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Schauder's fixed-point theorem: new applications and a new version for discontinuous operators

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Abstract

Schauder's fixed-point theorem, which applies for continuous operators, is used in this paper, perhaps unexpectedly, to prove existence of solutions to discontinuous problems. Moreover, we introduce a new version of Schauder's theorem for not necessarily continuous operators which implies existence of solutions for wider classes of problems. Leaning on an abstract fixed-point theorem, our approach is not limited to one-dimensional homogeneous Dirichlet problems, the only type of examples worked out in this paper for coherence and simplicity but yet novelty.

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Keywords: Schauder's theorem; fixed-point theorem; discontinuous differential equations

1 Introduction

This paper contains a probably unexpected application of Schauder's fixed-point theorem to a class of discontinuous problems, and a generalization of it that we have never seen before and proves useful in even more general contexts.

Our new version of Schauder's theorem yields novel existence results even for thoroughly studied problems such as

$$x'' = f(t,x), \qquad x(0) = x(1) = 0, \tag{1.1}$$

with a L^1-bounded nonlinearity f. The importance of our abstract result is that it allows f to be discontinuous with respect to the dependent variable and does not lean on monotonicity at all. This is a significant contribution to the available literature on existence of solutions to (1.1) with discontinuous f's which, roughly speaking, consists in rewriting $f(t,x) = g(t,x,x)$ for some function g which is continuous with respect to its second argument and monotone nonincreasing with respect to the third one. Essential references for this approach are [6, 12], and some more recent related results can be looked up in [5, 7, 13, 14].

Removing assumptions from the basic theory on (1.1) can only be useful in applications. Motions of particles in a force field, stationary distributions of temperatures, and many other phenomena can be modeled by means of equations of the form $x'' = f(t,x)$. In real life, external forces $f(t,x)$ often assume only a discrete set of more than one value, so they are often discontinuous (and not necessarily monotone).

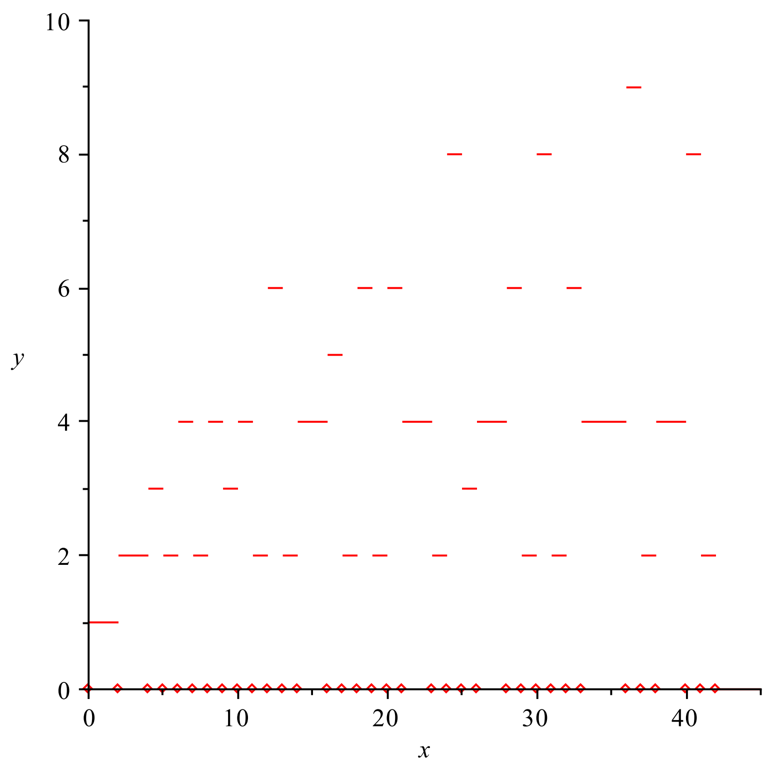

Figure 1 A plot of the 'counting divisors' function ϕ.

For completeness and later references, let us recall Schauder's theorem [16, Theorem 2.3.7].

Schauder's Fixed-Point Theorem *Let K be a nonempty, convex, and compact subset of a normed space.*

Any continuous operator $T : K \longrightarrow K$ has at least one fixed point.

