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Existence results for abstract semilinear evolution differential inclusions with nonlocal conditions

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Abstract
In this paper, we use a new method to study semilinear evolution differential inclusions with nonlocal conditions in Banach spaces. We derive conditions for F and g for the existence of mild solutions. The results obtained here improve and generalize many known results.
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Keywords: semilinear evolution differential inclusions; mild solutions; measure of noncompactness; upper semicontinuous

1 Introduction

In this paper, we discuss the nonlocal initial value problem

$$\begin{cases} x'(t) \in Ax(t) + F(t, x(t)), & t \in I = [0,1], \\ x(0) = g(x), \end{cases} \tag{1.1}$$

where A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators (*i.e.*, C_0-semigroup) $T(t)$ in a Banach space X, and $F : [0,1] \times X \to P_c(X)$, $g : C([0,1]; X) \to X$ are given X-valued functions.

The study of nonlocal evolution equations was initiated by Byszewski [1]. Since these represent mathematical models of various phenomena in physics, Byszewski's work was followed by many others [2–7]. Subsequently, many authors have contributed to the study of the differential inclusions (1.1). Differential inclusions (1.1) were first considered by Aizicovici and Gao [8] when g and $T(t)$ are compact. In [9–12] the semilinear evolution differential inclusions (1.1) were discussed when A generates a compact semigroup. Xue and Song [13] established the existence of mild solutions to the differential inclusions (1.1) when A generates an equicontinuous semigroup and $F(t, \cdot)$ is l.s.c. for a.e. $t \in [0,1]$. In [14] the author proved the existence of mild solutions of the differential inclusions (1.1) when A generates an equicontinuous semigroup and a Banach space X which is separable and uniformly smooth. In [15] Zhu and Li studied the differential inclusions (1.1) when F admits a strongly measurable selector. In [16] the differential inclusions (1.1) were discussed when $\{A(t)\}$ is a family of linear (not necessarily bounded) operators. In [17] local and global existence results for differential inclusions with infinite delay in a Banach space were considered. Benchohra and Ntouyas [18] studied the second-order initial value problems for delay integrodifferential inclusions. In [19, 20] the impulsive multivalued semilinear neutral functional differential inclusions were discussed in the case that the linear semigroup $T(t)$ is compact. The purpose of this paper is to continue the study of these authors. By using a new method, we prove the existence results of mild solutions for (1.1) under the following conditions of g and $T(t)$: g is either compact or Lipschitz continuous and $T(t)$ is an equicontinuous semigroup. So, our work extends and improves many main results such as those in [8–12, 14, 15].

The organization of this work is as follows. In Section 2, we recall some definitions and facts about set-valued analysis and the measure of noncompactness. In Section 3, we give the existence of mild solutions of the nonlocal initial value problem (1.1). In Section 4, an example is given to show the applications of our results.

2 Preliminaries

Let $(X, \|\cdot\|)$ be a real Banach space. Let $P_c(X) = \{A \subseteq X : \text{nonempty, closed, convex}\}$. A multivalued map $G : X \to X$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. We say that G is bounded on bounded sets if $G(B)$ is bounded in X for each bounded set B of X. The map G is called upper semicontinuous (u.s.c.) on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood M of x_0 such that $G(M) \subseteq N$. Also, G is said to be completely continuous if $G(B)$ is relatively compact for every bounded subset $B \subseteq X$. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (*i.e.*, $x_n \to x_0$, $y_n \to y_0$, $y_n \in G(x_n)$ imply $y_0 \in G(x_0)$). Moreover, the following conclusions hold. Let $D \subset X$ and $G(x)$ be closed for all $x \in D$, if G is u.s.c. and D is closed, then graph(G) is closed. If $\overline{G(D)}$ is compact and D is closed, then G is u.s.c. if and only if graph(G) is closed. Finally, we say that G has a fixed point if there exists $x \in X$ such that $x \in G(x)$.

We denote by $C([0,1];X)$ the space of X-valued continuous functions on $[0,1]$ with the norm $\|x\| = \sup\{\|x(t)\|; t \in [0,1]\}$, and by $L^1(0,1;X)$ the space of X-valued Bochner functions on $[0,1]$ with the norm $\|x\| = \int_0^1 \|x(s)\|\,ds$.

