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# Nonsimple material problems addressed by the Lagrange's identity

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**Abstract**
Our paper is concerned with some basic theorems for nonsimple thermoelastic materials. By using the Lagrange identity, we prove the uniqueness theorem and some continuous dependence theorems without recourse to any energy conservation law, or to any boundedness assumptions on the thermoelastic coefficients. Moreover, we avoid the use of positive definiteness assumptions on the thermoelastic coefficients.

**Keywords:** Lagrange identity; nonsimple materials; uniqueness; continuous dependence

## 1 Introduction

Even classical elasticity does not consider the inner structure, the material response of materials to stimuli depends in a relevant way on its internal structure. Thus, it has been needed to develop some new mathematical models for continuum materials where this kind of effects was taken into account. Some of them are nonsimple elastic solids. It is known that from a mathematical point of view, these materials are characterized by the inclusion of higher-order gradients of displacement in the basic postulates.

The theory of nonsimple elastic materials was first proposed by Toupin in his famous article [1]. Also, among the first studies devoted to this material, we must mention those belonging to Green and Rivlin [2] and Mindlin [3].

The interest to introduce high-order derivatives consists in the fact that the possible configurations of the materials are clarified more and more finely by the values of the successive higher gradients.

As it is known, the constitutive equations of nonsimple elastic solids are known to contain first- and second-order gradients, both contributing to dissipation. It is then interesting to understand the relevance of the two different dissipation mechanisms which can appear in the theory. In fact, the simultaneous presence of both mechanisms can be analyzed as well, with inessential changes in the proofs. In that situation, the behavior turns out to be the same as if only the higher-order dissipation appears in the equations.

In the last decade many studies have been devoted to nonsimple materials. We remember only three of them, differing in issues addressed, though in essence they are dedicated to nonsimple materials. So, in the paper of Pata and Quintanilla [4], the theory is linearized, and a uniqueness result is presented.

Also, the study [5] of Martinez and Quintanilla is devoted to study the incremental problem in the thermoelastic theory of nonsimple elastic materials.

A linearized theory of thermoelasticity for nonsimple materials is derived within the framework of extended thermodynamics in the paper of Ciarletta [6]. The theory is linearized, and a uniqueness result is presented. A Galerkin-type solution of the field equations and fundamental solutions for steady vibrations are also studied.

Previous papers on the uniqueness and continuous dependence in elasticity or thermoelasticity were based almost exclusively on the assumptions that the elasticity tensor or thermoelastic coefficients are positive definite (see, for instance, the paper [7]).

In other papers, authors recourse the energy conservation law in order to derive the uniqueness or continuous dependence of solutions. For instance, a uniqueness result was indicated in paper [8] of Green and Lindsay by supplementing the restrictions arising from thermodynamics with certain definiteness assumptions.

We want to outline that there are many papers which employ the various refinements of the Lagrange identity, of which we remember only a few, namely papers [9, 10] and [11]. Also, a lot of papers are dedicated to Cesaro means, as [12–14] and [9] for instance.

The objective of our study is to examine by a new approach the mixed initial-boundary value problem in the context of thermoelasticity of nonsimple materials. The approach is developed on the basis of Lagrange identity and its consequences. Therefore, we establish the uniqueness and continuous dependence of solutions with respect to body forces, body couple, generalized external body load and heat supply. We also deduce the continuous dependence of solutions of our problem with respect to initial data and, finally, with respect to thermoelastic coefficients. The results are obtained for bounded regions of the Euclidian three-dimensional space. We point out, again, that the results are obtained without recourse to the energy conservation law or to any boundedness assumptions on the thermoelastic coefficients. Also, we avoid the use of definiteness assumptions on the thermoelastic coefficients.

## 2 Basic equations

We assume that a bounded region $B$ of the three-dimensional Euclidian space $R^3$ is occupied by a nonsimple elastic body, referred to the reference configuration and a fixed system of rectangular Cartesian axes. Let $\bar{B}$ denote the closure of $B$ and call $\partial B$ the boundary of the domain $B$. We consider $\partial B$ to be a piecewise smooth surface and designate by $n_i$ the components of the outward unit normal to the surface $\partial B$. Letters in boldface stand for vector fields. We use the notation $v_i$ to designate the components of the vector $\mathbf{v}$ in the underlying rectangular Cartesian coordinates frame. Superposed dots stand for the material time derivative. We employ the usual summation and differentiation conventions: the subscripts are understood to range over integer $(1,2,3)$. Summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate.

The spatial argument and the time argument of a function will be omitted when there is no likelihood of confusion. We refer the motion of the body to a fixed system of rectangular Cartesian axes $Ox_i$, $i = 1,2,3$. Let us denote by $u_i$ the components of the displacement vector and by $\theta$ the temperature measured from the constant absolute temperature $\theta_0$ of the body in its reference state.

As usual, we denote by $\sigma_{ij}$ the components of the stress tensor and by $m_{ijk}$ the components of the hyperstress tensor over $B$.

We here will use the theory and the notation in the way developed by Iesan in his book, which tackles also nonsimple materials [15].

