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Existence of nonnegative solutions for a fractional m-point boundary value problem at resonance

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Abstract
We consider the fractional differential equation

$$D_{0+}^{q}u(t)=f(t,u(t)),\quad 0<t<1,$$

satisfying the boundary conditions

$$D_{0+}^{p}u(t)|_{t=0}=D_{0+}^{p-1}u(t)|_{t=0}=\cdots=D_{0+}^{p-n+1}u(t)|_{t=0}=0,\qquad u(1)=\sum_{i=1}^{m-2}\alpha_i u(\xi_i),$$

where D_{0+}^{q} is the Riemann-Liouville fractional order derivative. The parameters in the multi-point boundary conditions are such that the corresponding differential operator is a Fredholm map of index zero. As a result, the minimal and maximal nonnegative solutions for the problem are obtained by using a fixed point theorem of increasing operators.
MSC: 26A33; 34A08

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1 Introduction

Let us consider the fractional differential equation

$$D_{0+}^{q}u(t)=f(t,u(t)),\quad 0<t<1,\tag{1.1}$$

with the boundary conditions (BCs)

$$\begin{cases}D_{0+}^{p}u(t)|_{t=0}=D_{0+}^{p-1}u(t)|_{t=0}=\cdots=D_{0+}^{p-n+1}u(t)|_{t=0}=0,\\ u(1)=\sum_{i=1}^{m-2}\alpha_i u(\xi_i),\end{cases}\tag{1.2}$$

where $n\geq 1$, $\max\{q-2,0\}\leq p<q-1$, $n<q\leq n+1$, $\sum_{i=1}^{m-2}\alpha_i\xi_i^{q-1}=1$, $\alpha_i>0$, $0<\xi_1<\xi_2<\cdots<\xi_{m-2}<1$, $m\geq 3$. We assume that $f:[0,1]\times[0,\infty)\to[0,\infty)$ is continuous. A boundary value problem at resonance for ordinary or fractional differential equations has been studied by several authors, including the most recent works [1–7] and the references therein. In the most papers mentioned above, the coincidence degree theory was applied to establish existence theorems. But in [8], Wang obtained the minimal and maximal nonnegative solutions for a second-order m-point boundary value problem at resonance by using a new fixed point theorem of increasing operators, and in this paper we use this method of Wang to establish the existence theorem of equations (1.1) and (1.2).

For the convenience of the reader, we briefly recall some notations.

Let X, Z be real Banach spaces, $L : \mathrm{dom}(L) \subset X \to Z$ be a Fredholm map of index zero and $P : X \to X$, $Q : Z \to Z$ be continuous projectors such that $\mathrm{Im}(P) = \mathrm{Ker}(L)$, $\mathrm{Ker}(Q) = \mathrm{Im}(L)$ and $X = \mathrm{Ker}(L) \oplus \mathrm{Ker}(P)$, $Z = \mathrm{Im}(L) \oplus \mathrm{Im}(Q)$. It follows that $L|_{\mathrm{Ker}(P)\cap \mathrm{dom}(L)} : \mathrm{Ker}(P) \cap \mathrm{dom}(L) \to \mathrm{Im}(L)$ is invertible. We denote the inverse of the map by $K_P : \mathrm{Im}(L) \to \mathrm{Ker}(P) \cap \mathrm{dom}(L)$. Since $\dim \mathrm{Im}(Q) = \dim \mathrm{Ker}(L)$, there exists an isomorphism $J : \mathrm{Im}(Q) \to \mathrm{Ker}(L)$. Let Ω be an open bounded subset of X. The map $N : X \to Z$ will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ and $K_P(I - Q)(\overline{\Omega})$ are compact. We take $H = L + J^{-1}P$, then $H : \mathrm{dom}(L) \subset X \to Z$ is a linear bijection with bounded inverse and $(JQ + K_P(I - Q))(L + J^{-1}P) = (L + J^{-1}P)(JQ + K_P(I - Q)) = I$. We know from [9] that $K_1 = H(K \cap \mathrm{dom}(L))$ is a cone in Z.