A C_0-semigroup $T(t)$ is said to be compact if $T(t)$ is compact for any $t > 0$. If the semigroup $T(t)$ is compact, then $t \to T(t)x$ is equicontinuous at all $t > 0$ with respect to x in all bounded subsets of X; *i.e.*, the semigroup $T(t)$ is equicontinuous. If A is the generator of an analytic semigroup $T(t)$ or a differentiable semigroup $T(t)$, then $T(t)$ is an equicontinuous C_0-semigroup (see [21]). In this paper, we suppose that A generates an equicontinuous semigroup $T(t)$ on X. Since no confusion may occur, we denote by α the Hausdorff measure of noncompactness on both X and $C([0,1];X)$.

Definition 2.1 A function $x \in C([0,1];X)$ is a mild solution of (1.1) if

(1) $x(t) = T(t)g(x) + \int_0^t T(t-s)v(s)\,ds$,

(2) $x(0) = g(x)$, where $v \in S_{F,x} = \{v \in L^1(I,X) : v(t) \in F(t,x(t))\}$.

To prove the existence results in this paper, we need the following lemmas.

Lemma 2.2 [22] *If $W \subseteq C([0,1];X)$ is bounded, then $\alpha(W(t)) \le \alpha(W)$ for all $t \in [0,1]$, where $W(t) = \{x(t); x \in W\} \subseteq X$. Furthermore, if W is equicontinuous on $[0,1]$, then $\alpha(W(t))$ is continuous on $[0,1]$, and $\alpha(W) = \sup\{\alpha(W(t)); t \in [0,1]\}$.*

Lemma 2.3 [22] *If* $\{W_n\}_{n=1}^{+\infty}$ *is a decreasing sequence of bounded closed nonempty subsets of* X *and* $\lim_{n\to+\infty}\alpha(W_n)=0$, *then* $\bigcap_{n=1}^{+\infty} W_n$ *is nonempty and compact in* X.

Lemma 2.4 [23] *If* $\{u_n\}_{n=1}^{\infty}\subset L^1(0,1;X)$ *is uniformly integrable, then* $\alpha(\{u_n(t)\}_{n=1}^{\infty})$ *is measurable and*

$$\alpha\left(\left\{\int_0^t u_n(s)\,ds\right\}_{n=1}^{\infty}\right)\le 2\int_0^t \alpha\left(\left\{u_n(s)\right\}_{n=1}^{\infty}\right)ds. \tag{2.1}$$

Lemma 2.5 [24] *If the semigroup* $T(t)$ *is equicontinuous and* $\eta\in L^1(0,1;\Re^+)$, *then the set* $\{t\to\int_0^t T(t-s)x(s)\,ds; x\in L^1(0,1;\Re^+), \|x(s)\|\le\eta(s)$, *for a.e.* $s\in[0,1]\}$ *is equicontinuous on* $[0,1]$.

Lemma 2.6 [25] *If* W *is bounded, then for each* $\varepsilon>0$, *there is a sequence* $\{u_n\}_{n=1}^{\infty}\subseteq W$ *such that*

$$\alpha(W)\le 2\alpha\left(\{u_n\}_{n=1}^{\infty}\right)+\varepsilon. \tag{2.2}$$

A countable set $\{f_n\}_{n=1}^{\infty}\subset L^1(0,1;X)$ is said to be semicompact if

(a) it is integrably bounded: $\|f_n(t)\|\le\omega(t)$ for a.e. $t\in[0,1]$ and every $n\ge1$, where $\omega(\cdot)\in L^1(0,1;\Re^+)$;

(b) the set $\{f_n(t)\}_{n=1}^{\infty}$ is relatively compact for a.e. $t\in[0,1]$.

Lemma 2.7 [26] *Every semicompact set is weakly compact in the space* $L^1(0,1;X)$.

Lemma 2.8 [16, 26] *If* $\{f_n\}_{n=1}^{\infty}\subset L^1(0,1;X)$ *is semicompact, then* $\{\int_0^t T(t-s)f_n(s)\,ds\}_{n=1}^{\infty}$ *is relatively compact in* $C([0,1];X)$ *and, moreover, if* $f_n\rightharpoonup f_0$, *then*

$$\int_0^t T(t-s)f_n(s)\,ds\to\int_0^t T(t-s)f_0(s)\,ds$$

as $n\to\infty$.

The map $F: W\subseteq X\to X$ is said to be α contraction if there exists a positive constant $k<1$ such that

$$\alpha\left(F(Q)\right)\le k\alpha(Q)$$

for any bounded closed subset $Q\subseteq W$.