In order to illustrate our new application of Schauder's theorem, we shall construct some examples based on the following discontinuous and nonmonotone function: let us call $\phi(x)$ the number of divisors of the integer part of $x \in [0, +\infty)$. See Figure 1 for a plot of this function.

Obviously, ϕ is piecewise constant, discontinuous at infinitely many natural numbers, right-continuous at every $x \geq 0$, $\phi(x) \geq 1$ for all $x \geq 0$, and

$$\limsup_{x \to +\infty} \phi(x) = +\infty.$$

There exist arbitrarily big prime numbers, so we also have

$$\liminf_{x \to +\infty} \phi(x) = 2.$$

The following result can be proven by means of Schauder's theorem, as we shall see in Section 2.

Proposition 1.1 *There exists at least one solution in $W^{2,1}(0,1)$ to the Dirichlet problem*

$$x'' = \frac{1}{4}\phi\left(\frac{1}{\sqrt{t} + |x|}\right) \quad \textit{for almost all } t \in [0,1], \qquad x(0) = x(1) = 0, \tag{1.2}$$

where $\phi(z)$ is the number of divisors of the integer part of $z \in [0,+\infty)$.

Notice that the right-hand side of the differential equation in (1.2) has discontinuities with respect to the unknown in every neighborhood of the boundary condition $(t_0, x_0) = (0,0)$. This makes it surprising at first sight that Schauder's theorem can be applied.

The rest of this paper is organized as follows. In Section 2, we show how to apply Schauder's theorem to derive a pretty easy proof of existence of solutions for a class of discontinuous second-order scalar problems containing (1.2) as a particular case. While our existence result in Section 2 is quite general and has an easy proof, it is somewhat restricted in the type of discontinuities that it admits. In Section 3, we present a generalization of Schauder's theorem for not necessarily continuous operators which allows working with more general types of discontinuities, as we illustrate in Section 4. Our fixed-point type approach is not limited to second-order differential equations or to homogeneous Dirichlet conditions, which we have considered only for the sake of simplicity.

2 A new application of Schauder's theorem

One of the simplest and best known applications of Schauder's fixed-point theorem is the proof of existence of solutions to

$$x'' = f(t,x) \quad \text{for a.a. } t \in I = [0,1], \qquad x(0) = x(1) = 0, \tag{2.1}$$

under the so-called Carathéodory's conditions, namely,

(C1) For every $x \in \mathbb{R}$, the mapping $t \in I \mapsto f(t,x)$ is measurable;

(C2) For a.a. $t \in I$, the mapping $x \in \mathbb{R} \mapsto f(t,x)$ is continuous.

Further conditions are needed in order to apply Schauder's theorem, and the next one is conveniently simple so as not to hide the main contributions in this paper (which have to do with weak forms of (C2)):

(C3) There exists $M \in L^1(I)$ such that for a.a. $t \in I$ and all $x \in \mathbb{R}$ we have $|f(t,x)| \le M(t)$.

The following result is standard.

Proposition 2.1 *If f satisfies (C1), (C2), and (C3) then problem (2.1) has at least one solution $x \in W^{2,1}(I)$.*

Remark 2.1 We shall identify the set $W^{2,1}(I)$ with that of all real-valued functions having an absolutely continuous derivative on I.

One can prove Proposition 2.1 *via* Schauder's theorem using the set

$$K = \left\{ x \in \mathcal{C}^1(I) : x(0) = x(1) = 0, \left|x'(t) - x'(s)\right| \le \int_s^t M(r)\,dr (s \le t) \right\},$$

which, by the Ascoli-Arzelá theorem, is a compact subset of the Banach space $\mathcal{C}^1(I)$ equipped with the norm

$$\|x\|_{\mathcal{C}^1} = \max_{t\in I}\left|x(t)\right| + \max_{t\in I}\left|x'(t)\right|.$$

Obviously, K is also convex, and, moreover, any element $x \in K$ has an absolutely continuous derivative x' and

$$\left|x''(t)\right| \le M(t) \quad \text{for a.a. } t \in I.$$

Solutions of (2.1) coincide then with fixed points of the usual operator

$$Tx(t) = \int_0^1 G(t,s)f\bigl(s,x(s)\bigr)\,ds \quad (t \in I, x \in K), \tag{2.2}$$

where G is the Green's function[a] corresponding to problem (2.1).