The equations of motion from thermoelasticity of nonsimple materials are as follows (see also [16]):

$$\sigma_{ji,j} + m_{sji,sj} + \varrho f_i = \varrho \ddot{u}_i. \tag{1}$$

The equation of energy is given by

$$\theta_0 \dot{S} = -q_{i,i} + \varrho r. \tag{2}$$

For an anisotropic and homogeneous nonsimple thermoelastic material, the constitutive equations have the form

$$\begin{aligned}
&\sigma_{ij} = A_{ijrs}\varepsilon_{rs} + B_{ijpqr}\chi_{pqr} + E_{ij}\theta, \\
&m_{ijk} = B_{rsijk}\varepsilon_{rs} + C_{ijkmnr}\chi_{mnr} + D_{ijk}\theta, \\
&S = -E_{ij}\varepsilon_{ij} - D_{ijk}\chi_{ijk} + \frac{c}{T_0}\theta - b_i\theta_{,i}, \\
&q_i = \theta_0(b_i\dot{\theta} - k_{ij}\theta_{,j}).
\end{aligned} \tag{3}$$

The kinematic characteristics of the body (components of the strain tensors) $\varepsilon_{ij}$ and $\chi_{ijk}$ are defined by means of the geometric equations

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \qquad \chi_{ijk} = u_{k,ij}. \tag{4}$$

In the above equations, we have used the following notations:
- $f_i$ the components of body force;
- $\varrho$ is the reference constant mass density;
- $\sigma_{ij}$ and $m_{ijk}$ are the components of the stress;
- $S$ is the entropy per unit mass;
- $r$ is the heat supply per unit mass;
- $q_i$ are the components of heat flux vector.

Also, the coefficients $A_{ijrs}$, $B_{ijpqr}$, $C_{ijkmnr}$, $E_{ij}$, $D_{ijk}$, $b_i$, $a$ and $k_{ij}$ are the characteristic constitutive constants of the material and they satisfy the following symmetry relations:

$$\begin{aligned}
&A_{ijrs} = A_{rsij} = A_{jirs}, \qquad B_{ijpqr} = B_{jipqr} = B_{ijqpr}, \\
&C_{ijkpqr} = C_{pqrijk} = C_{jipqr}, \qquad E_{ij} = E_{ji}, \qquad D_{ijk} = D_{jik}, \qquad k_{ij} = k_{ji}.
\end{aligned} \tag{5}$$

Also, the second law of thermodynamics implies that

$$k_{ij}\xi_i\xi_j \geq 0, \quad \forall \xi_i.$$

We denote by $t_i$ the components of surface traction and by $q$ the heat flux. These quantities are defined by

$$t_i = \sigma_{ji}n_j, \qquad q = q_i n_i$$

at regular points of the surface $\partial B$. Here, $n_i$ are the components of the outward unit normal of the surface $\partial B$.

Along with the system of field equations (1)-(4), we consider the following initial conditions:

$$u_i(x,0) = c_i(x), \qquad \dot{u}_i(x,0) = d_i(x), \qquad \theta(x,0) = \sigma(x), \quad x \in \bar{B} \tag{6}$$

and the following prescribed boundary conditions:

$$\begin{aligned} &u_i = \bar{u}_i \quad \text{on } \partial B_1 \times [0,t_0), \qquad t_i = \sigma_{ji} n_j = \bar{t}_i \quad \text{on } \partial B_1^c \times [0,t_0), \\ &\theta = \bar{\theta} \quad \text{on } \partial B_2 \times [0,t_0), \qquad q = q_i n_i = \bar{q} \quad \text{on } \partial B_2^c \times [0,t_0), \end{aligned} \tag{7}$$

where $t_0$ is some instant that may be infinite.

Also, $\partial B_1$ and $\partial B_2$ with respective complements $\partial B_1^c$ and $\partial B_2^c$ are subsets of the surface $\partial B$ such that

$$\begin{aligned} &\partial B_1 \cap \partial B_1^c = \partial B_2 \cap \partial B_2^c = \emptyset, \\ &\partial B_1 \cup \partial B_1^c = \partial B_2 \cup \partial B_2^c = \partial B. \end{aligned}$$

Assume that $c_i$, $d_i$, $\sigma$, $\bar{u}_i$, $\bar{t}_i$, $\bar{\theta}$ and $\bar{q}$ are prescribed smooth functions in their domains.

To avoid repeating the regularity assumptions, we assume from the beginning that:

(i) all constitutive coefficients are continuously differentiable functions on $\bar{B}$;
(ii) $\varrho$ is continuous on $\bar{B}$;
(iii) $f_i$ and $r$ are continuous functions on $\bar{B} \times [0,t_0)$;
(iv) $c_i$, $d_i$ and $\sigma$ are continuous functions on $\bar{B}$;
(v) $\bar{u}_i$ and $\bar{\theta}$ are continuous functions on $\partial B_1 \times [0,t_0)$ and $\partial B_2 \times [0,t_0)$, respectively;
(vi) $\bar{t}_i$ and $\bar{q}$ are piecewise regular functions on $\partial B_1^c \times [0,t_0)$ and $\partial B_2^c \times [0,t_0)$, respectively, and continuous in time.

Taking into account the constitutive equations (3), from (1) and (2) we obtain the following system of equations:

$$\begin{aligned} \varrho \ddot{u}_i &= A_{ijrs} u_{r,sj} + B_{ijpqr} u_{r,pqj} + E_{ij}\theta_{,j} \\ &\quad + B_{mnsji} u_{m,nsj} + C_{sjimnr} u_{r,mnsj} + D_{sji}\theta_{,sj} + \varrho f_i, \\ c\dot{\theta} &+ \theta_0 (E_{ij}\dot{u}_{i,j} + D_{ijk}\dot{u}_{k,ij}) = (k_{ij}\theta_{,j})_{,i} + \varrho r. \end{aligned} \tag{8}$$

By a solution of the mixed initial boundary value problem of the theory of thermoelasticity of nonsimple bodies in the cylinder $\Omega_0 = B \times [0,t_0)$, we mean an ordered array $(u_i,\theta)$ which satisfies the system of equations (8) for all $(x,t) \in \Omega_0$, the boundary conditions (7) and the initial conditions (6).

