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Solutions to a boundary value problem of a fourth-order impulsive differential equation

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Abstract
This paper is concerned with the existence of solutions to a boundary value problem of a fourth-order impulsive differential equation with a control parameter λ. By employing some existing critical point theorems, we find the range of the control parameter in which the boundary value problem admits at least one solution. It is also shown that under certain conditions there exists an interval of the control parameter in which the boundary value problem possesses infinitely many solutions. The main results are also demonstrated with examples.

MSC: 34B15; 34B18; 34B37; 58E30

Keywords: critical point theorem; impulsive differential equations; boundary value problem

1 Introduction

Fourth-order two-point boundary value problems of ordinary differential equations are widely employed by engineers to describe the beam deflection with two simply supported ends [1–3]. One example is the following fourth-order two-point boundary value problem:

$$\begin{cases} u^{(iv)}(t) + Au''(t) + Bu(t) = \lambda f(t,u(t)), & t \in [0,1], \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \tag{1.1}$$

where $u^{(iv)}(t)$, $u'''(t)$, $u''(t)$ are the fourth, third, and second derivatives of $u(t)$ with respect to t, respectively, $f \in C([0,1] \times R, R)$, A and B are two real constants. System (1.1) has been studied in [4–7] and the references therein. For a beam, $t = 0$ and $t = 1$ in (1.1) refer to the two ends of the beam. At other locations of the beam, $t \in (0,1)$, there may be some sudden changes in loads placed on the beam, or some unexpected forces working on the beam. These sudden changes may result in impulsive effects for the governing differential equation. This motivates us to consider the following boundary value problem for a fourth-order impulsive differential equation:

$$\begin{cases} u^{(iv)}(t) + Au''(t) + Bu(t) = \lambda f(t,u(t)), & t \neq t_j, t \in [0,1], \\ \Delta u''(t_j) = I_{1j}(u'(t_j)), \quad -\Delta u'''(t_j) = I_{2j}(u(t_j)), & j = 1,2,\dots,m, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \tag{1.2}$$

where $I_{1j}, I_{2j} \in C(R,R)$, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = 1$, and the operator Δ is defined as $\Delta U(t_j) = U(t_j^+) - U(t_j^-)$, where $U(t_j^+)$ ($U(t_j^-)$) denotes the right-hand (left-hand) limit of U at t_j and $\lambda > 0$ is referred to as a control parameter.

We are mainly concerned with the existence of solutions of system (1.2). A function $u(t) \in C([0,1])$ is said to be a (classical) solution of (1.2) if $u(t)$ satisfies (1.2). In literature, tools employed to establish the existence of solutions of impulsive differential equations include fixed point theorems, the upper and lower solutions method, the degree theory, critical point theory and variational methods. See, for example, [8–20]. In this paper, we establish the existence of solutions of (1.2) by converting the problem to the existence of critical points of some variational structure. In this paper we regard λ as a parameter and find the ranges in which (1.2) admits at least one and infinitely many solutions, respectively. Note that when $\lambda = 1$ system (1.2) reduces to the one studied in [21]. Our results extend those ones in [21].

The rest of this paper is organized as follows. In Section 2 we present some preliminary results. Our main results and their proofs are given in Section 3.

2 Preliminaries

Throughout we assume that A and B satisfy

$$A \leq 0 \leq B. \tag{2.1}$$

Let

$$H_0^1([0,1]) = \{u \in L^2([0,1]) : u' \in L^2([0,1]), u(0) = u(1) = 0\},$$

and

$$H^2([0,1]) = \{u \in L^2([0,1]) : u', u'' \in L^2([0,1])\}.$$

Take $X := H^2([0,1]) \cap H_0^1([0,1])$ and define

$$\|u\|_X = \left(\int_0^1 \left(\left|u''(t)\right|^2 - A\left|u'\right|^2 + B|u|^2\right) dt\right)^{\frac{1}{2}}, \quad u \in X. \tag{2.2}$$

Since A and B satisfy (2.1), it is straightforward to verify that (2.2) defines a norm for the Sobolev space X and this norm is equivalent to the usual norm defined as follows:

$$\|u\| = \left(\int_0^1 u''(t)^2 \, dt\right)^{\frac{1}{2}}.$$

It follows from (2.1) that

$$\|u\| \leq \|u\|_X.$$

For the norm in $C^1([0,1])$,

$$\|u\|_\infty = \max\left(\max_{t\in[0,1]} \left|u(t)\right|, \max_{t\in[0,1]} \left|u'(t)\right|\right),$$

we have the following relation.