Theorem 1.1 [9] *$N(u) + J^{-1}P(u) = H(\tilde{u})$, where*

$$\tilde{u} = P(u) + JQN(u) + K_P(I - Q)N(u)$$

and $\tilde{u}$ is uniquely determined.

From the above theorem, the author [9] obtained that the assertions

(i) $P(u) + JQN(u) + K_P(I - Q)N(u) : K \cap \mathrm{dom}(L) \to K \cap \mathrm{dom}(L)$ and

(ii) $N(u) + J^{-1}P(u) : K \cap \mathrm{dom}(L) \to K_1$ are equivalent.

We also need the following definition and theorem.

Definition 1.1 [8] Let K be a normal cone in a Banach space X, $u_0 \le v_0$, and $u_0, v_0 \in K \cap \mathrm{dom}(L)$ are said to be coupled lower and upper solutions of the equation $Lx = Nx$ if

$$\begin{cases} Lu_0 \le Nu_0, \\ Lv_0 \ge Nv_0. \end{cases}$$

Theorem 1.2 [8] *Let $L : \mathrm{dom}(L) \subset X \to Z$ be a Fredholm operator of index zero, K be a normal cone in a Banach space X, $u_0, v_0 \in K \cap \mathrm{dom}(L)$, $u_0 \le v_0$, and $N : [u_0, v_0] \to Z$ be L-compact and continuous. Suppose that the following conditions are satisfied*:

(C_1) *u_0 and v_0 are coupled lower and upper solutions of the equation $Lx = Nx$*;

(C_2) *$N + J^{-1}P : K \cap \mathrm{dom}(L) \to K_1$ is an increasing operator.*

Then the equation $Lx = Nx$ has a minimal solution u^* and a maximal solution v^* in $[u_0, v_0]$. Moreover,

$$u^* = \lim_{n\to\infty} u_n, \qquad v^* = \lim_{n\to\infty} v_n,$$

where

$$u_n = \bigl(L + J^{-1}P\bigr)^{-1}\bigl(N + J^{-1}P\bigr)u_{n-1}, \qquad v_n = \bigl(L + J^{-1}P\bigr)^{-1}\bigl(N + J^{-1}P\bigr)v_{n-1},$$

$n = 1, 2, 3, \ldots$ and $u_0 \le u_1 \le u_2 \le \cdots \le u_n \le \cdots \le v_n \le \cdots \le v_2 \le v_1 \le v_0$.

2 Preliminaries

In this section, we present some necessary basic knowledge and definitions about fractional calculus theory.

Definition 2.1 (see Equation 2.1.1 in [10]) The R-L fractional integral $I_{0+}^{q}u$ of order $q \in R$ ($q > 0$) is defined by

$$I_{0+}^{q}u(t) := \frac{1}{\Gamma(q)} \int_0^t \frac{u(\tau)\,d\tau}{(t-\tau)^{1-q}} \quad (t > 0).$$

Here $\Gamma(q)$ is the gamma function.

Definition 2.2 (see Equation 2.1.5 in [10]) The R-L fractional derivative $D_{0+}^{q}u$ of order $q \in R$ ($q > 0$) is defined by

$$\begin{aligned} D_{0+}^{q}u(t) &= \left(\frac{d}{dt}\right)^n I_{0+}^{n-q}u(t) \\ &= \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{u(\tau)\,d\tau}{(t-\tau)^{q-n+1}} \quad \bigl(n = [q]+1, t > 0\bigr), \end{aligned}$$

where $[q]$ means the integral part of q.

Lemma 2.1 [11] *If $q_1, q_2 > 0$, $q > 0$, then, for $u(t) \in L_p(0,1)$, the relations*

$$I_{0+}^{q_1} I_{0+}^{q_2} u(t) = I_{0+}^{q_1+q_2} u(t)$$

and

$$D_{0+}^{q_1} I_{0+}^{q_1} u(t) = u(t)$$

hold a.e. on $[0,1]$.

Lemma 2.2 (see [11]) *Let $q > 0$, $n = [q]+1$, $D_{0+}^{q}u(t) \in L_1(0,1)$, then we have the equality*

$$I_{0+}^{q} D_{0+}^{q} u(t) = u(t) + \sum_{i=1}^{n} C_i t^{q-i},$$

where $C_i \in R$ ($i = 1, 2, \ldots, n$) are some constants.