Lemma 2.9 [27–30] (Fixed point theorem) *If* $W\subseteq X$ *is a nonempty, bounded, closed, convex and compact subset, the map* $F: W\to 2^W$ *is upper semicontinuous with* $F(x)$ *a nonempty, closed, convex subset of* W *for each* $x\in W$, *then* F *has at least one fixed point in* W.

Lemma 2.10 [26] (Fixed point theorem) *If* $W\subseteq X$ *is nonempty, bounded, closed and convex, the map* $F: W\to 2^W$ *is a closed* α *contraction map with* $F(x)$ *a nonempty, convex and compact subset of* W *for each* $x\in W$, *then* F *has at least one fixed point in* W.

3 Main results

In this section, by using the measure of noncompactness and fixed point theorems, we give the existence results of the nonlocal initial value problem (1.1). Here we list the following hypotheses.

(1) The C_0 semigroup $T(t)$ generated by A is equicontinuous. We denote $N = \sup\{\|T(t)\|; t \in [0,1]\}$.

(2) $g : C([0,1];X) \to X$ is continuous and compact, there exist positive constants c and d such that $\|g(x)\| \leq c\|x\| + d$, $\forall x \in C([0,1];X)$.

(3) The multivalued operator $F : [0,1] \times X \to P_c(X)$ satisfies the hypotheses:
$t \to F(t,x)$ is measurable for every $x \in X$;
$x \to F(t,x)$ is u.s.c. for a.e. $t \in [0,1]$;
the set $S_{F,x} = \{v \in L^1(I,X) : v(t) \in F(t,x(t));\ \text{for a.e. } t \in [0,1]\}$ is nonempty.

(4) There exists $L \in L^1(0,1;\Re^+)$ such that for any bounded $D \subset X$,

$$\alpha\bigl(F(t,D)\bigr) \leq L(t)\alpha(D)$$

for a.e. $t \in [0,1]$.

(5) There exist a function $m \in L^1(0,1;\Re^+)$ and a nondecreasing continuous function $\Omega : \Re^+ \to \Re^+$ such that

$$\bigl\|F(t,x)\bigr\| \leq m(t)\Omega\bigl(\|x\|\bigr)$$

for all $x \in X$, and a.e. $t \in [0,1]$.

Remark 3.1 If $\dim X < \infty$, then $S_{F,x} \neq \emptyset$ for each $x \in C([0,1];X)$ (see Lasota and Opial [31]). If $\dim X = \infty$ and $x \in C([0,1];X)$, then the set $S_{F,x}$ is nonempty if and only if the function $Y : [0,1] \to \Re$ defined by $Y(t) = \inf\{\|v\| : v \in F(t,x(t))\}$ belongs to $L^1(0,1;\Re^+)$ (see Hu and Papageorgiou [32]).

The following lemma plays a crucial role in the proof of the main theorem.

Lemma 3.2 [26] *Under assumptions* (3)-(5), *if we consider sequences* $\{x_n\}_{n=1}^{\infty} \subset C([0,1];X)$ *and* $\{v_n\}_{n=1}^{\infty} \subset L^1(0,1;X)$, *where* $v_n \in S_{F,x_n}$, *such that* $x_n \to x$, $v_n \rightharpoonup v$, *then* $v \in S_{F,x}$.

Now we give the existence results under the above hypotheses.

Theorem 3.3 *If* (1)-(5) *are satisfied, then there is at least one mild solution for* (1.1) *provided that there exists a constant R with*

$$\int_0^1 m(s)\,ds < \int_{N(cR+d)}^{R} \frac{1}{N\Omega(s)}\,ds. \tag{3.1}$$

Proof Define the operator $\Gamma : C([0,1];X) \to C([0,1];X)$ by

$$(\Gamma x)(t) = \left\{y(t) \in C\bigl([0,1];X\bigr) : y(t) = T(t)g(x) + \int_0^t T(t-s)v(s)\,ds; v \in S_{F,x}\right\}.$$

We shall show that the multivalued Γ has at least one fixed point. The fixed point is then a mild solution of the problem (1.1).

(1) We contract a bounded, convex, closed and compact set $W \subset C([0,1];X)$ such that Γ maps W into itself.

In view of (3.1), we know there exists a constant $\eta > 0$ such that

$$\int_0^1 m(s)\,ds < \int_{T_0+\eta}^{R} \frac{1}{N\Omega(s)}\,ds, \tag{3.2}$$

where $T_0 = N(cR + d)$.