## 3 Main result

Let us consider $f(t,x)$ and $g(t,x)$, two functions assumed to be twice continuously differentiable with respect to the time variable $t$. By direct calculations, it is easy to deduce that

$$\frac{d}{dt}(f\dot{g} - \dot{f}g) = \dot{f}\dot{g} + f\ddot{g} - \ddot{f}g - \dot{f}\dot{g} = f\ddot{g} - \ddot{f}g.$$

For the sake of simplicity, the spatial argument and the time argument of the functions $f(t,x)$ and $g(t,x)$ are omitted because there is no likelihood of confusion.

In the above equality, we substitute the functions $f(t,x)$ and $g(t,x)$ by the functions $U_i(x,t)$ and $V_i(x,t)$, respectively, which also are assumed to be twice continuously differentiable with respect to the time variable, and then we obtain the following well known Lagrange identity:

$$\begin{aligned}
&\int_B \varrho(x)\big[U_i(x,t)\dot{V}_i(x,t) - \dot{U}_i(x,t)V_i(x,t)\big]\,dV \\
&\quad = \int_0^t \int_B \varrho(x)\big[U_i(x,s)\ddot{V}_i(x,s) - \ddot{U}_i(x,s)V_i(x,s)\big]\,dV\,ds \\
&\qquad + \int_B \varrho(x)\big[U_i(x,0)\dot{V}_i(x,0) - \dot{U}_i(x,0)V_i(x,0)\big]\,dV.
\end{aligned} \tag{9}$$

Let us denote by $(u_i^{(\alpha)}, \theta^{(\alpha)})$ $(\alpha = 1,2)$ two solutions of the mixed initial boundary value problem defined by (8), (6) and (7) which correspond to the same boundary data and same initial data, but to different body forces and heat supplies,

$$\big(f_i^{(\alpha)}, r^{(\alpha)}\big) \quad (\alpha = 1,2),$$

respectively.

We introduce the following notations:

$$\begin{aligned}
&v_i = u_i^{(2)} - u_i^{(1)}, \qquad \chi = \theta^{(2)} - \theta^{(1)}, \\
&t_{ij} = \sigma_{ij}^{(2)} - \sigma_{ij}^{(1)}, \qquad m_{ijk} = \mu_{ijk}^{(2)} - \mu_{ijk}^{(1)}, \qquad \eta = S^{(2)} - S^{(1)}, \\
&p_i = q_i^{(2)} - q_i^{(1)}, \qquad \mathcal{F}_i = f_i^{(2)} - f_i^{(1)}, \qquad \mathcal{P} = r^{(2)} - r^{(1)}.
\end{aligned} \tag{10}$$

The constitutive equations become

$$\begin{aligned}
&t_{ij} = A_{ijrs}v_{r,s} + B_{ijpqr}v_{p,qr} + E_{ij}\chi, \\
&\mu_{ijk} = B_{rsijk}v_{r,s} + C_{ijkmnr}v_{r,mn} + D_{ijk}\chi, \\
&\eta = -E_{ij}v_{i,j} - D_{ijk}v_{k,ij} + \frac{c}{T_0}\chi - b_i\chi_{,i}, \\
&p_i = \theta_0(b_i\dot{\chi} - k_{ij}\chi_{,j}).
\end{aligned} \tag{11}$$

So, we deduce that the differences $(v_i, \chi)$ satisfy the following equations and conditions:

- the equations of motion

$$\begin{aligned}
\varrho\ddot{v}_i = {} & A_{ijrs}v_{r,sj} + B_{ijpqr}v_{r,pqj} + E_{ij}\chi_{,j} \\
& + B_{mnsji}v_{m,nsj} + C_{sjimnr}v_{r,mnsj} + D_{sji}\chi_{,sj} + \varrho\mathcal{F}_i;
\end{aligned} \tag{12}$$

- the equations of energy

$$c\dot{\chi} + \theta_0(E_{ij}\dot{v}_{i,j} + D_{ijk}\dot{v}_{k,ij}) = (k_{ij}\chi_{,j})_{,i} + \varrho\mathcal{P}; \tag{13}$$

- the initial conditions

$$v_i(x,0) = 0, \qquad \dot{v}_i(x,0) = 0, \qquad \chi(x,0) = 0, \quad x \in \bar{B}; \tag{14}$$

- the boundary conditions

$$\begin{aligned} &v_i(x,t) = 0 \quad \text{on } \partial B_1 \times [0,t_0), \qquad t_{ji}(x,t)n_j = 0 \quad \text{on } \partial B_1^c \times [0,t_0), \\ &\chi(x,t) = 0 \quad \text{on } \partial B_2 \times [0,t_0), \qquad p_i(x,t)n_i = 0 \quad \text{on } \partial B_2^c \times [0,t_0). \end{aligned} \tag{15}$$

We are now in a position to prove the first basic result.