Lemma 2.3 (see Corollary 2.1 in [10]) *Let $q > 0$ and $n = [q]+1$, the equation $D_{0+}^{q}u(t) = 0$ is valid if and only if $u(t) = \sum_{i=1}^{n} C_i t^{q-i}$, where $C_i \in R$ ($i = 1, 2, \ldots, n$) are arbitrary constants.*

Let $X = Z = C[0,1]$ with the norm $\|u\| = \sup_{t \in [0,1]} |u(t)|$, then X and Z are Banach spaces.

Let $K = \{u \in X : u(t) \geq 0, t \in [0,1]\}$. It follows from Theorem 1.1.1 in [12] that K is a normal cone.

Let $\operatorname{dom}(L) = \{u(t) \in X \mid D_{0+}^{q}u(t) \in Z, u(t)$ *satisfies BCs* (1.2)$\}$.

We define the operators $L : \operatorname{dom}(L) \to Z$ by

$$(Lu)(t) = D_{0+}^{q}u(t) \qquad (2.1)$$

and $N : K \to Z$ by

$$(Nu)(t) = f\big(t, u(t)\big),$$

then BVPs (1.1) and (1.2) can be written as $Lu = Nu$, $u \in K \cap \operatorname{dom}(L)$.

Lemma 2.4 *If the operator L is defined in* (2.1), *then*

(i) $\operatorname{Ker}(L) = \{c \cdot t^{q-1} \mid c \in R\}$,

(ii) $\operatorname{Im}(L) = \{y \in Z \mid \int_0^1 (1-s)^{q-2} \sum_{i=1}^{m-2} \alpha_i \xi_i^{q-1} \int_{\xi_i s}^s y(\tau)\,d\tau\,ds = 0\} =: \mathcal{L}$.

Proof (i) It can be seen from Lemma 2.3 and BCs (1.2) that $\operatorname{Ker}(L) = \{c \cdot t^{q-1} \mid c \in R\}$.

(ii) If $y \in \operatorname{Im}(L)$, then there exists a function $u \in \operatorname{dom}(L)$ such that $y(t) = D_{0+}^q u(t)$, by Lemma 2.2, we have

$$I_{0+}^q y(t) = u(t) + c_1 t^{q-1} + \cdots + c_n t^{q-n}.$$

It follows from BCs (1.2) and the equation $\sum_{i=1}^{m-2} \alpha_i \xi_i^{q-1} = 1$ that

$$I_{0+}^q y(1) = \sum_{i=1}^{m-2} I_{0+}^q \alpha_i y(\xi_i)$$

and noting the definition of I_{0+}^q, we have

$$I_{0+}^q y(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y(s)\,ds = \frac{q-1}{\Gamma(q)} \int_0^t (t-s)^{q-2} \int_0^s y(\tau)\,d\tau\,ds.$$

Thus,

$$\begin{aligned}
\frac{q-1}{\Gamma(q)} \int_0^1 (1-s)^{q-2} \int_0^s y(\tau)\,d\tau\,ds &= \frac{q-1}{\Gamma(q)} \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} (\xi_i - s)^{q-2} \int_0^s y(\tau)\,d\tau\,ds \\
&= \frac{q-1}{\Gamma(q)} \sum_{i=1}^{m-2} \alpha_i \xi_i \int_0^1 (\xi_i - \xi_i s)^{q-2} \int_0^{\xi_i s} y(\tau)\,d\tau\,ds \\
&= \frac{q-1}{\Gamma(q)} \sum_{i=1}^{m-2} \alpha_i \xi_i^{q-1} \int_0^1 (1-s)^{q-2} \int_0^{\xi_i s} y(\tau)\,d\tau\,ds,
\end{aligned}$$

which is

$$\int_0^1 (1-s)^{q-2} \sum_{i=1}^{m-2} \alpha_i \xi_i^{q-1} \int_{\xi_i s}^s y(\tau)\,d\tau\,ds = 0.$$

Then $y \in \mathcal{L}$, hence $\operatorname{Im}(L) \subset \mathcal{L}$.