The operator T is well defined, maps K into itself, and satisfies the conditions in Schauder's theorem by virtue of (C1), (C2), and (C3).

Remark 2.2 In fact, proving Proposition 2.1 can be made even easier by working in the Banach space $\mathcal{C}(I)$ instead of $\mathcal{C}^1(I)$; see, for instance, [9]. However, working in $\mathcal{C}^1(I)$ will be more adequate in next sections, and we have chosen the proof outlined in the previous paragraph because our generalizations will start exactly the same way.

Can we relax condition (C2) in Proposition 2.1 and still get existence of solutions by means of essentially the same proof? We are going to show that the answer is positive, and, moreover, that is the way we are going to generalize (C2) is really meaningful.

Before going into detail, let us recall that (C1) and (C2) imply

(H1) Any composition $t \in I \mapsto f(t,x(t))$ is measurable whenever $x \in \mathcal{C}(I)$.

We refer readers to [1] for more information on measurability of compositions. Proposition 3.2 in [8] may help when checking (H1) in practice.

Next, we present a nontrivial generalization of Proposition 2.1 which has a remarkably simple proof. For the convenience of readers, we recall the following technical result: if two absolutely continuous functions agree on a given measurable set, then their derivatives coincide almost everywhere in that set; see, for instance, [17, Exercise 5(i), p.332].

Theorem 2.2 *Assume that $f : I \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies (H1), (C3), and*

(C2)[*] There exist $W^{2,1}$ functions

$$\gamma_n : I_n = [a_n, b_n] \subset I \longrightarrow \mathbb{R} \quad (n \in \mathbb{N})$$

such that for a.a. $t \in I$ the mapping $x \mapsto f(t,x)$ is continuous on $\mathbb{R} \setminus \bigcup_{\{n: t\in I_n\}}\{\gamma_n(t)\}$. Moreover, for each $n \in \mathbb{N}$ and a.a. $t \in I_n$ we have

$$\left|\gamma_n''(t)\right| > M(t),$$

where M is as in (C3).

Then problem (2.1) has at least one solution $x \in W^{2,1}(I)$.

Proof Let $K \subset \mathcal{C}^1(I)$ and $T : K \longrightarrow K$ be as in (2.2). Operator T is well-defined and maps K into itself by (H1) and (C3).

To prove T has at least one fixed point by means of Schauder's theorem, it suffices to show that T is continuous. To do it, let $x_m \to x$ in K.

For every $n \in \mathbb{N}$, the set $\{t \in I_n : x(t) = \gamma_n(t)\}$ is a null-measure set, for otherwise we would have in a positive measure set

$$\left|\gamma_n''(t)\right| = \left|x''(t)\right| \le M(t) \quad \text{(because } x \in K\text{)},$$

a contradiction with (C2)*.

Hence, for a.a. $t \in I$, the mapping $f(t, \cdot)$ is continuous at $x(t)$ and, therefore,

$$\lim_{m\to\infty} f\big(t, x_m(t)\big) = f\big(t, x(t)\big) \quad \text{for a.a. } t \in I.$$

We now deduce that $Tx_m \to Tx$ in $\mathcal{C}^1(I)$ thanks to standard properties of the Green's function and a straightforward application of Lebesgue's dominated convergence theorem. □

As an example, we show that Proposition 1.1 is a particular case to Theorem 2.2.

Proof of Proposition 1.1 Solutions of (1.2), if any, are strictly convex, hence negative in the interval $(0, 1)$. Therefore, for all $t \in (0, 1]$ and all $x \le 0$, we define

$$f(t, x) = \frac{1}{4}\phi\left(\frac{1}{\sqrt{t} - x}\right),$$

and for $t \in (0, 1]$ and $x > 0$ we define $f(t, x) = \phi(t^{-1/2})/4$. This definition ensures that $f(t, \cdot)$ is continuous on $[0, +\infty)$ for each $t \in (0, 1]$, and the corresponding problem (2.1) can only have strictly convex solutions, which would then be solutions of (1.2).