**Theorem 1** *For the differences* $(v_i, \chi)$ *of two solutions of the mixed initial boundary value problem* (8), (6) *and* (7), *the Lagrange identity becomes*

$$\begin{aligned} &2\int_B \varrho v_i(t)\dot{v}_i(t)\,dV + \int_B \frac{1}{\theta_0}k_{ij}\left(\int_0^t \chi_{,i}(\xi)\,d\xi\right)\left(\int_0^t \chi_{,j}(\xi)\,d\xi\right)dV \\ &\quad = \int_0^t ds \int_B \varrho\left[v_i(2t-s)\mathcal{F}_i(s) - v_i(s)\mathcal{F}_i(2t-s)\right]dV \\ &\qquad + \int_0^t \int_B \frac{\varrho}{\theta_0}\left[\chi(s)\int_0^{2t-s}\mathcal{P}(\xi)\,d\xi \right. \\ &\qquad \left. - \chi(2t-s)\int_0^s \mathcal{P}(\xi)\,d\xi\right]dV\,ds, \quad t \in \left[0, \frac{t_0}{2}\right). \end{aligned} \tag{16}$$

*Proof* Because of the linearity of the problem defined by (8), (6) and (7), we deduce that the differences $(v_i, \chi)$ represent the solution of a mixed initial boundary value problem analogous to (8), (6) and (7), namely the problem consisting of equations (12) and (13) with loads $\mathcal{F}_i$, respectively $\mathcal{P}$, the initial conditions (14) and boundary conditions (15). By setting

$$U_i(x,s) = v_i(x,s), \qquad V_i(x,s) = v_i(x,2t-s), \quad s \in [0,2t], t \in \left[0, \frac{t_0}{2}\right),$$

then the identity (9), after some straightforward calculation, becomes

$$2\int_B \varrho v_i(t)v_i(t)\,dV = \int_0^t ds \int_B \varrho\left[v_i(2t-s)\ddot{v}_i(s) - \ddot{v}_i(2t-s)v_i(s)\right]dV, \tag{17}$$

where we have used the fact that the initial and boundary data are null.

We shall eliminate the inertial terms on the right-hand side of the relation (17) by means of the equations of motion for the differences $(v_i, \chi)$.

So, in view of equation (12), we have

$$\begin{aligned} &\varrho\left[v_i(2t-s)\ddot{v}_i(s) - \ddot{v}_i(2t-s)v_i(s)\right] \\ &\quad = \left\{v_i(2t-s)\left[A_{ijrs}v_{r,s}(s) + B_{ijpqr}v_{r,pq}(s) + E_{ij}\chi(s) \right.\right. \\ &\qquad \left.\left. + B_{rskji}v_{r,sk}(s) + C_{kjipqr}v_{r,pqk}(s) + D_{kji}\chi_{,k}(s)\right]\right\}_{,j} \\ &\qquad - \left\{v_i(s)\left[A_{ijrs}v_{r,s}(2t-s) + B_{ijpqr}v_{r,pq}(2t-s) + E_{ij}\chi(2t-s) \right.\right. \\ &\qquad \left.\left. + B_{rskji}v_{r,sk}(2t-s) + C_{kjipqr}v_{r,pqk}(2t-s) + D_{kji}\chi_{,k}(2t-s)\right]\right\}_{,j} \\ &\qquad - A_{ijrs}v_{r,s}(s)v_{i,j}(2t-s) - B_{ijpqr}v_{r,pq}(s)v_{i,j}(2t-s) - E_{ij}\chi(s)v_{i,j}(2t-s) \end{aligned}$$

$$
\begin{aligned}
&- B_{rskji} v_{r,sk}(s) v_{i,j}(2t-s) - C_{kjipqr} v_{r,pqk}(s) v_{i,j}(2t-s) - D_{kji} \chi_{,k}(s) v_{i,j}(2t-s) \\
&+ A_{ijrs} v_{r,s}(2t-s) v_{i,j}(s) + B_{ijpqr} v_{r,pq}(2t-s) v_{i,j}(s) + E_{ij} \chi(2t-s) v_{i,j}(s) \\
&+ B_{rskji} v_{r,sk}(2t-s) v_{i,j}(s) + C_{kjipqr} v_{r,pqk}(2t-s) v_{i,j}(s) + D_{kji} \chi_{,k}(2t-s) v_{i,j}(s) \\
&+ \varrho \bigl[\mathcal{F}_i(s) v_i(2t-s) - \mathcal{F}_i(2t-s) v_i(s)\bigr].
\end{aligned}
$$

After we use the symmetry relations (5), this equality takes on the form

$$
\begin{aligned}
&\varrho \bigl[v_i(2t-s)\ddot{v}_i(s) - \ddot{v}_i(2t-s) v_i(s)\bigr] \\
&\quad = \bigl\{v_i(2t-s)\bigl[A_{ijrs} v_{r,s}(s) + B_{ijpqr} v_{r,pq}(s) + E_{ij}\chi(s) \\
&\qquad + B_{rskji} v_{r,sk}(s) + C_{kjipqr} v_{r,pqk}(s) + D_{kji}\chi_{,k}(s)\bigr]\bigr\}_{,j} \\
&\qquad - \bigl\{v_i(s)\bigl[A_{ijrs} v_{r,s}(2t-s) + B_{ijpqr} v_{r,pq}(2t-s) + E_{ij}\chi(2t-s) \\
&\qquad + B_{rskji} v_{r,sk}(2t-s) + C_{kjipqr} v_{r,pqk}(2t-s) + D_{kji}\chi_{,k}(2t-s)\bigr]\bigr\}_{,j} \\
&\qquad + E_{ij}\bigl[\chi(2t-s) v_{i,j}(s) - \chi(s) v_{i,j}(2t-s)\bigr] \\
&\qquad + D_{kji}\chi_{,k}(2t-s) v_{i,j}(s) - D_{kji}\chi_{,k}(s) v_{i,j}(2t-s) \\
&\qquad + \varrho\bigl[\mathcal{F}_i(s) v_i(2t-s) - \mathcal{F}_i(2t-s) v_i(s)\bigr].
\end{aligned}
\tag{18}
$$

