mathbb{R}^N,$$

then problem (1)-(4) *has at least a solution*.

Proof Similar to the proof of Theorem 3.1, we omit it here. □

Theorem 4.2 *Suppose that* $\sigma = 1$, $\sum_{\ell=1}^{m-2}\alpha_\ell - \delta \neq 1$; *f satisfies the sub-*$(p^- - 1)$ *growth condition*; *and operators A and* $D = (D_1,\dots,D_k)$ *satisfy the following conditions*:

$$\sum_{i=1}^{k}\bigl|A_i(u,v)\bigr| \leq C_1\bigl(1+|u|+|v|\bigr)^{\frac{q^+-1}{p^+-1}},$$

$$\sum_{i=1}^{k}\bigl|D_i(u,v)\bigr| \leq C_2\bigl(1+|u|+|v|\bigr)^{\alpha_i^+}, \quad \forall (u,v) \in \mathbb{R}^N \times \mathbb{R}^N,$$

where

$$\alpha_i \leq \frac{q^+-1}{p(t_i)-1} \quad \textit{and} \quad p(t_i) - 1 \leq q^+ - \alpha_i, \quad i = 1,\dots,k,$$

then problem (1) *with* (2), (4) *and* (5) *has at least a solution*.

Proof Similar to the proof of Theorem 3.2, we omit it here. □

Theorem 4.3 *Suppose that* $\sigma = 1$, $\sum_{\ell=1}^{m-2}\alpha_\ell - \delta \neq 1$; f *satisfies the sub-*$(p^- - 1)$ *growth condition*; *and we assume that*

$$\sum_{i=1}^{k}\bigl|A_i(u,v)\bigr| \le C_1\bigl(1+|u|+|v|\bigr)^{\frac{q^+-1}{p^+-1}},$$

$$\sum_{i=1}^{k}\bigl|B_i(u,v)\bigr| \le C_2\bigl(1+|u|+|v|\bigr)^{q^+-1}, \quad \forall (u,v)\in \mathbb{R}^N\times\mathbb{R}^N,$$

then problem (26) *with* (2)-(4) *has at least one solution when parameter* ε *is small enough*.

Proof Similar to the proof of Theorem 3.3, we omit it here. □

Theorem 4.4 *Suppose that* $\sigma = 1$, $\sum_{\ell=1}^{m-2}\alpha_\ell - \delta \neq 1$; f *satisfies the sub-*$(p^- - 1)$ *growth condition*; *and we assume that*

$$\sum_{i=1}^{k}\bigl|A_i(u,v)\bigr| \le C_1\bigl(1+|u|+|v|\bigr)^{\frac{q^+-1}{p^+-1}},$$

$$\sum_{i=1}^{k}\bigl|D_i(u,v)\bigr| \le C_2\bigl(1+|u|+|v|\bigr)^{\alpha_i^+}, \quad \forall (u,v)\in \mathbb{R}^N\times\mathbb{R}^N,$$

where $\alpha_i \le \frac{q^+-1}{p(t_i)-1}$, *and* $p(t_i)-1 \le q^+ - \alpha_i$, $i=1,\ldots,k$, *then problem* (26) *with* (2), (4) *and* (5) *has at least one solution when parameter* ε *is small enough*.

Proof Similar to the proof of Theorem 3.2 and Theorem 3.3, we omit it here. □

5 Existence of solutions in Case (iii)

In this section, we apply Leray-Schauder's degree to deal with the existence of solutions and nonnegative solutions for system (1)-(4) or (1) with (2), (4) and (5) when $\sigma < 1$, $\sum_{\ell=1}^{m-2}\alpha_\ell - \delta < 1$.

When f satisfies the sub-$(p^- - 1)$ growth condition, we have the following theorem.