Lemma 2.1 *Let* $M_1 = 1 + \frac{1}{\pi}$. *Then* $\|u\|_\infty \le M_1\|u\|_X$, $\forall u \in X$.

Proof The proof follows easily from Wirtinger's inequality [22], Lemma 2.3 of [23] and Hölder's inequality. The detailed argument is similar to the proof of Lemma 2.2 in [21], and we thus omit it here. □

Define a functional φ_λ as

$$\varphi_\lambda(u) = \Phi(u) - \lambda\Psi(u), \quad u \in X, \tag{2.3}$$

where

$$\Phi(u) = \frac{1}{2}\|u\|_X^2 + \sum_{j=1}^m \int_0^{u'(t_j)} I_{1j}(s)\,ds + \sum_{j=1}^m \int_0^{u(t_j)} I_{2j}(s)\,ds \tag{2.4}$$

and

$$\Psi(u) = \int_0^1 F(t,u)\,dt, \tag{2.5}$$

with

$$F(t,u) = \int_0^{u(t)} f(t,s)\,ds.$$

Note that φ_λ is Fréchet differentiable at any $u \in X$, and for any $v \in X$ we have

$$\begin{aligned}\varphi_\lambda'(u)(v) &= \lim_{h\to 0}\frac{\varphi_\lambda(u+hv)-\varphi_\lambda(u)}{h}\\ &= \int_0^1 \big(u''(t)v''(t) - Au'(t)v'(t) + Bu(t)v(t)\big)\,dt + \sum_{j=1}^m I_{2j}\big(u(t_j)\big)v(t_j)\\ &\quad + \sum_{j=1}^m I_{1j}\big(u'(t_j)\big)v'(t_j) - \lambda\int_0^1 f\big(t,u(t)\big)v(t)\,dt. \end{aligned}\tag{2.6}$$

Next we show that a critical point of the functional φ_λ is a solution of system (1.2).

Lemma 2.2 *If $u \in X$ is a critical point of φ_λ, then u is a solution of system* (1.2).

Proof Suppose that $u \in X$ is a critical point of φ_λ. Then for any $v \in X$ one has

$$\begin{aligned}\lambda\int_0^1 f\big(t,u(t)\big)v(t)\,dt &= \int_0^1 \big(u''(t)v''(t) - Au'(t)v'(t) + Bu(t)v(t)\big)\,dt\\ &\quad + \sum_{j=1}^m I_{2j}\big(u(t_j)\big)v(t_j) + \sum_{j=1}^m I_{1j}\big(u'(t_j)\big)v'(t_j). \end{aligned}\tag{2.7}$$

For $j \in \{1,2,\dots,m\}$, choose $v \in X$ such that $v(t) = 0$ for $t \in [0,t_j] \cup [t_{j+1},1]$, then we have

$$\int_{t_j}^{t_{j+1}} \big(u^{(iv)} + Au''(t) + Bu(t)\big)v(t)\,dt = \lambda\int_{t_j}^{t_{j+1}} f\big(t,u(t)\big)v(t)\,dt.$$

Thus

$$u^{(iv)} + Au''(t) + Bu(t) = \lambda f\big(t, u(t)\big) \quad \text{a.e. } t \in (t_j, t_{j+1}).$$

Therefore, by (2.7) we have

$$\sum_{j=1}^{m}\big(\Delta u'''(t_j) + I_{2j}\big(u(t_j)\big)\big)v(t_j) - \sum_{j=1}^{m}\big(\Delta u''(t_j) - I_{1j}\big(u'(t_j)\big)\big)v'(t_j) = 0.$$

Next we show that u satisfies

$$-\Delta u'''(t_j) = I_{2j}\big(u(t_j)\big), \quad j = 1, 2, \ldots, m.$$

Suppose on the contrary that there exists some $j \in \{1, 2, \ldots, m\}$ such that

$$\Delta u'''(t_j) + I_{2j}\big(u(t_j)\big) \neq 0.$$

Pick

$$v(t) = \big(t^3 - 3t_j^2 t\big) \prod_{i=0,i\neq j}^{m+1} \left(\frac{1}{3}t^3 - \frac{1}{2}(t_j + t_i)t^2 + t_j t_i t + \frac{1}{6}t_i^3 - \frac{1}{2}t_j t_i^2\right),$$

then

$$\begin{aligned} v'(t) &= 3\big(t^2 - t_j^2\big) \prod_{i=0,i\neq j}^{m+1} \left(\frac{1}{3}t^3 - \frac{1}{2}(t_j + t_i)t^2 + t_j t_i t + \frac{1}{6}t_i^3 - \frac{1}{2}t_j t_i^2\right) \\ &\quad + \big(t^3 - 3t_j t\big) \sum_{k=0,k\neq j}^{m+1} \Bigg\{\big(t^2 - (t_k + t_j)t + t_k t_j\big) \\ &\quad \cdot \prod_{i=0,i\neq j,k}^{m+1} \left(\frac{1}{3}t^3 - \frac{1}{2}(t_j + t_i)t^2 + t_j t_i t + \frac{1}{6}t_i^3 - \frac{1}{2}t_j t_i^2\right)\Bigg\}. \end{aligned}$$