We integrate by parts equality (18), and after using boundary conditions (15), we get the equality

$$
\begin{aligned}
&\int_B \varrho\bigl[v_i(2t-s)\ddot{v}_i(s) - \ddot{v}_i(2t-s) v_i(s)\bigr]\,dV \\
&\quad = \int_0^t \int_B \bigl[E_{ij} v_{i,j}(s) + D_{kji} v_{i,kj}(s)\bigr]\chi(2t-s)\,dV\,ds \\
&\qquad - \int_0^t \int_B \bigl[E_{ij} v_{i,j}(2t-s) + D_{kji} v_{i,kj}(2t-s)\bigr]\chi(s)\,dV\,ds \\
&\qquad + \int_0^t \int_B \varrho\bigl[\mathcal{F}_i(s) v_i(2t-s) - \mathcal{F}_i(2t-s) v_i(s)\bigr]\,dV\,ds.
\end{aligned}
\tag{19}
$$

Now we integrate equation (13) on the interval $[0,s]$ and take into account the zero initial data in (14) so that we obtain the relation

$$
\begin{aligned}
&E_{ij} v_{i,j}(s) + D_{kji} v_{i,kj}(s) \\
&\quad = \frac{c}{\theta_0}\chi(s) - \frac{1}{\theta_0}\left(\int_0^s \chi_{,j}(z)\,dz\right)_{,i} - \frac{\varrho}{\theta_0}\int_0^s \mathcal{P}(z)\,dz, \quad s \in [0, t_0).
\end{aligned}
\tag{20}
$$

After we multiply in equality (20) by $\chi(2t-s)$ and use a similar result obtained for $E_{ij} v_{i,j}(2t-s) + D_{kji} v_{i,kj}(2t-s)$, multiplied by $\chi(s)$, we find

$$
\begin{aligned}
&\int_0^t \int_B \bigl[E_{ij} v_{i,j}(s) + D_{kji} v_{i,kj}(s)\bigr]\chi(2t-s)\,dV\,ds \\
&\qquad - \int_0^t \int_B \bigl[E_{ij} v_{i,j}(2t-s) + D_{kji} v_{i,kj}(2t-s)\bigr]\chi(s)\,dV\,ds
\end{aligned}
$$

$$
\begin{aligned}
&= \int_B \frac{1}{\theta_0}\left[k_{ij}\chi_{,i}(2t-s)\int_0^s \chi_{,j}(z)\,dz - k_{ij}\chi_{,i}(s)\int_0^{2t-s}\chi_{,j}(z)\,dz\right]dV \\
&\quad + \int_B \frac{\varrho}{\theta_0}\left[\chi(s)\int_0^{2t-s}\mathcal{P}(z)\,dz - \chi(2t-s)\int_0^s \mathcal{P}(z)\,dz\right]dV.
\end{aligned} \tag{21}
$$

If we introduce (21) into (19), we obtain

$$
\begin{aligned}
&\int_B \varrho\left[v_i(2t-s)\ddot{v}_i(s) - \ddot{v}_i(2t-s)v_i(s)\right]dV \\
&\quad = \int_B \frac{1}{\theta_0}\left[k_{ij}\chi_{,i}(2t-s)\int_0^s \chi_{,j}(z)\,dz - k_{ij}\chi_{,i}(s)\int_0^{2t-s}\chi_{,j}(z)\,dz\right]dV \\
&\qquad + \int_B \frac{\varrho}{\theta_0}\left[\chi(s)\int_0^{2t-s}\mathcal{P}(z)\,dz - \chi(2t-s)\int_0^s \mathcal{P}(z)\,dz\right]dV \\
&\qquad + \int_0^t \int_B \varrho\left[\mathcal{F}_i(s)v_i(2t-s) - \mathcal{F}_i(2t-s)v_i(s)\right]dV\,ds.
\end{aligned} \tag{22}
$$

Based on the symmetry of the tensor $k_{ij}$, we get

$$
\begin{aligned}
&\int_0^t \int_B \frac{1}{\theta_0}k_{ij}\frac{d}{ds}\left[\left(\int_0^s \chi_{,i}(\xi)\,d\xi\right)\left(\int_0^{2t-s}\chi_{,j}(\xi)\,d\xi\right)\right]dV\,ds \\
&\quad = \int_B \frac{1}{\theta_0}k_{ij}\int_0^t \frac{d}{ds}\left[\left(\int_0^s \chi_{,i}(\xi)\,d\xi\right)\left(\int_0^{2t-s}\chi_{,j}(\xi)\,d\xi\right)\right]ds\,dV \\
&\quad = \int_B \frac{1}{\theta_0}k_{ij}\left(\int_0^t \chi_{,i}(\xi)\,d\xi\right)\left(\int_0^t \chi_{,j}(\xi)\,d\xi\right)dV.
\end{aligned} \tag{23}
$$

On the other hand, integrating by parts, we obtain

$$
\begin{aligned}
&\int_0^t ds\int_B \frac{1}{\theta_0}k_{ij}\left[\chi_{,i}(s)\int_0^{2t-s}\chi_{,j}(\xi)\,d\xi - \chi_{,j}(2t-s)\int_0^s \chi_{,i}(\xi)\,d\xi\right]dV \\
&\quad = \int_0^t ds\int_B \frac{1}{\theta_0}k_{ij}\frac{d}{ds}\left[\left(\int_0^s \chi_{,i}(\xi)\,d\xi\right)\left(\int_0^{2t-s}\chi_{,j}(\xi)\,d\xi\right)\right]dV.
\end{aligned} \tag{24}
$$

From (23) and (24) we deduce

$$
\begin{aligned}
&\int_0^t ds\int_B \frac{1}{\theta_0}k_{ij}\left[\chi_{,i}(2t-s)\int_0^s \chi_{,j}(\xi)\,d\xi - \chi_{,i}(2t-s)\int_0^{2t-s}\chi_{,j}(\xi)\,d\xi\right]dV \\
&\quad = -\int_B \frac{1}{\theta_0}k_{ij}\left[\left(\int_0^s \chi_{,i}(\xi)\,d\xi\right)\left(\int_0^{2t-s}\chi_{,j}(\xi)\,d\xi\right)\right]dV.
\end{aligned} \tag{25}
$$

We now substitute (25) in (22), and so we are led to equality (16). With this, the proof of Theorem 1 is completed. □

**Remark** It is important to note that the identity (16) is just like in the classical thermoelasticity (see [17]).

The identity (16) constitutes the basis on which we shall prove the uniqueness and the continuous dependence results.