Then there exists a positive integer K such that

$$\int_{T_0+\eta}^{T_0+K\eta} \frac{1}{N\Omega(s)}\,ds < \int_0^1 m(s)\,ds \le \int_{T_0+\eta}^{T_0+(K+1)\eta} \frac{1}{N\Omega(s)}\,ds. \tag{3.3}$$

Hence, we get $0 = t_0 < t_1 < t_2 < \cdots < t_{K-1} < t_K = 1$ such that

$$\begin{aligned}
&\int_0^{t_1} m(s)\,ds = \int_{T_0+\eta}^{T_0+2\eta} \frac{1}{N\Omega(s)}\,ds, \\
&\int_{t_1}^{t_2} m(s)\,ds = \int_{T_0+2\eta}^{T_0+3\eta} \frac{1}{N\Omega(s)}\,ds, \\
&\cdots, \\
&\int_{t_{K-2}}^{t_{K-1}} m(s)\,ds = \int_{T_0+(K-1)\eta}^{T_0+K\eta} \frac{1}{N\Omega(s)}\,ds, \\
&\int_{t_{K-1}}^{1} m(s)\,ds \le \int_{T_0+K\eta}^{T_0+(K+1)\eta} \frac{1}{N\Omega(s)}\,ds.
\end{aligned}$$

We denote $W_0 = \{x \in C([0,1];X), \|x(t)\|_i = \sup\{\|x(t)\| : t \in [t_{i-1}, t_i]\} \le T_0 + i\eta, i = 1,2,\ldots,K\}$, then $W_0 \subseteq C([0,1];X)$ is nonempty, bounded, closed and convex.

For any $x \in W_0$, we have

$$(\Gamma x)(t) = \left\{ y(t) : y(t) = T(t)g(x) + \int_0^t T(t-s)v(s)\,ds;\, v(t) \in S_{F,x} \right\}.$$

Therefore

$$\begin{aligned}
\|y(t)\| &\le \|T(t)g(x)\| + \left\| \int_0^t T(t-s)v(s)\,ds \right\| \le N\bigl(c\|x\| + d\bigr) + N\int_0^t m(s)\Omega\bigl(\|x(s)\|\bigr)\,ds \\
&\le N\bigl(c(T_0 + K\eta) + d\bigr) + N\int_0^t m(s)\Omega\bigl(\|x(s)\|\bigr)\,ds \\
&\le N(cR + d) + N\int_0^t m(s)\Omega\bigl(\|x(s)\|\bigr)\,ds \le T_0 + N\int_0^t m(s)\Omega\bigl(\|x(s)\|\bigr)\,ds,
\end{aligned}$$

and

$$\begin{aligned}
\|y\|_i &= \sup\bigl\{\|y(t)\| : t \in [t_{i-1}, t_i]\bigr\} \\
&\le \sup\left\{ T_0 + N\int_0^t m(s)\Omega\bigl(\|x(s)\|\bigr)\,ds : t \in [t_{i-1}, t_i] \right\}
\end{aligned}$$

$$
\begin{aligned}
&\le T_0 + N\int_0^{t_i} m(s)\Omega\bigl(\|x(s)\|\bigr)\,ds \\
&\le T_0 + N\biggl[\int_0^{t_1} m(s)\Omega\bigl(\|x(s)\|\bigr)\,ds + \int_{t_1}^{t_2} m(s)\Omega\bigl(\|x(s)\|\bigr)\,ds \\
&\quad + \cdots + \int_{t_{i-1}}^{t_i} m(s)\Omega\bigl(\|x(s)\|\bigr)\,ds\biggr] \\
&\le T_0 + N\biggl[\int_0^{t_1} m(s)\,ds\Omega(T_0+\eta) + \int_{t_1}^{t_2} m(s)\,ds\Omega(T_0+2\eta) \\
&\quad + \cdots + \int_{t_{i-1}}^{t_i} m(s)\,ds\Omega(T_0+i\eta)\biggr] \\
&\le T_0 + N\biggl[\int_{T_0+\eta}^{T_0+2\eta} \frac{1}{N\Omega(s)}\,ds\Omega(T_0+\eta) + \int_{T_0+2\eta}^{T_0+3\eta} \frac{1}{N\Omega(s)}\,ds\Omega(T_0+2\eta) \\
&\quad + \cdots + \int_{T_0+i\eta}^{T_0+(i+1)\eta} \frac{1}{N\Omega(s)}\,ds\Omega(T_0+i\eta)\biggr] \\
&\le T_0 + i\eta,
\end{aligned}
$$

which implies $\Gamma : W_0 \to 2^{W_0}$ is a bounded operator.