Clearly, $v \in X$. Simple calculations show that $v(t_i) = 0$, $i = 1, 2, \ldots, j-1, j+1, \ldots, m+1$, $v(t_j) \neq 0$ and $v'(t_i) = 0$, $i = 1, 2, \ldots, m+1$. Thus

$$\frac{1}{3}t_j^3\big(\Delta u'''(t_j) + I_{2j}\big(u(t_j)\big)\big) \prod_{i=0,i\neq j}^{m+1} (t_i - t_j)^3 = 0,$$

which is a contradiction. Similarly, one can show that $\Delta u''(t_j) = I_{1j}(u'(t_j))$, $j = 1, 2, \ldots, m$. Therefore, u is a solution of (1.2). □

For $r_1, r_2 \in \mathbb{R}$ with $r_1 < r_2$, we define

$$\alpha(r_1, r_2) = \sup_{v\in\Phi^{-1}((r_1,r_2))} \frac{\Psi(v) - \sup_{u\in\Phi^{-1}((-\infty,r_1))}\Psi(u)}{\Phi(v) - r_1} \tag{2.8}$$

and

$$\beta(r_1,r_2) = \inf_{v\in\Phi^{-1}((r_1,r_2))} \frac{\sup_{u\in\Phi^{-1}((r_1,r_2))}\Psi(u) - \Psi(v)}{r_2 - \Phi(v)}. \tag{2.9}$$

For $r \in \mathbb{R}$, we define

$$\rho_1(r) = \inf_{v\in\Phi^{-1}((-\infty,r))} \frac{\sup_{u\in\Phi^{-1}((-\infty,r))}\Psi(u) - \Psi(v)}{r - \Phi(v)}, \tag{2.10}$$

$$\rho_2(r) = \sup_{v\in\Phi^{-1}((r,+\infty))} \frac{\Psi(v) - \sup_{u\in\Phi^{-1}((-\infty,r])}\Psi(u)}{\Phi(v) - r}. \tag{2.11}$$

3 Main results

3.1 Existence of at least one solution

In this section we derive conditions under which system (1.2) admits at least one solution. For this purpose, we introduce the following assumption.

(H1) Assume that there exist two positive constants k_1 and k_2 such that for each $u \in X$

$$0 \le \sum_{j=1}^{m} \int_0^{u'(t_j)} I_{1j}(s)\,ds \le k_1 \max_{j\in\{1,2,\dots,m\}} \left|u'(t_j)\right|^2 \tag{3.1}$$

and

$$0 \le \sum_{j=1}^{m} \int_0^{u(t_j)} I_{2j}(s)\,ds \le k_2 \max_{j\in\{1,2,\dots,m\}} \left|u(t_j)\right|^2. \tag{3.2}$$

Let $k_0 = 2 - \frac{A}{6} + \frac{B}{60}$ and $k_3 = k_0 + k_1 + \frac{1}{4}k_2$ with k_1 given in (3.1) and k_2 given in (3.2). For constants c_1, c_2, and c satisfying

$$c_1 < \sqrt{2k_0}M_1 c < \sqrt{2k_3}M_1 c < c_2, \tag{3.3}$$

we define

$$a(c_2,c) = \frac{\int_0^1 \max_{|u|\le c_2} F(t,u)\,dt - \int_0^1 F(t,u_1(t))\,dt}{c_2^2 - 2k_3M_1^2c^2} \tag{3.4}$$

and

$$b(c_1,c) = \frac{\int_0^1 F(t,u_1(t))\,dt - \int_0^1 \max_{|u|\le c_1} F(t,u)\,dt}{2k_3M_1^2c^2 - c_1^2}, \tag{3.5}$$

where

$$u_1(t) = ct(1-t). \tag{3.6}$$

Note that for every $c > 0$ and $t \in [0,1]$ we have $|u_1(t)| = ct(1-t) \le \frac{c}{4} < c$. Since $A \le 0 \le B$, then $k_0 > 2$. Thus, if c and c_2 satisfy (3.3), then $c_2 > c$ and $\int_0^1 \max_{|u|\le c_2} F(t,u)\,dt - \int_0^1 F(t,u_1(t))\,dt > 0$ and hence $a(c_2,c) > 0$.