Theorem 5.1 *Suppose that* $\sigma < 1$, $\sum_{\ell=1}^{m-2}\alpha_\ell - \delta < 1$ *and* α_ℓ, g, h *satisfy one of the following*:

(1^0) $\sum_{\ell=1}^{m-2}\alpha_\ell \le 1$, $g(t)(1-\sum_{\ell=1}^{m-2}\alpha_\ell + \delta) + h(t)(1-\sigma) \ge 0$;
(2^0) $h(t)\ge 0$ *on* $[\xi_1, 1]$, $\alpha_\ell \ge \int_{\xi_\ell}^{\xi_{\ell+1}} h(t)\,dt$ $(\ell = 1,\ldots,m-2)$ *and* $h(t)\le 0$ *on* $[0,\xi_1]$;

when f *satisfies the sub-*$(p^- - 1)$ *growth condition*; *and operators* A *and* B *satisfy the following conditions*:

$$\sum_{i=1}^{k}\bigl|A_i(u,v)\bigr| \le C_1\bigl(1+|u|+|v|\bigr)^{\frac{q^+-1}{p^+-1}},$$

$$\sum_{i=1}^{k}\bigl|B_i(u,v)\bigr| \le C_2\bigl(1+|u|+|v|\bigr)^{q^+-1}, \quad \forall (u,v)\in \mathbb{R}^N\times\mathbb{R}^N,$$

then problem (1)-(4) *has at least a solution*.

Proof Similar to the proof of Theorem 3.1, we omit it here. □

Theorem 5.2 *Suppose that* $\sigma < 1$, $\sum_{\ell=1}^{m-2} \alpha_\ell - \delta < 1$ *and* α_ℓ, g, h *satisfy one of the following*:

(1^0) $\sum_{\ell=1}^{m-2} \alpha_\ell \leq 1$, $g(t)(1 - \sum_{\ell=1}^{m-2} \alpha_\ell + \delta) + h(t)(1-\sigma) \geq 0$;
(2^0) $h(t) \geq 0$ *on* $[\xi_1, 1]$, $\alpha_\ell \geq \int_{\xi_\ell}^{\xi_{\ell+1}} h(t)\,dt$ $(\ell = 1, \ldots, m-2)$ *and* $h(t) \leq 0$ *on* $[0, \xi_1]$;

when f satisfies the sub-$(p^- - 1)$ *growth condition*; *and operators A and* $D = (D_1, \ldots, D_k)$ *satisfy the following conditions*:

$$\sum_{i=1}^{k} \left|A_i(u,v)\right| \leq C_1 \left(1 + |u| + |v|\right)^{\frac{q^+-1}{p^+-1}},$$

$$\sum_{i=1}^{k} \left|D_i(u,v)\right| \leq C_2 \left(1 + |u| + |v|\right)^{\alpha_i^+}, \quad \forall (u,v) \in \mathbb{R}^N \times \mathbb{R}^N,$$

where

$$\alpha_i \leq \frac{q^+ - 1}{p(t_i) - 1} \quad \textit{and} \quad p(t_i) - 1 \leq q^+ - \alpha_i, \quad i = 1, \ldots, k,$$

then problem (1) *with* (2), (4) *and* (5) *has at least a solution.*

Proof Similar to the proof of Theorem 3.2, we omit it here. □

Theorem 5.3 *Suppose that* $\sigma < 1$, $\sum_{\ell=1}^{m-2} \alpha_\ell - \delta < 1$ *and* α_ℓ, g, h *satisfy one of the following*:

(1^0) $\sum_{\ell=1}^{m-2} \alpha_\ell \leq 1$, $g(t)(1 - \sum_{\ell=1}^{m-2} \alpha_\ell + \delta) + h(t)(1-\sigma) \geq 0$;
(2^0) $h(t) \geq 0$ *on* $[\xi_1, 1]$, $\alpha_\ell \geq \int_{\xi_\ell}^{\xi_{\ell+1}} h(t)\,dt$ $(\ell = 1, \ldots, m-2)$ *and* $h(t) \leq 0$ *on* $[0, \xi_1]$;

when f satisfies the sub-$(p^- - 1)$ *growth condition*; *and we assume that*

$$\sum_{i=1}^{k} \left|A_i(u,v)\right| \leq C_1 \left(1 + |u| + |v|\right)^{\frac{q^+-1}{p^+-1}},$$

$$\sum_{i=1}^{k} \left|B_i(u,v)\right| \leq C_2 \left(1 + |u| + |v|\right)^{q^+-1}, \quad \forall (u,v) \in \mathbb{R}^N \times \mathbb{R}^N,$$

then problem (26) *with* (2)-(4) *has at least one solution when parameter* ε *is small enough.*