We proceed first to obtain the uniqueness of the solution of the mixed initial boundary value problem defined by (8), (6) and (7).

**Theorem 2** *Assume that the conductivity tensor $k_{ij}$ is positive definite in the sense there exists a positive constant $k_0$ such that*

$$k_{ij}\xi_i\xi_j \geq k_0\xi_i\xi_i, \quad \forall\xi_i.$$

*Also, we suppose that the symmetry relations* (5) *are satisfied. If $\partial B_2$ is not empty or $c(x) \neq 0$ on $B$, then the mixed initial boundary value problem in thermoelastodynamics of nonsimple materials has at most one solution.*

*Proof* Suppose, by contrary, that our mixed problem defined by (8), (6) and (7) has two solutions $(u_i^{(\alpha)}, \theta^{(\alpha)})$ $(\alpha = 1, 2)$ that correspond to the same initial and boundary data, to the same body force and the same heat supply.

If we denote by

$$v_i = u_i^{(2)} - u_i^{(1)}, \qquad \chi = \theta_i^{(2)} - \theta_i^{(1)}, \tag{26}$$

then we shall prove that

$$\begin{aligned} &v_i(x,t) = 0, \qquad \psi_i(x,t) = 0, \\ &\delta(x,t) = 0, \qquad \chi(x,t) = 0, \quad \forall(x,t) \in B \times [0, t_0). \end{aligned} \tag{27}$$

It is clear that the differences $(v_i, \psi_i, \delta, \chi)$ from (26) also represent a solution of our problem but with null body force and null heat supply. If we write the identity (16) for this particular case, we have

$$2\int_B \varrho v_i(t)\dot{v}_i(t)\, dV + \int_B \frac{1}{\theta_0} k_{ij} \left(\int_0^t \chi_{,i}(\xi)\, d\xi\right)\left(\int_0^t \chi_{,j}(\xi)\, d\xi\right) dV = 0.$$

Now, we integrate this equality on the interval $[0, s]$, $s \in [0, t_0/2)$ and obtain

$$\int_B \varrho v_i(s) v_i(s)\, dV + \int_0^s \int_B \frac{1}{\theta_0} k_{ij} \left(\int_0^\tau \chi_{,i}(\xi)\, d\xi\right)\left(\int_0^\tau \chi_{,j}(\xi)\, d\xi\right) dV\, d\tau = 0.$$

Taking into account the properties of $\varrho$ and $k_{ij}$, the above identity proves that

$$v_i(x,t) = 0, \qquad \chi_{,i}(x,t) = 0, \quad \forall(x,t) \in B \times [0, t_0/2). \tag{28}$$

If $\partial B_2$ is not empty, considering the boundary conditions (7), then from (28) we deduce that (27) holds. If $a(x) \neq 0$, from the equation of energy (written for the differences), we get $\dot{\chi} = 0$. However, $\chi$ vanishes initially, such that (27) again holds true.

If $t_0$ is infinite, then the proof of Theorem 2 is complete. If $t_0$ is finite, then we set

$$v_i\left(x, \frac{t_0}{2}\right) = \dot{v}_i\left(x, \frac{t_0}{2}\right) = 0, \qquad \chi\left(x, \frac{t_0}{2}\right) = 0$$

and repeat the above procedure on the interval $[t_0/2, t_0/2 + t_0/4]$ such that we extend the conclusion (27) on $B \times [0, 3t_0/4)$, and so on.

Finally, we obtain (27) on $B \times [0, t_0)$ and this concludes the proof of Theorem 2. □

We are ready to state and prove the continuous dependence theorem with regard to body force and heat supply on the compact subintervals of the interval $[0, t_0)$ for the solution of the mixed initial boundary value problem defined by the system of equations (8), the initial conditions (6) and the boundary conditions (7).

**Theorem 3** *Suppose the same conditions as in Theorem* 2. *Let* $(u_i^{(\alpha)}, \theta^{(\alpha)})$ $(\alpha = 1,2)$ *be two solutions of our mixed problem which correspond to the same initial and boundary data but to different body force and different heat supply*, $(\mathcal{F}_i^{(\alpha)}, \mathcal{P}^{(\alpha)})$ $(\alpha = 1,2)$, *where*

$$\mathcal{F}_i = f_i^2 - f_i^1, \qquad \mathcal{P} = r^2 - r^1.$$

*Moreover, we suppose that there exists* $t_* \in (0, t_0)$ *such that*

$$\begin{aligned} &\int_0^{t_*} \int_B \varrho \mathcal{F}_i(t) \mathcal{F}_i(t)\, dV\, dt \le M_1^2, \qquad \int_0^{t_*} \int_B \frac{\varrho}{\theta_0} \left( \int_0^t \mathcal{P}(\xi)\, d\xi \right)^2 dV\, dt \le M_2^2, \\ &\int_0^{t_*} \int_B \varrho v_i(t) v_i(t)\, dV\, dt \le K^2, \qquad \int_0^{t_*} \int_B \frac{\varrho}{\theta_0} \chi^2(t)\, dV\, dt \le Q^2. \end{aligned} \tag{29}$$