On the other hand, if $y \in \mathcal{L}$, let $u(t) = I_{0+}^q y(t)$, then $u \in \operatorname{dom}(L)$, and $D_{0+}^q u(t) = D_{0+}^q I_{0+}^q y(t) = y(t)$, which implies that $y \in \operatorname{Im}(L)$, thus $\mathcal{L} \subset \operatorname{Im}(L)$. In general $\operatorname{Im}(L) = \mathcal{L}$. Clearly, $\operatorname{Im}(L)$ is closed in Z and $\dim \operatorname{Ker}(L) = \operatorname{codim} \operatorname{Im}(L) = 1$, thus L is a Fredholm operator of index zero. This completes the proof. □

In what follows, some property operators are defined. We define continuous projectors $P : X \to X$ by

$$(Pu)(t) = q\int_0^1 u(s)\,ds \cdot t^{q-1}$$

and $Q : Z \to Z$ by

$$(Qu)(t) = \frac{1}{\gamma_0}\int_0^1 (1-s)^{q-2}\sum_{i=1}^{m-2}\alpha_i\xi_i^{q-1}\int_{\xi_i s}^{s} u(\tau)\,d\tau\,ds,$$

where

$$\begin{aligned}\gamma_0 &= \int_0^1 (1-s)^{q-2}\sum_{i=1}^{m-2}\alpha_i\xi_i^{q-1}\int_{\xi_i s}^{s} d\tau\,ds \\ &= \int_0^1 s(1-s)^{q-2}\,ds\left(1-\sum_{i=1}^{m-2}\alpha_i\xi_i^{q}\right) \\ &= B(2,q-1)\left(1-\sum_{i=1}^{m-2}\alpha_i\xi_i^{q}\right) > 0.\end{aligned}$$

$B(x,y)$ is the beta function defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt.$$

By calculating, we easily obtain $P^2 = P$, $Q^2 = Q$, and $X = \operatorname{Ker}(L) \oplus \operatorname{Ker}(P)$, $Z = \operatorname{Im}(L) \oplus \operatorname{Im}(Q)$. We also define $J : \operatorname{Im}(Q) \to \operatorname{Ker}(L)$ by

$$J(c) = ct^{q-1}, \quad \forall c \in R$$

and $K_P : \operatorname{Im}(L) \to \operatorname{dom}(L) \cap \operatorname{Ker}(P)$ by

$$\bigl(K_P(u)\bigr)(t) = \bigl(I_{0+}^q u\bigr)(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}u(s)\,ds,$$

thus

$$\bigl(QN(u)\bigr)(t) = \frac{1}{\gamma_0}\int_0^1 (1-s)^{q-2}\sum_{i=1}^{m-2}\alpha_i\xi_i^{q-1}\int_{\xi_i s}^{s} f\bigl(\tau,u(\tau)\bigr)\,d\tau\,ds$$

and

$$\begin{aligned}&\bigl(K_P(I-Q)N(u)\bigr)(t) \\ &\quad= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}f\bigl(s,u(s)\bigr)\,ds \\ &\qquad - \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\frac{1}{\gamma_0}\int_0^1 (1-\tilde{s})^{q-2}\sum_{i=1}^{m-2}\alpha_i\xi_i^{q-1}\int_{\xi_i\tilde{s}}^{\tilde{s}} f\bigl(\tau,u(\tau)\bigr)\,d\tau\,d\tilde{s}\,ds.\end{aligned}$$

Lemma 2.5 *Let Ω be any open bounded subset of $K \cap \operatorname{dom}(L)$, then $QN(\overline{\Omega})$ and $K_P(I - Q)N(\overline{\Omega})$ are compact, which implies that N is L-compact on $\overline{\Omega}$ for any open bounded set $\Omega \subset K \cap \operatorname{dom}(L)$.*