Proof Similar to the proof of Theorem 3.3, we omit it here. □

Theorem 5.4 *Suppose that* $\sigma < 1$, $\sum_{\ell=1}^{m-2} \alpha_\ell - \delta < 1$ *and* α_ℓ, g, h *satisfy one of the following*:

(1^0) $\sum_{\ell=1}^{m-2} \alpha_\ell \leq 1$, $g(t)(1 - \sum_{\ell=1}^{m-2} \alpha_\ell + \delta) + h(t)(1-\sigma) \geq 0$;
(2^0) $h(t) \geq 0$ *on* $[\xi_1, 1]$, $\alpha_\ell \geq \int_{\xi_\ell}^{\xi_{\ell+1}} h(t)\,dt$ $(\ell = 1, \ldots, m-2)$ *and* $h(t) \leq 0$ *on* $[0, \xi_1]$;

when f satisfies the sub-$(p^- - 1)$ *growth condition*; *and we assume that*

$$\sum_{i=1}^{k} \left|A_i(u,v)\right| \leq C_1 \left(1 + |u| + |v|\right)^{\frac{q^+-1}{p^+-1}},$$

$$\sum_{i=1}^{k} \left|D_i(u,v)\right| \leq C_2 \left(1 + |u| + |v|\right)^{\alpha_i^+}, \quad \forall (u,v) \in \mathbb{R}^N \times \mathbb{R}^N,$$

where $\alpha_i \le \frac{q^+-1}{p(t_i)-1}$, *and* $p(t_i)-1 \le q^+ - \alpha_i$, $i = 1,\dots,k$, *then problem* (26) *with* (2), (4) *and* (5) *has at least one solution when parameter* ε *is small enough*.

Proof Similar to the proof of Theorem 3.2 and Theorem 3.3, we omit it here. □

In the following, we will consider the existence of nonnegative solutions. For any $x = (x^1,\dots,x^N) \in \mathbb{R}^N$, the notation $x \ge 0$ means $x^j \ge 0$ for any $j = 1,\dots,N$.

Theorem 5.5 *Suppose that* $\sigma < 1$, $\sum_{\ell=1}^{m-2}\alpha_\ell - \delta < 1$, $\sum_{\ell=1}^{m-2}\alpha_\ell \le 1$, $g(t)(1-\sum_{\ell=1}^{m-2}\alpha_\ell + \delta) + h(t)(1-\sigma) \ge 0$. *We also assume*:

(1^0) $f(t,x,y,s,z) \le 0$, $\forall(t,x,y,s,z) \in J \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$;
(2^0) *For any* $i = 1,\dots,k$, $B_i(u,v) \le 0$, $\forall(u,v) \in \mathbb{R}^N \times \mathbb{R}^N$;
(3^0) *For any* $i = 1,\dots,k$, $j = 1,\dots,N$, $A_i^j(u,v)v^j \ge 0$, $\forall(u,v) \in \mathbb{R}^N \times \mathbb{R}^N$;
(4^0) $h(t) \le 0$.

Then every solution of (1)-(4) *is nonnegative*.

Proof Let u be a solution of (1)-(4). From Lemma 2.10, we have

$$u(t) = u(0) + \sum_{t_i<t} A_i + F\left\{\varphi^{-1}\left[t, (w(t))^{-1}\left(\rho_3(u) + \sum_{t_i<t} B_i + F(N_f(u))\right)\right]\right\}(t), \quad \forall t \in J.$$

We claim that $\rho_3(u) \ge 0$. If it is false, then there exists some $j \in \{1,\dots,N\}$ such that $\rho_3^j(u) < 0$.