*Then we have the following estimate*:

$$\begin{aligned} &\int_B \varrho v_i(s) v_i(s)\, dV + \int_0^s \int_B \frac{1}{\theta_0} k_{ij} \left( \int_0^t \chi_{,i}(\xi)\, d\xi \right) \left( \int_0^t \chi_{,j}(\xi)\, d\xi \right) dV\, dt \\ &\quad \le t_* K \left[ \int_0^{t_*} \int_B \varrho \mathcal{F}_i(t) \mathcal{F}_i(t)\, dV\, dt \right]^{1/2} \\ &\qquad + t_* Q \left[ \int_0^{t_*} \int_B \frac{\varrho}{\theta_0} \left( \int_0^t \mathcal{P}(\xi)\, d\xi \right)^2 dV\, dt \right]^{1/2}, \end{aligned} \tag{30}$$

*where* $v_i(s)$ *and* $\chi(s)$ *are the differences defined in* (26) *and* $s \in [0, t_*/2)$.

*Proof* We will use the identity (16). On the right-hand side of this identity, we employ the Schwarz inequality for each integral.

For instance, we have

$$\begin{aligned} &\int_0^t \int_B \varrho v_i(2t - s) \mathcal{F}_i(s)\, dV\, ds \\ &\quad \le \left[ \int_0^t \int_B \varrho \mathcal{F}_i(s) \mathcal{F}_i(s)\, dV\, ds \right]^{1/2} \left[ \int_0^{t_*} \int_B \varrho v_i(2t - s) v_i(2t - s)\, dV\, ds \right]^{1/2} \\ &\quad = \left[ \int_0^t \int_B \varrho \mathcal{F}_i(s) \mathcal{F}_i(s)\, dV\, ds \right]^{1/2} \left[ \int_t^{2t} \int_B \varrho v_i(s) v_i(s)\, dV\, ds \right]^{1/2} \\ &\quad \le K \left[ \int_0^{t_*} \int_B \varrho \mathcal{F}_i(s) \mathcal{F}_i(s)\, dV\, ds \right]^{1/2}, \end{aligned}$$

where, at last, we use the substitution $2t - s \to s$.

We proceed analogously with other integrals in the identity (16). Finally, we integrate the resulting inequality over $[0,s]$, $s \in [0,t_*/2]$ and we obtain the inequality (30) and the proof of Theorem 3 is complete. □

In the following theorem, we use the estimate (30) in order to deduce a continuous dependence result upon initial data.

**Theorem 4** *Assume that the symmetry relations* (5) *are satisfied. Consider*

$$\left(u_i^1, \theta^1\right), \qquad \left(u_i^1 + v_i, \theta^1 + \chi\right)$$

*two solutions of the mixed initial boundary value problem defined by* (8), (6) *and* (7) *which correspond to the same body force and heat supply and to the same boundary data, but to different initial data*

$$\left(u_{0i}^1, u_{1i}^1, \theta^1\right), \qquad \left(u_{0i}^2, u_{1i}^2, \theta^2\right),$$

*where*

$$u_{0i}^2 = u_{0i}^1 + a_i^0, \qquad u_{1i}^2 = u_{1i}^1 + a_i^1, \qquad \theta^2 = \theta^1 + d^0.$$

*Here the perturbations* $(a_i^0, a_i^1, d^0)$ *obey the following restrictions*:

$$\int_B \varrho\left(a_i^0 a_i^0 + a_i^1 a_i^1\right) dV \le M_3^2, \qquad \int_B \frac{T_0}{\varrho} \eta_0^2 \, dV \le M_4^2,$$

*where we used the notation*

$$\eta_0(x) = \frac{c(x)}{\theta_0} d^0(x) - E_{ij}(x) a_{i,j}^0(x) - D_{ijk}(x) a_{k,ij}^0(x).$$

*Using perturbation* $v_i$ *and* $\chi$, *we define the functions* $U_i(x,t)$ *and* $\Theta(x,t)$ *by*

$$U_i(x,t) = \int_0^t \int_0^s v_i(x,\tau)\, d\tau\, ds, \qquad \Theta(x,t) = \int_0^t \int_0^s \chi(x,\tau)\, d\tau\, ds. \tag{31}$$

*If the functions* $(U_i, \Theta)$ *satisfy the conditions* (29), *then we have the following estimate*:

$$\begin{aligned}
&\int_B \varrho U_i(t) U_i(t)\, dV + \int_0^t \int_B \frac{1}{\theta_0} k_{ij} \left(\int_0^s \Theta_{,i}(\xi)\, d\xi\right)\left(\int_0^s \Theta_{,j}(\xi)\, d\xi\right) dV\, ds \\
&\quad \le t_* K \left[\left(t_* + t_*^2/2\right) \int_B \varrho a_i^0 a_i^0 \, dV + \left(t_*^2/2 + t_*^3/3\right) \int_B \varrho a_i^1 a_i^1 \, dV\right]^{1/2} \\
&\quad\quad + t_*^{7/2} Q \frac{1}{\sqrt{20}} \left(\int_B \frac{T_0}{\varrho} \eta_0^2 \, dV\right)^{1/2}, \quad t \in \left[0, \frac{t_*}{2}\right].
\end{aligned} \tag{32}$$

*Proof* Integrating by parts in (31), we deduce

$$U_i(x,t) = \int_0^t (t-s) v_i(x,s)\, ds, \qquad \Theta(x,t) = \int_0^t (t-s) \chi(x,s)\, ds.$$