It suffices to show that f satisfies every condition in Theorem 2.2.

The definition of ϕ ensures that $\phi(z) \le z$ for all $z \ge 0$, hence f satisfies (C3) with

$$M(t) = \frac{1}{4\sqrt{t}} \quad \big(t \in I = (0, 1]\big).$$

To show that f satisfies (H1) and (C2)*, we use $\{n_k\}_{k\in\mathbb{N}}$ the sequence of all discontinuity points of ϕ, and we define $n_0 = 0$.

First, for every measurable function $\gamma : I \longrightarrow [0, \infty)$ and every $t \in I$ we have

$$\phi\big(\gamma(t)\big) = \sum_{k=1}^{\infty} \phi(n_{k-1})\chi_{J_k}(t),$$

where $J_k = \gamma^{-1}([n_{k-1}, n_k))$ is measurable for each $k \in \mathbb{N}$. Therefore, $\phi \circ \gamma$ is measurable whenever γ is measurable and nonnegative, and then the composition $t \mapsto \phi((\sqrt{t} + |x(t)|)^{-1})$ is measurable for any continuous function $x = x(t)$. Hence, (H1) is satisfied.

To check (C2)*, we note that for each $t \in (0, 1)$ all possible discontinuities of $f(t, \cdot)$ are located at those $x \in \mathbb{R}$, $x < 0$, satisfying $(\sqrt{t} - x)^{-1} = n_k$. This suggest solving for x to define, for each $k \in \mathbb{N}$, a function

$$\gamma_k(t) = \sqrt{t} - n_k^{-1} \quad \big(t \in I_k = \big[0, n_k^{-2}\big]\big).$$

For every $k \in \mathbb{N}$ and a.a. $t \in I_k$, we have

$$\left|\gamma_k''(t)\right| = \frac{1}{4\sqrt{t^3}} > M(t),$$

and $f(t,\cdot)$ is continuous on $\mathbb{R} \setminus \bigcup_{\{k:t\in I_k\}}\{\gamma_k(t)\}$. □

Condition (C2)* is restrictive because discontinuities must be located on graphs of curves with big absolute curvature. We can still revise our proof of Theorem 2.2 to generalize it further, but it is much better to note that the very Schauder's theorem can be extended to a class of discontinuous operators which allow more interferences between the second derivative of discontinuity curves and the values of the right-hand sides in the differential equations. This extension is carried out in the next section and will then be applied to deduce existence of solutions for greater classes of problems among which we find the following one, which we shall study in detail as an example:

$$x'' = \phi^{1/3}\left(\frac{1}{t^2 + |x|}\right) \quad \text{a.e. in } [0,1], \qquad x(0) = x(1) = 0. \tag{2.3}$$

3 Schauder's theorem for discontinuous operators

This section is devoted to introducing and proving a new fixed-point result of Schauder's type for not necessarily continuous operators. Despite its important implications (one of which we illustrate in Section 4) it is nothing but a straightforward corollary of Kakutani's fixed-point theorem for multivalued upper semicontinuous operators; see [16, Theorem 9.2.2] or [2, Theorem 3, p.232].

Theorem 3.1 *Let K be a nonempty, convex, and compact subset of a normed space X.*

Any mapping $T : K \longrightarrow K$ has at least one fixed point provided that for every $x \in K$ we have

$$\{x\} \cap \bigcap_{\varepsilon>0} \overline{\mathrm{co}}\, T\big(B_\varepsilon(x) \cap K\big) \subset \{Tx\}, \tag{3.1}$$

where $B_\varepsilon(x)$ stands for the closed ball in X with center x and radius $\varepsilon > 0$, and $\overline{\mathrm{co}}$ denotes the closed convex hull.

Proof Let us consider the well-known (see [10, Example 1.2]) multivalued mapping

$$\mathbb{T}x = \bigcap_{\varepsilon>0} \overline{\mathrm{co}}\, T\big(B_\varepsilon(x) \cap K\big) \quad (x \in K), \tag{3.2}$$

whose values are nonempty, convex, and compact subsets of K.