Proof For a positive integer n, let $\Omega = \{u \in K \cap \operatorname{dom}(L) : \|u\| \leq n\}$, $M = \sup_{(t,u)} f(t, u(t))$, $(t,u) \in [0,1] \times [0,n]$. It is easy to see that $QN(\overline{\Omega})$ is compact. Now, we prove that $K_P(I - Q)N(\overline{\Omega})$ is compact. For $\forall u \in \overline{\Omega}$, we have

$$\begin{aligned}
&\bigl\| \bigl(K_P(I-Q)N(u)\bigr)(t) \bigr\| \\
&= \sup_{t\in[0,1]} \Biggl| \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f\bigl(s,u(s)\bigr)\,ds \\
&\quad - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \frac{1}{\gamma_0} \int_0^1 (1-\tilde{s})^{q-2} \sum_{i=1}^{m-2} \alpha_i \xi_i^{q-1} \int_{\xi_i \tilde{s}}^{\tilde{s}} f\bigl(\tau, u(\tau)\bigr)\,d\tau\,d\tilde{s}\,ds \Biggr| \\
&\leq \sup_{t\in[0,1]} \biggl| \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f\bigl(s,u(s)\bigr)\,ds \biggr| \\
&\quad + \sup_{t\in[0,1]} \Biggl| \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \frac{1}{\gamma_0} \int_0^1 (1-\tilde{s})^{q-2} \sum_{i=1}^{m-2} \alpha_i \xi_i^{q-1} \int_{\xi_i \tilde{s}}^{\tilde{s}} f\bigl(\tau, u(\tau)\bigr)\,d\tau\,d\tilde{s}\,ds \Biggr| \\
&\leq \frac{2M}{\Gamma(q)} \sup_{t\in[0,1]} \biggl| \int_0^t (t-s)^{q-1}\,ds \biggr| \\
&= \frac{2M}{\Gamma(q+1)},
\end{aligned}$$

which implies that $K_P(I - Q)N(\overline{\Omega})$ is bounded.

Moreover, for each $u \in \overline{\Omega}$, let $t_1, t_2 \in [0,1]$ and $t_1 > t_2$, then

$$\begin{aligned}
&\bigl\| \bigl(K_P(I-Q)N(u)\bigr)(t_1) - \bigl(K_P(I-Q)N(u)\bigr)(t_2) \bigr\| \\
&\leq \biggl| \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1-s)^{q-1} f\bigl(s,u(s)\bigr)\,ds - \frac{1}{\Gamma(q)} \int_0^{t_2} (t_2-s)^{q-1} f\bigl(s,u(s)\bigr)\,ds \biggr| \\
&\quad + \Biggl| \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1-s)^{q-1} \frac{1}{\gamma_0} \int_0^1 (1-\tilde{s})^{q-2} \sum_{i=1}^{m-2} \alpha_i \xi_i^{q-1} \int_{\xi_i \tilde{s}}^{\tilde{s}} f\bigl(\tau, u(\tau)\bigr)\,d\tau\,d\tilde{s}\,ds \\
&\quad - \frac{1}{\Gamma(q)} \int_0^{t_2} (t_2-s)^{q-1} \frac{1}{\gamma_0} \int_0^1 (1-\tilde{s})^{q-2} \sum_{i=1}^{m-2} \alpha_i \xi_i^{q-1} \int_{\xi_i \tilde{s}}^{\tilde{s}} f\bigl(\tau, u(\tau)\bigr)\,d\tau\,d\tilde{s}\,ds \Biggr| \\
&\leq \biggl| \frac{1}{\Gamma(q)} \int_0^{t_2} (t_1-s)^{q-1} f\bigl(s,u(s)\bigr)\,ds - \frac{1}{\Gamma(q)} \int_0^{t_2} (t_2-s)^{q-1} f\bigl(s,u(s)\bigr)\,ds \biggr| \\
&\quad + \biggl| \frac{1}{\Gamma(q)} \int_{t_2}^{t_1} (t_1-s)^{q-1} f\bigl(s,u(s)\bigr)\,ds \biggr| \\
&+ \Biggl| \frac{1}{\Gamma(q)} \int_0^{t_2} (t_1-s)^{q-1} \frac{1}{\gamma_0} \int_0^1 (1-\tilde{s})^{q-2} \sum_{i=1}^{m-2} \alpha_i \xi_i^{q-1} \int_{\xi_i \tilde{s}}^{\tilde{s}} f\bigl(\tau, u(\tau)\bigr)\,d\tau\,d\tilde{s}\,ds \\
&\quad - \frac{1}{\Gamma(q)} \int_0^{t_2} (t_2-s)^{q-1} \frac{1}{\gamma_0} \int_0^1 (1-\tilde{s})^{q-2} \sum_{i=1}^{m-2} \alpha_i \xi_i^{q-1} \int_{\xi_i \tilde{s}}^{\tilde{s}} f\bigl(\tau, u(\tau)\bigr)\,d\tau\,d\tilde{s}\,ds \Biggr|
\end{aligned}$$