Theorem 3.1 *Assume that* (H1) *is satisfied. If there exist constants* c_1, c_2, *and* c *satisfying* (3.3) *and*

$$0 < a(c_2, c) < b(c_1, c), \tag{3.7}$$

then, for each $\lambda \in (\lambda_1, \lambda_2)$, *system* (1.2) *admits at least one solution* u *and* $\|u\|_X < \frac{c_2}{M_1}$, *where* $\lambda_1 = \frac{1}{2M_1^2 b(c_1,c)}$ *and* $\lambda_2 = \frac{1}{2M_1^2 a(c_2,c)}$.

Proof By Lemma 2.2, it suffices to show the functional φ_λ defined in (2.3) has at least one critical point. We prove this by verifying the conditions given in [10, Theorem 5.1]. Note that Φ defined in (2.4) is a nonnegative Gâteaux differentiable, coercive, and sequentially weakly lower semicontinuous functional, and its Gâteaux derivative admits a continuous inverse on X^*. Moreover, Ψ defined in (2.5) is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Set

$$r_1 = \frac{c_1^2}{2M_1^2}, \qquad r_2 = \frac{c_2^2}{2M_1^2}.$$

Note that $u_1(t) = ct(1-t) \in X$. It then follows from (H1) that

$$\begin{aligned}
\Phi(u_1) &= \frac{1}{2}\|u_1\|_X^2 + \sum_{j=1}^m \int_0^{u_1'(t_j)} I_{1j}(s)\,ds + \sum_{j=1}^m \int_0^{u_1(t_j)} I_{2j}(s)\,ds \\
&= \left(2 - \frac{A}{6} + \frac{B}{60}\right)c^2 + \sum_{j=1}^m \int_0^{c-2ct_j} I_{1j}(s)\,ds + \sum_{j=1}^m \int_0^{ct_j(1-t_j)} I_{2j}(s)\,ds \\
&= k_0 c^2 + \sum_{j=1}^m \int_0^{c-2ct_j} I_{1j}(s)\,ds + \sum_{j=1}^m \int_0^{ct_j(1-t_j)} I_{2j}(s)\,ds \\
&\le k_0 c^2 + k_1 \max_j |c - 2ct_j|^2 + k_2 \max_j \left|ct_j(1-t_j)\right|^2 \\
&\le k_0 c^2 + k_1 c^2 + \frac{1}{4} k_2 c^2 \\
&= k_3 c^2
\end{aligned} \tag{3.8}$$

and

$$\Phi(u_1) \ge \frac{1}{2}\|u_1\|_X^2 = k_0 c^2. \tag{3.9}$$

By (3.3) we have

$$r_1 = \frac{c_1^2}{2M_1^2} < k_0 c^2 \le \Phi(u_1) \le k_3 c^2 < \frac{c_2^2}{2M_1^2} = r_2.$$

For $u \in X$ satisfying $\Phi(u) < r_2$, by Lemma 2.1, one has

$$|u|^2 \le \|u\|_\infty^2 \le M_1^2 \|u\|_X^2 \le 2M_1^2 \Phi(u) < 2M_1^2 r_2 = c_2^2, \quad t \in [0,1],$$

which implies that

$$\Psi(u) = \int_0^1 F\bigl(t, u(t)\bigr)\,dt \le \int_0^1 \max_{|u|\le c_2} F(t,u)\,dt.$$

Hence

$$\sup_{u\in\Phi^{-1}((r_1,r_2))} \Psi(u) \le \sup_{u\in\Phi^{-1}((-\infty,r_2))} \Psi(u) \le \int_0^1 \max_{|u|\le c_2} F(t,u)\,dt. \tag{3.10}$$

For $u \in X$ with $\Phi(u) < r_1$, one can similarly obtain

$$\sup_{u\in\Phi^{-1}((-\infty,r_1))} \Psi(u) \le \int_0^1 \max_{|u|\le c_1} F(t,u)\,dt. \tag{3.11}$$

It follows from the definition of $\beta(r_1, r_2)$ that

$$\beta(r_1,r_2) \le \frac{\sup_{u\in\Phi^{-1}((-\infty,r_2))} \Psi(u) - \Psi(u_1)}{r_2 - \Phi(u_1)}. \tag{3.12}$$

Note that $\Phi(u_1) < r_2$. By (3.10) one has

$$\beta(r_1,r_2) \le \frac{\int_0^1 \max_{|u|\le c_2} F(t,u)\,dt - \int_0^1 F(t,u_1(t))\,dt}{r_2 - \Phi(u_1)}.$$