It is just routine to check that $\mathbb{T}$ is upper semicontinuous, *i.e.*, if $x_n \to x$ in K, $y_n \in \mathbb{T}x_n$ for all $n \in \mathbb{N}$, and $y_n \to y$, then we have $y \in \mathbb{T}x$.

Kakutani's fixed-point theorem guarantees that $\mathbb{T}$ has at least one fixed point, *i.e.*, at least one $x \in K$ such that $x \in \mathbb{T}x$. Now condition (3.1) trivially implies that x is a fixed point of T. □

Remark 3.1 One of the referees correctly pointed out that condition (3.1) can be rephrased simply as follows: either x is a fixed point of T, or $x \notin \mathbb{T}x$, where $\mathbb{T}$ is defined as in (3.2). In applications of Theorem 3.1, we should then prove that every $x \in \mathbb{T}x$ is a fixed point of T.

However, in order to highlight the roles of the different types of admissible discontinuity curves (which we shall define in our next section), we are going to use a different, not so simple, reformulation of (3.1).

Notice that the definition of $\mathbb{T}$ ensures that $\mathbb{T}x = \{Tx\}$ when T is continuous at x, so (3.1) is also equivalent to the following condition: for each $x \in K$ either T is continuous at x, or $x \notin \mathbb{T}x$, or $\{x\} \cap \mathbb{T}x = \{Tx\}$. We shall consider separately these three situations in our application of Theorem 3.1 in the proof of Theorem 4.4.

Finally, note also that many known fixed-point theorems could be extended exactly the same way we generalized Schauder's to Theorem 3.1.

For each $x \in K$, the set $\mathbb{T}x$ defined in (3.2) contains Tx along with, roughly speaking, every limit value $z \leftarrow Ty$ when $y \to x$, and every limit of convex combinations of the previous elements. For example, if $K = [a,b]$ with $a, b \in \mathbb{R}$, $a < b$, then for every $x \in [a,b]$ we have

$$\mathbb{T}x = \left[\min\left\{T(x), \liminf_{y\to x} T(y)\right\}, \max\left\{T(x), \limsup_{y\to x} T(y)\right\}\right],$$

considering the corresponding side limits for $x \in \{a,b\}$.

It is difficult to have a view on how $\mathbb{T}$ is in higher dimensions. Let us content ourselves with the following analytical characterization. The proof is trivial.

Proposition 3.2 *In the conditions of Theorem 3.1, let $x, y \in K$ be fixed.*

The following two statements are equivalent:

1. *$y \in \mathbb{T}x$ as defined in (3.2);*
2. *For every $\varepsilon > 0$ and every $\rho > 0$ there exists a finite family of vectors $x_i \in B_\varepsilon(x) \cap K$ and coefficients $\lambda_i \in [0,1]$ $(i = 1, 2, \ldots, m)$ such that $\sum \lambda_i = 1$ and*

$$\left\| y - \sum_{i=1}^{m} \lambda_i T x_i \right\|_X < \rho.$$

4 Application to Dirichlet problems

In this section, we illustrate the applicability of Theorem 3.1 to deduce the existence of $W^{2,1}$-solutions to the Dirichlet problem (2.1) with a function $f : I \times \mathbb{R} \longrightarrow \mathbb{R}$ which may be discontinuous with respect to both arguments.

Basically, we allow f to be discontinuous over countably many graphs of functions in the conditions of the following definition. The reader is referred to [3, 8, 15] for similar ideas for first-order problems.

Definition 4.1 An admissible discontinuity curve for the differential equation $x'' = f(t,x)$ is a $W^{2,1}$ function $\gamma : [a,b] \subset I \longrightarrow \mathbb{R}$ satisfying one of the following conditions: either $\gamma''(t) = f(t, \gamma(t))$ for a.a. $t \in [a,b]$ (and we then say that γ is viable for the differential equation), or there exist $\varepsilon > 0$ and $\psi \in L^1(a,b)$, $\psi(t) > 0$ for a.a. $t \in [a,b]$, such that either

$$\gamma''(t) + \psi(t) < f(t,y) \quad \text{for a.a. } t \in I \text{ and all } y \in \bigl[\gamma(t) - \varepsilon, \gamma(t) + \varepsilon\bigr], \tag{4.1}$$

or

$$\gamma''(t) - \psi(t) > f(t,y) \quad \text{for a.a. } t \in I \text{ and all } y \in \bigl[\gamma(t) - \varepsilon, \gamma(t) + \varepsilon\bigr]. \tag{4.2}$$

We say that the admissible discontinuity curve γ is inviable for the differential equation if it satisfies (4.1) or (4.2).