$$
\begin{aligned}
&\quad + \left| \frac{1}{\Gamma(q)} \int_{t_2}^{t_1} (t_1 - s)^{q-1} \frac{1}{\gamma_0} \int_0^1 (1-\tilde{s})^{q-2} \sum_{i=1}^{m-2} \alpha_i \xi_i^{q-1} \int_{\xi_i \tilde{s}}^{\tilde{s}} f\bigl(\tau, u(\tau)\bigr)\, d\tau\, d\tilde{s}\, ds \right| \\
&\le \frac{2M}{\Gamma(q)} \left| \int_0^{t_1} (t_1 - s)^{q-1}\, ds - \int_0^{t_2} (t_2 - s)^{q-1}\, ds \right| + \frac{2M}{\Gamma(q)} \left| \int_{t_2}^{t_1} (t_1 - s)^{q-1}\, ds \right| \\
&\le \frac{2M}{\Gamma(q)} \left| \int_0^{t_1} (t_1 - s)^{q-1}\, ds - \int_0^{t_2} (t_2 - s)^{q-1}\, ds \right| + \frac{2M}{\Gamma(q)} |t_1 - t_2| \\
&= \frac{2M}{\Gamma(q)} \left| t_1 \int_0^1 (t_1 - t_1 s)^{q-1}\, ds - t_2 \int_0^1 (t_2 - t_2 s)^{q-1}\, ds \right| + \frac{2M}{\Gamma(q)} |t_1 - t_2| \\
&= \frac{2M}{\Gamma(q+1)} \bigl| t_1^q - t_2^q \bigr| + \frac{2M}{\Gamma(q)} |t_1 - t_2| \\
&= \frac{2M}{\Gamma(q+1)} \bigl| q\eta^{q-1} \bigr| \cdot |t_1 - t_2| + \frac{2M}{\Gamma(q)} |t_1 - t_2|, \quad \eta = t_1 + \theta(t_2 - t_1), 0 < \theta < 1 \\
&\le \frac{(2^q + 2)M}{\Gamma(q)} |t_1 - t_2|.
\end{aligned}
$$

Thus

$$
\forall \varepsilon > 0, \quad \exists \delta = \frac{\Gamma(q)}{(2^q + 2)M} \varepsilon
$$

such that

$$
\bigl\| K_P(I-Q)N(u)(t_1) - K_P(I-Q)N(u)(t_2) \bigr\| < \varepsilon
$$

for

$$
|t_1 - t_2| < \delta
$$

and each

$$
u \in \overline{\Omega}.
$$

It is concluded that N is L-compact on $\overline{\Omega}$. This completes the proof. □

3 Main result

In this section, we establish the existence of the nonnegative solution to equations (1.1) and (1.2).

Theorem 3.1 *Suppose*

(H_1) *There exist* $u_0, v_0 \in K \cap \operatorname{dom}(L)$ *such that* $u_0 \le v_0$ *and*

$$
\begin{cases}
D_{0+}^q u_0(t) \le f(t, u_0(t)), & \forall t \in [0,1], \\
D_{0+}^q v_0(t) \ge f(t, v_0(t)), & \forall t \in [0,1].
\end{cases}
$$

(H_2) *For any* $x, y \in K \cap \operatorname{dom}(L)$, *satisfying*

$$
f\bigl(t, x(t)\bigr) - f\bigl(t, y(t)\bigr) \ge -q\left(\int_0^1 x(t)\, dt - \int_0^1 y(t)\, dt \right),
$$

where $\forall t \in [0,1]$ *and* $u_0(t) \le y(t) \le x(t) \le v_0(t)$, *then problems* (1.1) *and* (1.2) *have a minimal solution* u^* *and a maximal solution* v^* *in* $[u_0, v_0]$, *respectively.*

Proof By condition (H_1), we know that

$$Lu_0 \le Nu_0, \qquad Lv_0 \ge Nv_0,$$

so condition (C_1) in Theorem 1.1 holds.