Define $W_1 = \overline{conv}\Gamma(W_0)$, where $\overline{conv}$ means the closure of the convex hull in $C([0,1];X)$. Then $W_1 \subset C([0,1];X)$ is nonempty bounded closed convex on $[0,1]$ with $W_1 \subseteq W_0$. Let $W_{n+1} = \overline{conv}\Gamma(W_n)$ for all $n \ge 1$. Similarly to the above discussions, we know that $W_{n+1} \subseteq W_n$ for $n = 1,2,\ldots$ as $W_1 \subseteq W_0$ and $W_1, W_2, \ldots$ are both nonempty, closed, bounded and convex. Thus $\{W_n\}_{n=1}^{+\infty}$ is a decreasing sequence consisting of subsets of $C([0,1];X)$. Moreover, set

$$
W = \bigcap_{n=1}^{+\infty} W_n,
$$

then W is a convex, closed and bounded subset of $C([0,1];X)$ and $\Gamma(W) \subseteq W$.

Now, we claim that W is nonempty and compact in $C([0,1];X)$. To do so, from Lemma 2.6, we know for arbitrary given $\epsilon > 0$, there exist sequences $\{v_n\}_{n=1}^{+\infty} \subset S_{F,W_n}$ such that

$$
\begin{aligned}
\alpha\bigl(W_{n+1}(t)\bigr) &= \alpha\bigl((\Gamma W_n)(t)\bigr) \\
&\le 2\alpha\biggl(\int_0^t T(t-s)v_n(s)_{n=1}^{\infty}\,ds\biggr) + \varepsilon \\
&\le 4\int_0^t \alpha\bigl(T(t-s)v_n(s)_{n=1}^{\infty}\bigr)\,ds + \varepsilon \\
&\le 4N\int_0^t \alpha\bigl(v_n(s)_{n=1}^{\infty}\bigr)\,ds + \varepsilon \\
&\le 4N\int_0^t \alpha(F\bigl(s, W_n(s)\bigr)\,ds + \varepsilon \\
&\le 4N\int_0^t L(s)\alpha\bigl(W_n(s)\bigr)\,ds + \varepsilon.
\end{aligned}
$$

Since this is true for arbitrary $\epsilon > 0$, we have

$$\alpha\big(W_{n+1}(t)\big) \leq 4N\int_0^t L(s)\alpha\big(W_n(s)\big)\,ds.$$

Because W_n is decreasing for n, we can define

$$\mu(t) = \lim_{n\to+\infty}\alpha\big((W_n)(t)\big).$$

Let $n \to +\infty$, we have

$$\mu(t) \leq 4N\int_0^t L(s)\mu(s)\,ds.$$

It implies that $\mu(t) = 0$ for all $t \in [0,1]$. By Lemma 2.2, we know that $\lim_{n\to+\infty}\alpha(W_n) = 0$. Using Lemma 2.3, we obtain $W = \bigcap_{n=1}^{+\infty} W_n$ is nonempty and compact in $C([0,1];X)$.

(2) We shall show that Γ is closed on W with closed convex values. It is very easy to see that Γ has convex values.

Let us now verity that graph(Γ) is closed. Let $\{x_n\}_{n=1}^{\infty} \subset W$ with $x_n \to x$ in $C([0,1];X)$, and $y_n \in \Gamma x_n$ with $y_n \to y$ in $C([0,1];X)$. Moreover, let $\{v_n\}_{n=1}^{\infty} \subset L^1(0,1;X)$ be a sequence such that $v_n \in S_{F,x_n}$ for any $n \geq 1$, and

$$y_n(t) = T(t)g(x_n) + \int_0^t T(t-s)v_n(s)\,ds.$$

As $x_n \to x$ in $C([0,1];X)$, we know that $\{x_n\}_{n=1}^{\infty}$ is a bounded set of $C([0,1];X)$, we denote $R_x = \sup\{\|x_n\| : n = 1,2,\ldots\}$.