Remark 4.1 It should be already apparent that this paper owes many ideas to set-valued analysis and viability theory. It is therefore fair (and reasonable) to acknowledge it by using the adjectives viable or inviable for our admissible discontinuity curves.

Roughly, inviable curves push solutions away from them, and viable curves allow solutions slide over them.

If function f were continuous, then inviable admissible discontinuity curves would be just strict lower (or upper) solutions on subintervals of I. Of course, the interest of admissible discontinuity curves is Theorem 4.4 below, which concerns discontinuous f's.

Viable admissible discontinuity curves are nothing but solutions of the differential equation. It can be reasonably argued that viable discontinuity curves are unlikely to be found in applications. It will, however, remain clear in our final examples that viable curves can be in some cases more useful in applications than inviable ones.

Discontinuity curves in Theorem 2.2 cannot be viable but, curiously, need not be inviable.

Working with admissible discontinuity curves involves some technicalities gathered in the next lemma and its subsequent corollaries. Here, we mimic the ideas in [11, Lemma 2.3]. In the sequel m stands for the Lebesgue measure in $\mathbb{R}$.

Lemma 4.1 *Let $a, b \in \mathbb{R}$, $a < b$, and let $g, h \in L^1(a,b)$, $g \geq 0$ a.e., and $h > 0$ a.e. in (a,b).*

For every measurable set $J \subset (a,b)$ with $m(J) > 0$ there is a measurable set $J_0 \subset J$ with $m(J \setminus J_0) = 0$ such that for every $\tau_0 \in J_0$ we have

$$\lim_{t \to \tau_0^+} \frac{\int_{[\tau_0,t]\setminus J} g(s)\,ds}{\int_{\tau_0}^t h(s)\,ds} = 0 = \lim_{t \to \tau_0^-} \frac{\int_{[t,\tau_0]\setminus J} g(s)\,ds}{\int_t^{\tau_0} h(s)\,ds}. \tag{4.3}$$

Proof We define absolutely continuous functions

$$G_J(t) = \int_a^t g(s)\chi_{(a,b)\setminus J}(s)\,ds \quad \text{and} \quad H(t) = \int_a^t h(s)\,ds \quad \bigl(t \in [a,b]\bigr).$$

A classical result ensures the existence of a measurable set $J_0 \subset J$, with $m(J \setminus J_0) = 0$, such that for every $\tau_0 \in J_0$ there exist

$$G_J'(\tau_0) = g(\tau_0)\chi_{(a,b)\setminus J}(\tau_0) = 0 \quad \text{and} \quad H'(\tau_0) = h(\tau_0) > 0.$$

For $\tau_0 \in J_0$ and $t \in (a,b)$, $t > \tau_0$, we have

$$\frac{\int_{[\tau_0,t]\setminus J} g(s)\,ds}{\int_{\tau_0}^t h(s)\,ds} = \frac{\int_{\tau_0}^t g(s)\chi_{(a,b)\setminus J}(s)\,ds}{\int_{\tau_0}^t h(s)\,ds} = \frac{G_J(t)-G_J(\tau_0)}{H(t)-H(\tau_0)},$$

so taking limit when $t \to \tau_0^+$ we obtain the first identity in (4.3). The second identity admits a similar proof. □

Corollary 4.2 *Let $a,b \in \mathbb{R}$, $a < b$, and let $h \in L^1(a,b)$ be such that $h > 0$ a.e. in (a,b).*

For every measurable set $J \subset (a,b)$ with $m(J) > 0$ there is a measurable set $J_0 \subset J$ with $m(J \setminus J_0) = 0$ such that for all $\tau_0 \in J_0$ we have

$$\lim_{t\to\tau_0^+} \frac{\int_{[\tau_0,t]\cap J} h(s)\,ds}{\int_{\tau_0}^t h(s)\,ds} = 1 = \lim_{t\to\tau_0^-} \frac{\int_{[t,\tau_0]\cap J} h(s)\,ds}{\int_t^{\tau_0} h(s)\,ds}. \tag{4.4}$$