In addition, for each $u \in K$,

$$\begin{aligned}
&\big(P(u) + JQN(u) + K_P(I-Q)N(u)\big)(t) \\
&\quad = q\int_0^1 u(s)\,ds \cdot t^{q-1} + \frac{1}{\gamma_0}\int_0^1 (1-s)^{q-2}\sum_{i=1}^{m-2}\alpha_i\xi_i^{q-1}\int_{\xi_i s}^{s} f\big(\tau, u(\tau)\big)\,d\tau\,ds \cdot t^{q-1} \\
&\qquad + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f\big(s, u(s)\big)\,ds \\
&\qquad - \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\frac{1}{\gamma_0}\int_0^1 (1-\tilde{s})^{q-2}\sum_{i=1}^{m-2}\alpha_i\xi_i^{q-1}\int_{\xi_i\tilde{s}}^{\tilde{s}} f\big(\tau, u(\tau)\big)\,d\tau\,d\tilde{s}\,ds \\
&\quad = q\int_0^1 u(s)\,ds \cdot t^{q-1} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f\big(s, u(s)\big)\,ds \\
&\qquad + \frac{1}{\gamma_0}\int_0^1 (1-s)^{q-2}\sum_{i=1}^{m-2}\alpha_i\xi_i^{q-1}\int_{\xi_i s}^{s} f\big(\tau, u(\tau)\big)\,d\tau\,ds\left(t^{q-1} - \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\,ds\right) \\
&\quad \ge \frac{1}{\gamma_0}\int_0^1 (1-s)^{q-2}\sum_{i=1}^{m-2}\alpha_i\xi_i^{q-1}\int_{\xi_i s}^{s} f\big(\tau, u(\tau)\big)\,d\tau\,ds\left(t^{q-1} - \frac{t^q}{\Gamma(q+1)}\right) \ge 0.
\end{aligned}$$

Thus $(P + JQN + K_P(I-Q)N)(K) \subset K$, that is, $N + J^{-1}P : K \cap \operatorname{dom}(L) \to K_1$ by virtue of the equivalence. From condition (H_2), we have that $N + J^{-1}P : K \cap \operatorname{dom}(L) \to K_1$ is a monotone increasing operator. Then, in accordance with Lemma 2.5 and Theorem 1.2, we obtain a minimal solution u^* and a maximal solution v^* in $[u_0, v_0]$ for problems (1.1) and (1.2). Thus we can define iterative sequences $\{u_n(t)\}$ and $\{v_n(t)\}$ by

$$\begin{aligned}
u_n &= \big(L + J^{-1}P\big)^{-1}\big(N + J^{-1}P\big)u_{n-1} = \big(JQ + K_P(I-Q)\big)\big(N + J^{-1}P\big)u_{n-1} \\
&= \big(JQ + K_P(I-Q)\big)\left(f\big(t, u_{n-1}(t)\big) + q\int_0^1 u_{n-1}(s)\,ds\right) \\
&= \frac{1}{\gamma_0}\int_0^1 (1-s)^{q-2}\sum_{i=1}^{m-2}\alpha_i\xi_i^{q-1}\int_{\xi_i s}^{s}\left(f\big(\tau, u_{n-1}(\tau)\big) + q\int_0^1 u_{n-1}(\hat{s})\,d\hat{s}\right)d\tau\,ds \cdot t^{q-1} \\
&\quad + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left(f\big(s, u_{n-1}(s)\big) + q\int_0^1 u_{n-1}(\tilde{s})\,d\tilde{s}\right)ds \\
&\quad - \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\frac{1}{\gamma_0}\int_0^1 (1-\tilde{s})^{q-2}\sum_{i=1}^{m-2}\alpha_i\xi_i^{q-1} \\
&\quad \cdot \int_{\xi_i\tilde{s}}^{\tilde{s}}\left(f\big(\tau, u_{n-1}(\tau)\big) + q\int_0^1 u_{n-1}(\hat{s})\,d\hat{s}\right)d\tau\,d\tilde{s}\,ds
\end{aligned}$$