It follows from (1^0) and (2^0) that

$$\left[\rho_3(u) + \sum_{t_i<t} B_i + F(N_f(u))(t)\right]^j < 0, \quad \forall t \in J. \tag{29}$$

Thus (29) and condition (3^0) hold

$$A_i^j \le 0, \quad i = 1,\dots,k. \tag{30}$$

Similar to the proof before Lemma 2.8, from the boundary value conditions, we have

$$\begin{aligned} 0 = {}& \frac{1}{(1-\sigma)} \int_0^1 g(t)\left(F\left\{\varphi^{-1}\left[t, (w(t))^{-1}\left(\rho_3 + \sum_{t_i<t} B_i + F(N_f(u))\right)\right]\right\}(t) + \sum_{t_i<t} A_i\right) dt \\ &+ \frac{\sum_{\ell=1}^{m-2}\alpha_\ell\{\sum_{\xi_\ell \le t_i} A_i + \int_{\xi_\ell}^1 \varphi^{-1}[t, (w(t))^{-1}(\rho_3 + \sum_{t_i<t} B_i + F(N_f(u)))]\,dt\}}{1-\sum_{i=1}^{m-2}\alpha_\ell + \delta} \\ &+ \frac{\sum_{i=1}^{k} A_i(1-\sum_{\ell=1}^{m-2}\alpha_\ell)}{1-\sum_{i=1}^{m-2}\alpha_\ell + \delta} \\ &+ \frac{(1-\sum_{\ell=1}^{m-2}\alpha_\ell)\int_0^1 \varphi^{-1}[t, (w(t))^{-1}(\rho_3 + \sum_{t_i<t} B_i + F(N_f(u)))]\,dt}{1-\sum_{\ell=1}^{m-2}\alpha_\ell + \delta} \\ &+ \frac{\int_0^1 h(t)(F\{\varphi^{-1}[t, (w(t))^{-1}(\rho_3 + \sum_{t_i<t} B_i + F(N_f(u)))]\}(t) + \sum_{t_i<t} A_i)\,dt}{1-\sum_{\ell=1}^{m-2}\alpha_\ell + \delta}. \end{aligned} \tag{31}$$

From (29) and (30), we get a contradiction to (31). Thus $\rho_3(u) \geq 0$.

We claim that

$$\rho_3(u) + \sum_{i=1}^{k} B_i + F(N_f)(1) \leq 0. \tag{32}$$

If it is false, then there exists some $j \in \{1, \ldots, N\}$ such that

$$\left[\rho_3(u) + \sum_{i=1}^{k} B_i + F(N_f)(1)\right]^j > 0.$$

It follows from (1^0) and (2^0) that

$$\left[\rho_3(u) + \sum_{t_i<t} B_i + F\big(N_f(u)\big)(t)\right]^j > 0, \quad \forall t \in J. \tag{33}$$

Thus (33) and condition (3^0) hold

$$A_i^j \geq 0, \quad i = 1, \ldots, k. \tag{34}$$

From (33), (34), we get a contradiction to (31). Thus (32) is valid.

Denote $\Theta(t) = \rho_3(u) + \sum_{t_i<t} B_i + F(N_f(u))(t)$, $\forall t \in J'$.

Obviously, $\Theta(0) = \rho_3 \geq 0$, $\Theta(1) \leq 0$, and $\Theta(t)$ is decreasing, *i.e.*, $\Theta(t') \leq \Theta(t'')$ for any $t', t'' \in J$ with $t' \geq t''$. For any $j = 1, \ldots, N$, there exist $\zeta_j \in J$ such that

$$\Theta^j(t) \geq 0, \quad \forall t \in (0, \zeta_j), \quad \text{and} \quad \Theta^j(t) \leq 0, \quad \forall t \in (\zeta_j, T).$$

It follows from condition (3^0) that $u^j(t)$ is increasing on $[0, \zeta_j]$ and $u^j(t)$ is decreasing on $(\zeta_j, T]$. Thus $\min\{u^j(0), u^j(1)\} = \inf_{t\in J} u^j(t)$, $j = 1, \ldots, N$.

For any fixed $j \in \{1, \ldots, N\}$, if

$$u^j(0) = \inf_{t\in J} u^j(t), \tag{35}$$

from (4) and (35), we have $(1 - \sigma)u^j(0) \geq 0$. Then $u^j(0) \geq 0$.

If

$$u^j(1) = \inf_{t\in J} u^j(t), \tag{36}$$

from (4), (36) and condition (4^0), we have $(1 - \sum_{i=1}^{m-2} \alpha_\ell + \delta)u^j(1) \geq 0$. Then $u^j(1) \geq 0$.