It is easy to prove that the difference functions $(v_i, \chi)$ satisfy the equations of motion and the equation of energy as in (8), but with null loads

$$f_i = r = 0.$$

Also, by direct calculations, we deduce that the difference functions satisfy the initial conditions in the form

$$v_i(x,0) = a_i^0(x), \qquad \dot{v}_i(x,0) = a_i^1(x), \qquad \chi(x,0) = d^0(x), \quad \forall x \in B.$$

Then, a straightforward calculation proves that the functions $(U_i, \Theta)$ defined in (31) satisfy the equations of motion and the equation of energy as in (8), but with the following body force and heat supply:

$$\begin{aligned} f_i(x,t) &= a_i^0(x) + t a_i^1(x), \\ r(x,t) &= \frac{\theta_0}{\varrho} t \left[ \frac{c(x)}{\theta_0} d^0(x) - E_{ij}(x) a_{i,j}^0(x) - D_{ijk}(x) a_{k,ij}^0(x) \right]. \end{aligned}$$

By using these specifications, the estimate (32) follows from (30) and Theorem 4 is concluded. □

Finally, we obtain a continuous dependence result of the solution to problems (8), (6) and (7) upon the thermoelastic coefficients, again as a consequence of Theorem 3.

**Theorem 5** *Assume that the symmetry relations* (5) *are satisfied and consider*

$$\left(u_i^1, \theta^1\right), \qquad \left(u_i^1 + v_i, \theta^1 + \chi\right)$$

*two solutions of the mixed initial boundary value problem defined by* (8), (6) *and* (7) *which correspond to the same body force and heat supply and to the same boundary and initial data, but to different thermoelastic coefficients*

$$\begin{aligned} &\left(A_{ijrs}^{(1)}, B_{ijpqr}^{(1)}, C_{ijkmnr}^{(1)}, E_{ij}^{(1)}, D_{ijk}^{(1)}, b_i^{(1)}, k_{ij}^{(1)}, c^{(1)}\right), \\ &\left(A_{ijrs}^{(1)} + \mathcal{A}_{ijrs}, B_{ijpqr}^{(1)} + \mathcal{B}_{ijpqr}, C_{ijkmnr}^{(1)} + \mathcal{C}_{ijkmnr}, \right. \\ &\quad \left. E_{ij}^{(1)} + \mathcal{E}_{ij}, D_{ijk}^{(1)} + \mathcal{D}_{ijk}, b_i^{(1)} + \mathcal{B}_i, k_{ij}^{(1)} + \mathcal{K}_{ij}, c^{(1)} + \mathcal{C}\right). \end{aligned}$$

*Suppose that the perturbations* $(v_i, \chi)$ *satisfy the conditions* (29). *Then any solution* $(u_i, \theta)$ *of the initial boundary value problem defined by* (8), (6) *and* (7) *that satisfies the condition*

$$\int_0^{t_*} \int_B \left(u_{i,j} u_{i,j} + u_{i,jk} u_{i,jk} + \dot{u}_{i,j} \dot{u}_{i,j} + \theta_{,j} \theta_{,j} + \theta_{,jk} \theta_{,jk} + \dot{\theta}^2\right) dV\, ds \le M_5^2$$

*depends continuously on the thermoelastic coefficients on the interval* $[0, t_*/2]$ *in*

$$\int_B \varrho v_i(t) v_i(t) + \int_0^t \int_B \frac{1}{\theta_0} k_{ij} \left( \int_0^s \chi_{,i}(\xi)\, d\xi \right) \left( \int_0^s \chi_{,j}(\xi)\, d\xi \right) dV\, ds.$$

*Proof* A straightforward calculation proves that the perturbations $(v_i, \chi)$ of two solutions verify the equations of motion and the equation of energy with the following body force load and heat supply:

$$\begin{aligned}\varrho\mathcal{F}_i &= \mathcal{A}_{ijrs}u^{(2)}_{r,sj} + \mathcal{B}_{ijpqr}u^{(2)}_{r,pqj} + \mathcal{E}_{ij}\theta^{(2)}_{,j} \\ &\quad + \mathcal{B}_{rskji}u^{(2)}_{r,skj} + \mathcal{C}_{kjipqr}u^{(2)}_{r,pqkj} + \mathcal{D}_{kji}\theta^{(2)}_{,kj}, \\ \varrho\mathcal{P} &= \bigl(\mathcal{K}_{ij}\theta^{(2)}_{,j}\bigr)_{,i} - \theta_0\left(\frac{\mathcal{C}}{\theta_0}\dot{\theta}^2 - \mathcal{E}_{ij}u^{(2)}_{i,j} - \mathcal{D}_{ijk}u^{(2)}_{k,ij}\right).\end{aligned}$$

Thus the problem is analogous to the problem from Theorem 4. Therefore, according to the estimates (32) and (30), we obtain the desired result. □

## 4 Concluding remarks

The uniqueness theorem and the continuous dependence theorems were proved without recourse to any conservation laws or to any boundedness assumptions on the thermoelastic coefficients. In various papers, the existence of the solution to the mixed initial boundary value problem defined by (8), (6) and (7) is obtained by assuming some strong restrictions. For instance, in the paper [18] the existence of the solution is obtained under assumption that the internal energy density is positive definite.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
MIM proposed main results of the paper and verified all calculations and demonstrations. RPA proposed the method of demonstration of results, without using a sophisticated mathematical apparatus. Also, he controlled the final shape of the paper. SRM performed all calculations and demonstrations and took into account the suggestions given by MIM.

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