Making use of $\Phi(u_1) < r_2$, $\int_0^1 \max_{|u|\le c_2} F(t,u)\,dt - \int_0^1 F(t,u_1(t))\,dt > 0$, and (3.8), we obtain

$$\begin{aligned}\beta(r_1,r_2) &\le \frac{\int_0^1 \max_{|u|\le c_2} F(t,u)\,dt - \int_0^1 F(t,u_1(t))\,dt}{\frac{c_2^2}{2M_1^2} - k_3c^2} \\ &= 2M_1^2 \frac{\int_0^1 \max_{|u|\le c_2} F(t,u)\,dt - \int_0^1 F(t,u_1(t))\,dt}{c_2^2 - 2k_3M_1^2c^2} \\ &= 2M_1^2 a(c_2,c) > 0.\end{aligned}$$

By (2.8), and note that $\Phi(u_1) > r_1$, one has

$$\alpha(r_1,r_2) \ge \frac{\Psi(u_1) - \sup_{u\in\Phi^{-1}((-\infty,r_1))} \Psi(u)}{\Phi(u_1) - r_1}.$$

By (3.11) we have

$$\alpha(r_1,r_2) \ge \frac{\int_0^1 F(t,u_1(t))\,dt - \int_0^1 \max_{|u|\le c_1} F(t,u)\,dt}{\Phi(u_1) - r_1}.$$

Note that (3.7) implies that

$$\int_0^1 F\bigl(t,u_1(t)\bigr)\,dt - \int_0^1 \max_{|u|\le c_1} F(t,u)\,dt > 0,$$

which, together with (3.8), gives

$$\begin{aligned}\alpha(r_1,r_2) &\geq \frac{\int_0^1 F(t,u_1(t))\,dt - \int_0^1 \max_{|u|\leq c_1} F(t,u)\,dt}{k_3c^2 - r_1} \\ &= 2M_1^2\frac{\int_0^1 F(t,u_1(t))\,dt - \int_0^1 \max_{|u|\leq c_1} F(t,u)\,dt}{2k_3M_1^2c^2 - c_1^2} \\ &= 2M_1^2 b(c_1,d).\end{aligned}$$

Therefore, $\beta(r_1,r_2) \leq 2M_1^2 a(c_2,c) < 2M_1^2 b(c_1,d) \leq \alpha(r_1,r_2)$. Thus all the conditions in [10, Theorem 5.1] are verified, and hence for each $\lambda \in (\lambda_1,\lambda_2)$ the functional $\varphi_\lambda = \Phi - \lambda\Psi$ admits at least one critical point u such that $r_1 < \Phi(u) < r_2$. Consequently, system (1.2) admits at least one solution u and $\|u\|_X < \frac{c_2}{M_1}$. □

In particular, if we take $c_1 = 0$, then (3.4) and (3.5) become

$$a(c_2,c) = \frac{\int_0^1 \max_{|u|\leq c_2} F(t,u)\,dt - \int_0^1 F(t,u_1(t))\,dt}{c_2^2 - 2k_3M_1^2c^2}$$

and

$$b(c_1,c) = b(0,c) = \frac{\int_0^1 F(t,u_1(t))\,dt}{2k_3M_1^2c^2}.$$

Correspondingly, conditions (3.3) and (3.7) reduce to

$$\sqrt{2k_3}M_1c < c_2 \tag{3.13}$$

and

$$\int_0^1 \max_{|u|\leq c_2} F(t,u)\,dt < \frac{c_2^2}{2k_3M_1^2c^2}\int_0^1 F\big(t,u_1(t)\big)\,dt. \tag{3.14}$$

If (3.13) and (3.14) hold, then

$$\lambda_1 = \frac{k_3c^2}{\int_0^1 F(t,u_1(t))\,dt} := \hat{\lambda}_1$$

and

$$\begin{aligned}\lambda_2 &= \frac{c_2^2 - 2k_3M_1^2c^2}{2M_1^2(\int_0^1 \max_{|u|\leq c_2} F(t,u)\,dt - \int_0^1 F(t,u_1(t))\,dt)} \geq \frac{c_2^2}{2M_1^2\int_0^1 \max_{|u|\leq c_2} F(t,u)\,dt} \\ &:= \hat{\lambda}_2 > \hat{\lambda}_1.\end{aligned}$$

As a consequence, we have the following result.