and

$$\begin{aligned}
v_n &= \left(L + J^{-1}P\right)^{-1}\left(N + J^{-1}P\right)v_{n-1} = \left(JQ + K_P(I-Q)\right)\left(N + J^{-1}P\right)v_{n-1} \\
&= \left(JQ + K_P(I-Q)\right)\left(f\left(t, v_{n-1}(t)\right) + q\int_0^1 v_{n-1}(s)\,ds\right) \\
&= \frac{1}{\gamma_0}\int_0^1 (1-s)^{q-2}\sum_{i=1}^{m-2}\alpha_i\xi_i^{q-1}\int_{\xi_i s}^{s}\left(f\left(\tau, v_{n-1}(\tau)\right) + q\int_0^1 v_{n-1}(\hat{s})d\hat{s}\right)d\tau\,ds\cdot t^{q-1} \\
&\quad + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left(f\left(s, v_{n-1}(s)\right) + q\int_0^1 v_{n-1}(\tilde{s})\,d\tilde{s}\right)ds \\
&\quad - \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\frac{1}{\gamma_0}\int_0^1 (1-\tilde{s})^{q-2}\sum_{i=1}^{m-2}\alpha_i\xi_i^{q-1} \\
&\quad \cdot\int_{\xi_i\tilde{s}}^{\tilde{s}}\left(f\left(\tau, v_{n-1}(\tau)\right) + q\int_0^1 v_{n-1}(\hat{s})\,d\hat{s}\right)d\tau\,d\tilde{s}\,ds, \quad n = 1,2,3,\ldots
\end{aligned}$$

Then from Theorem 1.2 we get $\{u_n\}$ and $\{v_n\}$ converge uniformly to $u^*(t)$ and $v^*(t)$, respectively. Moreover,

$$u_0 \leq u_1 \leq u_2 \leq \cdots \leq u_n \leq \cdots \leq v_n \leq \cdots \leq v_2 \leq v_1 \leq v_0. \qquad \square$$

4 Example

We consider the following problem:

$$D_{0+}^{\frac{3}{2}}u(t) = \left(\frac{u^2}{u^2+1} + t\right)^m, \quad 0 < t < 1, m > 0, \tag{4.1}$$

subject to BCs

$$D_{0+}^{\frac{1}{4}}u(t)|_{t=0} = 0, \qquad u(1) = \sqrt{2}u\left(\frac{1}{2}\right). \tag{4.2}$$

We can choose

$$u_0(t) = \frac{1}{\Gamma(\frac{3}{2})}\int_0^t (t-s)^{\frac{1}{2}}s^m\,ds + t^{\frac{1}{2}} \leq \frac{1}{\Gamma(\frac{3}{2})}\int_0^t (t-s)^{\frac{1}{2}}(s+1)^m\,ds + t^{\frac{1}{2}} = v_0(t),$$

then

$$D_{0+}^{\frac{3}{2}}u_0(t) = t^m \leq \left(\frac{u^2}{u^2+1} + t\right)^m \leq (t+1)^m = D_{0+}^{\frac{3}{2}}v_0(t).$$

Let $\operatorname{dom}(L) = \{u(t) \in X \mid D_{0+}^{\frac{3}{2}}u(t) \in Z, u(t) \text{ satisfies BCs (4.2)}\}$, then for any $x, y \in K \cap \operatorname{dom}(L)$, we have

$$\left(\frac{x^2}{x^2+1} + t\right)^m - \left(\frac{y^2}{y^2+1} + t\right)^m \geq -\frac{3}{2}\left(\int_0^1 x(t)\,dt - \int_0^1 y(t)\,dt\right),$$

where $u_0(t) \leq y(t) \leq x(t) \leq v_0(t)$. Finally, by Theorem 3.1, equation (4.1) with BCs (4.2) has a minimal solution u^* and a maximal solution v^* in $[u_0, v_0]$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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