Thus $u(t) \geq 0$, $\forall t \in [0, T]$. The proof is completed. □

Corollary 5.6 *Under the conditions of Theorem* 5.1, *we also assume*:

(1^0) $f(t, x, y, s, z) \leq 0$, $\forall (t, x, y, s, z) \in J \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$ *with* $x, s, z \geq 0$;

(2^0) *For any* $i = 1, \ldots, k$, $B_i(u, v) \leq 0$, $\forall (u, v) \in \mathbb{R}^N \times \mathbb{R}^N$ *with* $u \geq 0$;

(3^0) *For any* $i = 1, \ldots, k$, $j = 1, \ldots, N$, $A_i^j(u, v)v^j \geq 0$, $\forall (u, v) \in \mathbb{R}^N \times \mathbb{R}^N$ *with* $u \geq 0$;

(4^0) $h(t) \leq 0$;
(5^0) *For any* $t \in [0,1]$ *and* $s \in [0,1]$, $k_*(t,s) \geq 0$, $h_*(t,s) \geq 0$.

Then (1)-(4) *has a nonnegative solution.*

Proof Define $M(u) = (M_\#(u^1), \dots, M_\#(u^N))$, where

$$M_\#(u) = \begin{cases} u, & u \geq 0, \\ 0, & u < 0. \end{cases}$$

Denote

$$\widetilde{f}(t,u,v,S(u),T(u)) = f(t,M(u),v,S(M(u)),T(M(u))), \quad \forall (t,u,v) \in J \times \mathbb{R}^N \times \mathbb{R}^N,$$

then $\widetilde{f}(t,u,v,S(u),T(u))$ satisfies the Caratheodory condition, and $\widetilde{f}(t,u,v,S(u),T(u)) \leq 0$ for any $(t,u,v) \in J \times \mathbb{R}^N \times \mathbb{R}^N$.

For any $i = 1, \dots, k$, we denote

$$\widetilde{A}_i(u,v) = A_i(M(u),v), \qquad \widetilde{B}_i(u,v) = B_i(M(u),v), \quad \forall (u,v) \in \mathbb{R}^N \times \mathbb{R}^N,$$

then $\widetilde{A}_i$ and $\widetilde{B}_i$ are continuous and satisfy

$$\widetilde{B}_i(u,v) \leq 0, \quad \forall (u,v) \in \mathbb{R}^N \times \mathbb{R}^N \text{ for any } i = 1, \dots, k,$$
$$\widetilde{A}_i^j(u,v)v^j \geq 0, \quad \forall (u,v) \in \mathbb{R}^N \times \mathbb{R}^N \text{ for any } i = 1, \dots, k, j = 1, \dots, N.$$

It is not hard to check that

(2^0)′ $\lim_{|u|+|v|\to+\infty}(\widetilde{f}(t,u,v,S(u),T(u))/(|u|+|v|)^{q(t)-1}) = 0$ for $t \in J$ uniformly, where $q(t) \in C(J,\mathbb{R})$, and $1 < q^- \leq q^+ < p^-$;
(3^0)′ $\sum_{i=1}^k |\widetilde{A}_i(u,v)| \leq C_1(1+|u|+|v|)^{\frac{q^+-1}{p^+-1}}$, $\forall (u,v) \in \mathbb{R}^N \times \mathbb{R}^N$;
(4^0)′ $\sum_{i=1}^k |\widetilde{B}_i(u,v)| \leq C_2(1+|u|+|v|)^{q^+-1}$, $\forall (u,v) \in \mathbb{R}^N \times \mathbb{R}^N$.