Corollary 3.2 *Assume that* (H1) *is satisfied. If there exist two constants c and c_2 satisfying* (3.13) *and* (3.14), *then for each $\lambda \in (\hat{\lambda}_1,\hat{\lambda}_2)$ system* (1.2) *admits at least one nontrivial solution u.*

Example 3.1 Consider the boundary value problem

$$\begin{cases} u^{(iv)}(t) = \lambda t, \quad t \neq t_1, t \in [0,1], \\ \Delta u''(t_1) = \frac{1}{4}u'(t_1), \quad t_1 = \frac{1}{2}, \\ -\Delta u'''(t_1) = \frac{1}{5}u(t_1), \quad t_1 = \frac{1}{2}, \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases} \tag{3.15}$$

Here, $f(t,u) = t$, $I_{11}(s) = \frac{1}{4}s$, $I_{21}(s) = \frac{1}{5}s$, $A = B = 0$ and $m = 1$. It is easy to verify that (H1) is satisfied with $k_1 = \frac{1}{8}$ and $k_2 = \frac{1}{10}$. Direct calculations give $F(t,u) = tu$, $k_0 = 2$, $k_3 = \frac{43}{20}$ and $M_1 = 1 + \frac{1}{\pi} \approx 1.318$. Let $c_1 = 0.01$, $c = 12$, $c_2 = 1{,}500$, then c_1, c, c_2 satisfy (3.3) and $a(c_2,c) \approx 3.33 \times 10^{-4} < b(c_1,c) \approx 9.25 \times 10^{-4}$. Thus, it follows from Theorem 3.1 that system (3.15) has at least one solution for $\lambda \in (\lambda_1,\lambda_2) = (311.2, 864.4)$.

3.2 Existence of infinitely many solutions

In this section, we derive some conditions under which system (1.2) admits infinitely many distinct solutions. To this end, we need the following assumptions.

(H2) Assume that

$$\{t_1, t_2, \dots, t_m\} \subseteq \left[\frac{1}{4}, \frac{3}{4}\right].$$

(H3) Assume that

$$F(t,u) \geq 0, \quad \text{for } (t,u) \in \left(\left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right]\right) \times \mathbb{R}.$$

Let $k_4 = 2{,}048(\frac{3}{8} - \frac{9}{10 \cdot 4^4}A + \frac{79}{14 \cdot 4^8}B)$ and $k_5 = k_4 + k_2$ with k_2 given in (3.2). We define

$$\gamma_1 := \liminf_{r \to +\infty} \rho_1(r), \qquad \gamma_2 := \liminf_{r \to (\inf_X \Phi)^+} \rho_1(r), \tag{3.16}$$

where $\rho_1(r)$ is given (2.10). Let

$$\mu_1 = 2M_1^2 \liminf_{\xi \to +\infty} \frac{\int_0^1 \max_{|u| \leq \xi} F(t,u)\, dt}{\xi^2}, \qquad \mu_2 = \frac{1}{k_5} \limsup_{\xi \to +\infty} \frac{\int_{\frac{1}{4}}^{\frac{3}{4}} F(t,\xi)\, dt}{\xi^2}.$$

Theorem 3.3 *Assume that* (H1), (H2), *and* (H3) *are satisfied. If*

$$\mu_1 < \mu_2 \tag{3.17}$$

holds, then for each $\lambda \in (\frac{1}{\mu_2}, \frac{1}{\mu_1})$ *system* (1.2) *has an unbounded sequence of solutions in* X.

Proof We apply [5, Theorem 2.1] to show that the functional φ_λ defined in (2.3) has an unbounded sequence of critical points.

We first show that $\gamma_1 < +\infty$. Let $\{\xi_n\}$ be a sequence of positive numbers such that $\xi_n \to +\infty$ as $n \to \infty$ and

$$\lim_{n \to +\infty} \frac{\int_0^1 \max_{|u| \leq \xi_n} F(t,u)\, dt}{\xi_n^2} = \liminf_{\xi \to +\infty} \frac{\int_0^1 \max_{|u| \leq \xi} F(t,\xi)\, dt}{\xi^2}.$$