From hypothesis (5), we obtain

$$\big\|v_n(t)\big\| \leq \big\|F(t,x_n(t)\big\| \leq m(t)\Omega\big(\|x_n\|\big) \leq m(t)\Omega(R_x).$$

Then we have the set $\{v_n\}_{n=1}^{\infty}$ is integrably bounded for a.e. $t \in [0,1]$.

From hypothesis (4), we know

$$\alpha\big(\{v_n(t)\}_{n=1}^{\infty}\big) \leq \alpha\big(F\big(t,\{x_n(t)\}_{n=1}^{\infty}\big)\big) \leq L(t)\alpha\big(\{x_n(t)\}_{n=1}^{\infty}\big) = 0$$

for a.e. $t \in [0,1]$. Then the set $\{v_n(t)\}_{n=1}^{\infty}$ is relatively compact for a.e. $t \in [0,1]$.

So, the set $\{v_n(t)\}_{n=1}^{\infty}$ is semicompact. By applying Lemma 2.7, it yields that $\{v_n(t)\}_{n=1}^{\infty}$ is weakly compact in $L^1(0,1;X)$. We get that there exists $v \in L^1(0,1;X)$ such that $v_n \rightharpoonup v$. Therefore, we infer that

$$\int_0^t T(t-s)v_n(s)\,ds \to \int_0^t T(t-s)v(s)\,ds.$$

Further, we have

$$y_n(t) \to T(t)g(x) + \int_0^t T(t-s)v(s)\,ds,$$

and hence

$$y(t) = T(t)g(x) + \int_0^t T(t-s)v(s)\,ds.$$

By Lemma 3.2, it implies that $v \in S_{F,x}$, *i.e.*, $y \in \Gamma(x)$. Therefore graph(Γ) is closed. And hence Γ has closed values on W.

(4) Γ is u.s.c. on W.

Since $\overline{\Gamma W} \subseteq W$ is compact, W is closed and graph(Γ) is closed, we can come to the conclusion that Γ is u.s.c. (see [30]).

Finally, due to fixed point Lemma 2.9, Γ has at least one point $x \in \Gamma(x)$, and x is a mild solution to the semilinear evolution differential inclusions with the nonlocal conditions (1.1). Thus the proof is complete. □

Remark 3.4 In [8–12] the authors discuss the nonlocal initial value problem (1.1) when $T(t)$ is compact. In [14] the existence of mild solutions of the differential inclusions (1.1) is proved when A generates an equicontinuous semigroup and Banach space X is separable and uniformly smooth. In this paper, by using a new method, we prove the operator Γ maps compact set W into itself. We do not impose any restriction on the coefficient $L(t)$, and we only require $T(t)$ to be an equicontinuous semigroup. So, Theorem 3.3 generalizes and improves the related results in [8–12, 14].

Theorem 3.5 [15] *If* (1)-(5) *are satisfied, then there is at least one mild solution for* (1.1) *provided that there exists a constant* $R > 0$ *such that*

$$N(cR + d) + N\int_0^1 m(s)\,ds\Omega(R) \le R. \tag{3.4}$$

Proof In view of (3.4), we get

$$\int_0^1 m(s)\,ds \le \frac{R - N(cR+d)}{N\Omega(R)} < \int_{N(cR+d)}^R \frac{1}{N\Omega(s)}\,ds.$$

From Theorem 3.3, the nonlocal initial value problem (1.1) has at least one mild solution. □

Remark 3.6 If $N = 1$, $c = \frac{1}{3}$, $d = 0$, $\Omega(x) = x$ and $\int_0^1 m(s)\,ds = 1$. We cannot obtain a constant R such that

$$\frac{1}{3}R + R \le R.$$

By using Theorem 3.5, we do not know whether or not equation (1.1) has a mild solution. But we know there exists a constant $R = 1$ such that

$$\int_0^1 m(s)\,ds = 1 < \ln 3 = \int_{N(cR+d)}^R \frac{1}{N\Omega(s)}\,ds.$$

So, Theorem 3.3 is better than Theorem 3.5.

Theorem 3.7 [12] *If* (1)-(5) *are satisfied and* $\|g(x)\| \leq d$, *then there is at least one mild solution for* (1.1) *provided that*

$$\int_0^1 m(s)\,ds < \int_d^{+\infty} \frac{1}{N\Omega(s)}\,ds. \tag{3.5}$$

Proof In view of (3.5), we get there exists a constant R such that

$$\int_0^1 m(s)\,ds < \int_{0R+d}^{R} \frac{1}{N\Omega(s)}\,ds.$$

By Theorem 3.3, we complete the proof of this theorem. □

Next, we give the existence result for (1.1) when g is Lipschitz continuous.