Let us consider

$$\left.\begin{array}{l} (w(t)\varphi_{p(t)}(u'(t)))' = \widetilde{f}(t,u,(w(t))^{\frac{1}{p(t)-1}}u',S(u),T(u)), \quad t \in J', \\ \lim_{t\to t_i^+} u(t) - \lim_{t\to t_i^-} u(t_i) \\ \quad = \widetilde{A}_i(\lim_{t\to t_i^-} u(t), \lim_{t\to t_i^-}(w(t))^{\frac{1}{p(t)-1}}u'(t)), \quad i = 1, \dots, k, \\ \lim_{t\to t_i^+} w(t)\varphi_{p(t)}(u'(t)) - \lim_{t\to t_i^-} w(t)\varphi_{p(t)}(u'(t)) \\ \quad = \widetilde{B}_i(\lim_{t\to t_i^-} u(t), \lim_{t\to t_i^-}(w(t))^{\frac{1}{p(t)-1}}u'(t)), \quad i = 1, \dots, k, \\ u(0) = \int_0^1 g(t)u(t)\,dt, \qquad u(1) = \sum_{\ell=1}^{m-2} \alpha_\ell u(\xi_\ell) - \int_0^1 h(t)u(t)\,dt. \end{array}\right\} \tag{37}$$

It follows from Theorem 5.1 and Theorem 5.5 that (37) has a nonnegative solution u. Since $u \geq 0$, we have $M(u) = u$, and then

$$\widetilde{f}(t,u,(w(t))^{\frac{1}{p(t)-1}}u',S(u),T(u)) = f(t,u,(w(t))^{\frac{1}{p(t)-1}}u',S(u),T(u)),$$

$$\widetilde{A}_i\Big(\lim_{t\to t_i^-} u(t), \lim_{t\to t_i^-}(w(t))^{\frac{1}{p(t)-1}}u'(t)\Big) = A_i\Big(\lim_{t\to t_i^-} u(t), \lim_{t\to t_i^-}(w(t))^{\frac{1}{p(t)-1}}u'(t)\Big),$$

$$\widetilde{B}_i\Big(\lim_{t\to t_i^-} u(t), \lim_{t\to t_i^-}(w(t))^{\frac{1}{p(t)-1}}u'(t)\Big) = B_i\Big(\lim_{t\to t_i^-} u(t), \lim_{t\to t_i^-}(w(t))^{\frac{1}{p(t)-1}}u'(t)\Big).$$

Thus u is a nonnegative solution of (1)-(4). This completes the proof. □

Note (i) Similarly, we can get the existence of nonnegative solutions of (26) with (2)-(4).

(ii) Similarly, under the conditions of Case (ii), we can discuss the existence of nonnegative solutions.

6 Examples

Example 6.1 Consider the existence of solutions of (1)-(4) under the following assumptions:

$$\begin{aligned} & f\left(t,u,\left(w(t)\right)^{\frac{1}{p(t)-1}}u',S(u),T(u)\right) \\ & \quad = |u|^{q(t)-2}u + \left(w(t)\right)^{\frac{q(t)-1}{p(t)-1}}\left|u'\right|^{q(t)-2}u' \\ & \qquad + \left(S(u)\right)^{q(t)-1} + \left(T(u)\right)^{q(t)-1}, \quad t\in(0,1), t\neq t_i = \frac{i}{k+\pi}, \\ & A_i(u,v) = |u|^{-1/2}u + |v|^{-1/2}v, \quad i=1,\dots,k, \\ & B_i(u,v) = |u|^2u + |v|^2v, \quad i=1,\dots,k, \\ & g(t) = \frac{1}{1+t^2}, \qquad \alpha_\ell = \frac{\ell+1}{\ell}, \qquad \xi_\ell = \frac{\ell}{m}, \qquad h(t) = \begin{cases} 0, & 0\le t\le \frac{1}{m}, \\ \frac{1}{1+t}, & \frac{1}{m}\le t\le 1, \end{cases} \end{aligned}$$

where $(Su)(t) = \int_0^1 e^{t+s}u(s)\,ds$, $(T(u))(t) = \int_0^t (t^2+s^2)u(s)\,ds$, $p(t) = 6 + 3^{-t}\cos 3t$, $q(t) = 3 + 2^{-t}\cos t$.

Obviously, $q(t)\le 4 < 5 \le p(t)$; $h(t) = 0$ when $0\le t\le \frac{1}{m} = \xi_1$; $\alpha_\ell \ge \int_{\xi_\ell}^{\xi_{\ell+1}} h(t)\,dt$ ($\ell = 1,\dots,m-2$); then the conditions of Theorem 3.1 are satisfied, then (1)-(4) has a solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors typed, read and approved the final manuscript.

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