For any positive integer n, we let $r_n = \frac{\xi_n^2}{2M_1^2}$. For $u \in X$ satisfying $\Phi(u) < r_n$, similar to the proof of Theorem 3.1, one can show that

$$\|u\|_\infty^2 \le 2M_1^2\Phi(u) < \xi_n^2, \quad t \in [0,1],$$

which implies that

$$\Psi(u) = \int_0^1 F(t,u)\,dt \le \int_0^1 \max_{|u|\le\xi_n} F(t,u)\,dt.$$

Note that $\Psi(0) = \Phi(0) = 0$, thus we have

$$\begin{aligned}
\rho_1(r_n) &= \inf_{v\in\Phi^{-1}((-\infty,r_n))} \frac{\sup_{u\in\Phi^{-1}((-\infty,r_n))}\Psi(u) - \Psi(v)}{r_n - \Phi(v)} \\
&\le \frac{\sup_{u\in\Phi^{-1}((-\infty,r_n))}\Psi(u) - \Psi(0)}{r_n - \Phi(0)} \\
&= \frac{\sup_{u\in\Phi^{-1}((-\infty,r_n))}\Psi(u)}{r_n} \\
&\le \frac{2M_1^2\int_0^1 \max_{|u|\le\xi_n} F(t,u)\,dt}{\xi_n^2},
\end{aligned}$$

which, together with (3.16), gives us

$$\gamma_1 \le 2M_1^2 \liminf_{\xi\to+\infty} \frac{\int_0^1 \max_{|u|\le\xi} F(t,u)\,dt}{\xi^2} = \mu_1 < +\infty.$$

This shows that $(\frac{1}{\mu_2},\frac{1}{\mu_1}) \subseteq (0,\frac{1}{\gamma_1})$. For any fixed $\lambda \in (\frac{1}{\mu_2},\frac{1}{\mu_1})$, it follows from [5, Theorem 2.1] that either $\varphi_\lambda = \Phi - \lambda\Psi$ has a global minimum or there is a sequence $\{u_n\}$ of critical points (local minima) of φ_λ such that $\lim_{n\to+\infty}\|u_n\|_X = +\infty$.

Next we show that the functional φ_λ has no global minimum for $\lambda \in (\frac{1}{\mu_2},\frac{1}{\mu_1})$. Since $\lambda > \frac{1}{\mu_2} = k_5/\limsup_{\xi\to+\infty}\frac{\int_{\frac14}^{\frac34} F(t,\xi)\,dt}{\xi^2}$, we can choose a constant M such that, for each $n \in N = \{1,2,\ldots\}$,

$$\sup_{\xi\ge n}\frac{\int_{\frac14}^{\frac34} F(t,\xi)\,dt}{\xi^2} > M > \frac{k_5}{\lambda}.$$

Thus, there exists $\xi_n \ge n$ such that

$$\frac{\int_{\frac14}^{\frac34} F(t,\xi_n)\,dt}{\xi_n^2} > M.$$

Define $u_n(t)$ as follows:

$$u_n(t) = \begin{cases} 64\xi_n(t^3 - \frac34 t^2 + \frac{3}{16}t), & t \in [0,\frac14), \\ \xi_n, & t \in [\frac14,\frac34], \\ 64\xi_n(-t^3 + \frac94 t^2 - \frac{27}{16}t + \frac{7}{16}), & t \in (\frac34,1]. \end{cases}$$

This, together with (H2), yields

$$\begin{aligned}\Phi(u_n) &= 2{,}048\left(\frac{3}{8}-\frac{9}{10\cdot 4^4}A+\frac{79}{14\cdot 4^8}B\right)\xi_n^2+\sum_{j=1}^{m}\int_0^{\xi_n} I_{2j}(s)\,ds\\ &\le k_4\xi_n^2+k_2\xi_n^2\\ &= k_5\xi_n^2.\end{aligned}$$

It then follows from (H3) that

$$\begin{aligned}\varphi_\lambda(u_n) &= \Phi(u_n)-\lambda\Psi(u_n)\\ &\le k_5\xi_n^2-\lambda\int_{\frac{1}{4}}^{\frac{3}{4}} F(t,\xi_n)\,dt\\ &\le \xi_n^2(k_5-\lambda M).\end{aligned}$$

Note that $k_5-\lambda M<0$. Thus the functional φ_λ is unbounded from below and hence it has no global minimum and the proof is complete. □

Corollary 3.4 *Assume that* (H1), (H2), *and* (H3) *are satisfied. If*

$$\liminf_{\xi\to+\infty}\frac{\int_0^1 \max_{|u|\le\xi} F(t,u)\,dt}{\xi^2}<\frac{1}{2M_1^2}$$

and

$$\limsup_{\xi\to+\infty}\frac{\int_{\frac{1}{4}}^{\frac{3}{4}} F(t,\xi)\,dt}{\xi^2}>k_5$$

hold, then (1.2) *has an unbounded sequence of solutions in* X.