We suppose that:

(6) There exists a constant $c \in \Re^+$ such that $\|g(u) - g(v)\| \leq c\|u - v\|$ for all $u, v \in C([0,1];X)$. Therefore, $\|g(x)\| \leq c\|x\| + d$, where $d = \|g(0)\|$.

Theorem 3.8 *If* (1) *and* (3)-(6) *are satisfied and*

$$Nc + 4N \int_0^1 L(s)\,ds < 1,$$

then there is at least one mild solution for (1.1) *provided that there exists a constant R satisfying*

$$\int_0^1 m(s)\,ds < \int_{T_0}^{+\infty} \frac{1}{N\Omega(s)}\,ds. \tag{3.6}$$

Proof With the same arguments as given in the first portion of the proof of Theorem 3.3, we know $\Gamma : W_0 \to 2^{W_0}$ is a bounded map with convex values and is closed on W_0.

Now, we prove the values of Γ are compact in $C([0,1];X)$.

Let $x \in C([0,1];X)$ and $y_n \in \Gamma(x)$. To prove that $\Gamma(x)$ is compact, we have to show that y_n has a subsequence converging to a point $y \in \Gamma(x)$. We have $v_n \in S_{F,x}$ such that

$$y_n(t) = T(t)g(x) + \int_0^t T(t-s)v_n(s)\,ds.$$

From hypothesis (5), we obtain

$$\|v_n(t)\| \leq \|F(t, x(t))\| \leq m(t)\Omega(\|x\|).$$

Then we have the set $\{v_n\}_{n=1}^{\infty}$ is integrably bounded for a.e. $t \in [0,1]$.

From hypothesis (4), we know

$$\alpha(\{v_n(t)\}_{n=1}^{\infty}) \leq \alpha(F(t, x(t))) \leq L(t)\alpha(x(t)) = 0$$

for a.e. $t \in [0,1]$. Then the set $\{v_n(t)\}_{n=1}^{\infty}$ is relatively compact for a.e. $t \in [0,1]$.

So, the set $\{v_n(t)\}_{n=1}^{\infty}$ is semicompact. By applying Lemma 2.7, it yields that $\{v_n(t)\}_{n=1}^{\infty}$ is weakly compact in $L^1(0,1;X)$. We get that there exists $v \in L^1(0,1;X)$ such that $v_n \rightharpoonup v$. Therefore, we infer that

$$\int_0^t T(t-s)v_n(s)\,ds \to \int_0^t T(t-s)v(s)\,ds,$$

and

$$\lim_{n\to+\infty} y_n(t) = T(t)g(x) + \int_0^t T(t-s)v(s)\,ds = y(t).$$

By Lemma 3.2, it implies that $v \in S_{F,x}$, *i.e.*,$y \in \Gamma(x)$. Therefore Γ has compact values.

Next, we prove Γ is an α contraction map. For any $B \subseteq W_0$, we have

$$\begin{aligned}\alpha\big((\Gamma B)(t)\big) &= \alpha\left(T(t)g(B) + \int_0^t T(t-s)S_{F,B}\,ds\right)\\ &\le Nc\alpha(B) + \alpha\left(\int_0^t T(t-s)S_{F,B}\,ds\right).\end{aligned}$$

From Lemma 2.6, we know for arbitrary given $\epsilon > 0$, there exist sequences $\{v_n\}_{n=1}^{+\infty} \subset S_{F,B}$ such that

$$\begin{aligned}\alpha\left(\int_0^t T(t-s)S_{F,B}\,ds\right) &= 2\alpha\left(\int_0^t T(t-s)v_n(s)_{n=1}^{\infty}\,ds\right) + \varepsilon\\ &\le 4\int_0^t \alpha\big(T(t-s)v_n(s)_{n=1}^{\infty}\big)\,ds + \varepsilon\\ &\le 4N\int_0^t \alpha\big(v_n(s)_{n=1}^{\infty}\big)\,ds + \varepsilon\\ &\le 4N\int_0^t \alpha\big(F\big(s,B(s)\big)\big)\,ds + \varepsilon\\ &\le 4N\int_0^t L(s)\alpha\big(B(s)\big)\,ds + \varepsilon\\ &\le 4N\alpha(B)\int_0^1 L(s)\,ds + \varepsilon.\end{aligned}$$

Since this is true for arbitrary $\epsilon > 0$, we have

$$\alpha\left(\int_0^t T(t-s)S_{F,B}\,ds\right) \le 4N\alpha(B)\int_0^1 L(s)\,ds.$$

Therefore, we obtain

$$\alpha(\Gamma B) \le \left(Nc + 4N\int_0^1 L(s)\,ds\right)\alpha(B).$$

Noting $Nc + 4N\int_0^t L(s)\,ds < 1$, therefore Γ is an α contraction map.