Let

$$\mu_3=2M_1^2\liminf_{\omega\to 0^+}\frac{\int_0^1 \max_{|u|\le\omega} F(t,u)\,dt}{\omega^2},\qquad \mu_4=\frac{1}{k_5}\limsup_{\omega\to 0^+}\frac{\int_{\frac{1}{4}}^{\frac{3}{4}} F(t,\omega)\,dt}{\omega^2}.$$

Theorem 3.5 *Assume that* (H1), (H2), *and* (H3) *are satisfied. If*

$$\mu_3<\mu_4 \tag{3.18}$$

holds, then for each $\lambda\in(\frac{1}{\mu_4},\frac{1}{\mu_3})$ *system* (1.2) *has a sequence of non-zero solutions in* X, *which weakly converges to* 0.

Proof The proof is similar to that of Theorem 3.3 by showing that $\gamma_2<+\infty$ and 0 is not a local minimum of the functional $\varphi_\lambda=\Phi-\lambda\Psi$. □

Example 3.2 Consider

$$\begin{cases} u^{(iv)}(t) - 2u''(t) + u = \lambda f(t,u), & t \neq t_1, t \in [0,1], \\ \Delta u''(t_1) = \frac{1}{10} u'(t_1), & t_1 = \frac{1}{2}, \\ -\Delta u'''(t_1) = \frac{1}{20} u(t_1), & t_1 = \frac{1}{2}, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \tag{3.19}$$

where $f(t,u) = 4tu(1+\sin u) + 2tu^2 \cos u$.

Here $I_{11}(s) = \frac{1}{10}s$, $I_{21}(s) = \frac{1}{20}s$, $A = -2$, $B = 1$ and $m = 1$. Note that

$$\begin{aligned} &\int_0^{u'(t_1)} I_{11}(s)\,ds = \int_0^{u'(t_1)} \frac{1}{10} s\,ds = \frac{1}{20}\left|u'(t_1)\right|^2, \\ &\int_0^{u(t_1)} I_{21}(s)\,ds = \int_0^{u(t_1)} \frac{1}{20} s\,ds = \frac{1}{40}\left|u(t_1)\right|^2, \\ &\quad t_1 = \frac{1}{2} \in \left[\frac{1}{4}, \frac{3}{4}\right], \\ &F(t,u) = 2t(1+\sin u)u^2, \end{aligned}$$

so (H1), (H2), and (H3) are satisfied. Moreover, we have $k_5 \approx 782.6$, and

$$\lim_{\xi\to+\infty} \inf \frac{\int_0^1 \max_{|u|\leq\xi} F(t,u)\,dt}{\xi^2} = 0, \qquad \lim_{\xi\to+\infty} \sup \frac{\int_{\frac{1}{4}}^{\frac{3}{4}} F(t,\xi)\,dt}{\xi^2} = 1.$$

Therefore, condition (3.17) holds and Theorem 3.3 applies: For $\lambda \in (782.6, +\infty)$, (3.19) admits an unbounded sequence of solutions in X.

Example 3.3 Consider the boundary value problem

$$\begin{cases} u^{(iv)}(t) - 2u''(t) + u = \lambda f(t,u), & t \neq t_1, t \in [0,1], \\ \Delta u''(t_1) = \frac{1}{5} u'(t_1), & t_1 = \frac{1}{2}, \\ -\Delta u'''(t_1) = \frac{1}{8} u(t_1), & t_1 = \frac{1}{2}, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \tag{3.20}$$

where

$$f\big(t,u(t)\big) = \begin{cases} 4tu(0.5001 + \frac{1}{2}\cos(\ln|u|) - \frac{1}{4}\sin(\ln(|u|))) & \text{if } u \neq 0, \\ 0 & \text{if } u = 0. \end{cases}$$

In this example, $I_{11}(s) = \frac{1}{5}s$, $I_{21}(s) = \frac{1}{8}s$, $A = -2$ and $B = 1$. The assumptions (H1), (H2), and (H3) clearly hold.

Direct calculations give

$$F\big(t,u(t)\big) = \begin{cases} 2tu^2(0.5001 + \frac{1}{2}\cos(\ln|u|)) & \text{if } u \neq 0, \\ 0 & \text{if } u = 0, \end{cases}$$

$k_5 \approx 762.64$ and

$$\lim_{w\to 0^+} \inf \frac{\int_0^1 \max_{|u|\le w} F(t,u)\,dt}{w^2} = 0.0001, \qquad \lim_{w\to 0^+} \sup \frac{\int_{\frac{1}{4}}^{\frac{3}{4}} F(t,w)\,dt}{w^2} = 0.50005.$$

Hence (3.18) holds. Therefore it follows from Theorem 3.5 that (3.20) admits a sequence of distinct solutions in X provided that $\lambda \in (1525.1, 2877.0)$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors made equal contribution. Both authors read and approved the final manuscript.

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