Finally, due to Lemma 2.10, Γ has at least one fixed point. This completes the proof. □

4 An example

In this section, as an application of our main results, an example is presented. We consider the following partial differential equation:

$$\begin{cases} \frac{\partial u(\zeta,t)}{\partial t} + \Sigma_{|\alpha|\le 2m} a_\alpha(\zeta) D^\alpha u(\zeta,t) \in f(t,u(\zeta,t)), & (\zeta,t) \in \Omega \times [0,1], \\ u(\zeta,t) = 0, & (\zeta,t) \in \partial\Omega \times [0,1], \\ u(\zeta,0) = \int_\Omega \int_0^1 k(t,\zeta,\eta,u(\eta,t))\,dt\,d\eta, & \zeta \in \Omega, \end{cases} \tag{4.1}$$

where Ω is a bounded domain in $\Re^n$ with a smooth boundary $\partial\Omega$, $a_\alpha(\zeta)$ is a smooth real function on $\bar{\Omega}$.

We suppose that

(a) The differential operator $\Sigma_{|\alpha|\le 2m} a_\alpha(\zeta) D^\alpha$ is strongly elliptic [21].

(b) The function $k : [0,1] \times \Omega \times \Omega \times \Re \to \Re$ satisfies the following conditions:

(b_1) $k(t,\zeta,\eta,r)$ is a continuous function about r for a.e. $(t,\zeta,\eta) \in [0,1] \times \Omega \times \Omega$.

(b_2) $k(t,\zeta,\eta,r)$ is measurable about (t,ζ,η) for each fixed $r \in \Re$.

(b_3) For any $R > 0$, there is $\beta_R \in L^1([0,1] \times \Omega \times \Omega \times \Re; \Re^+)$ such that

$$\left|k(t,\zeta,\eta,r) - k\left(t,\zeta',\eta,r\right)\right| \le \beta_R\left(t,\zeta,\zeta',\eta\right)$$

for all $(t,\zeta,\eta,r), (t,\zeta',\eta,r) \in ([0,1] \times \Omega \times \Omega \times \Re)$ with $|r| \le R$, and

$$\lim_{\Delta\zeta \to 0} \int_\Omega \int_0^1 \beta_R(t,\zeta,\Delta\zeta,\eta)\,dt\,d\eta = 0$$

uniformly for $\zeta \in \Omega$.

(b_4) There exist $a(\cdot) \in L(0,1)$ and $d(\cdot) \in L^2([0,1] \times \Omega \times \Omega, \Re^+)$ such that

$$\left|k(t,\zeta,\eta,r)\right| \le a(t)r + d(t,\zeta,\eta)$$

for all $(t,\zeta,\eta,r) \in ([0,1] \times \Omega \times \Omega \times \Re)$.

Let $D(A) = H^{2m} \cap H_0^m(\Omega)$ and $Au(\zeta) = -\Sigma_{|\alpha|\le 2m} a_\alpha(\zeta) D^\alpha u(\zeta,\cdot)$, then A generates an analytic semigroup on $X = L^2(\Omega)$ ([21]). We suppose

$$g(u)(\zeta) = \int_\Omega \int_0^1 k\left(t,\zeta,\eta,u(\eta,t)\right)dt\,d\eta.$$

From [33], we obtain g satisfies hypothesis (2).

Then equation (4.1) can be regarded as the following nonlocal semilinear evolution equation:

$$\begin{cases} u'(t) \in Au(t) + f(t,u(t)), & t \in I = [0,1], \\ u(0) = g(u). \end{cases} \tag{4.2}$$

By using Theorem 3.3, the problem (4.1) has at least one mild solution $u \in C([0,1]; L^2(\Omega))$ provided that hypotheses (3)-(5) and (3.1) hold.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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