Proof Let $J_0 \subset J$ be the set given by Lemma 4.1 when $g = h$. For every $\tau_0 \in J_0$, we compute

$$\lim_{t\to\tau_0^+} \frac{\int_{[\tau_0,t]\cap J} h(s)\,ds}{\int_{\tau_0}^t h(s)\,ds} = \lim_{t\to\tau_0^+} \frac{\int_{\tau_0}^t h(s)\,ds - \int_{[\tau_0,t]\setminus J} h(s)\,ds}{\int_{\tau_0}^t h(s)\,ds} = 1,$$

and the other identity can be proven in the same way. □

A second consequence of Lemma 4.1 has independent interest (notice that the set A in our next corollary need not be an interval).

Corollary 4.3 *Let $a,b \in \mathbb{R}$, $a < b$, and let $f, f_n : [a,b] \longrightarrow \mathbb{R}$ be absolutely continuous functions on $[a,b]$ ($n \in \mathbb{N}$), such that $f_n \to f$ uniformly on $[a,b]$ and for a measurable set $A \subset [a,b]$ with $m(A) > 0$ we have*

$$\lim_{n\to\infty} f_n'(t) = g(t) \quad \text{for a.a. } t \in A.$$

If there exists $M \in L^1(a,b)$ such that $|f'(t)| \le M(t)$ a.e. in $[a,b]$ and also $|f_n'(t)| \le M(t)$ a.e. in $[a,b]$ ($n \in \mathbb{N}$), then $f'(t) = g(t)$ for a.a. $t \in A$.

Proof Reasoning by contradiction, we assume that for some $r > 0$ the measurable set

$$A_r = \{t \in A : f'(t) > g(t) + r\}$$

has positive Lebesgue measure.

By virtue of Egorov's theorem, $f_n' \to g$ uniformly in some set $B \subset A_r$ with $m(B) > 0$. Hence, there exists $N \in \mathbb{N}$ such that for all $n \ge N$ and all $t \in B$ we have

$$f'(t) > f_n'(t) + \frac{r}{2}. \tag{4.5}$$

We deduce from Lemma 4.1 and Corollary 4.2 with $g = M$ and $h = 1$, that we can find $\tau_0 \in B$ and $\tau_1 > \tau_0$ such that

$$\frac{r}{4} m\big([\tau_0,\tau_1] \cap B\big) > \int_{[\tau_0,\tau_1]\setminus B} M(t)\,dt. \tag{4.6}$$

Now for all $n \in \mathbb{N}$, $n \geq N$, we have

$$\begin{aligned}
\frac{r}{2} m\big([\tau_0,\tau_1]\cap B\big) &\leq \int_{[\tau_0,\tau_1]\cap B} \big(f'(t) - f_n(t)\big)\,dt \quad \big(\text{by } (4.5)\big) \\
&= \int_{\tau_0}^{\tau_1} \big(f'(t) - f_n'(t)\big)\,dt - \int_{[\tau_0,\tau_1]\setminus B} \big(f'(t) - f_n'(t)\big)\,dt \\
&\leq f(\tau_1) - f(\tau_0) - f_n(\tau_1) + f_n(\tau_0) + 2\int_{[\tau_0,\tau_1]\setminus B} M(t)\,dt \\
&\leq 2\|f - f_n\|_0 + 2\int_{[\tau_0,\tau_1]\setminus B} M(t)\,dt,
\end{aligned}$$

which implies that $\|f - f_n\|_0 = \max\{|f(t) - f_n(t)| : t \in [a,b]\}$ does not tend to zero because of (4.6), a contradiction.

One can prove by means of analogous arguments that

$$m\big(\{t \in A : f'(t) < g(t) - r\}\big) = 0 \quad \text{for all } r > 0,$$

and therefore $f' = g$ a.e. in A. □

We are now ready for the proof of the main result in this section.

Theorem 4.4 *Problem (2.1) has at least one solution in* $W^{2,1}(I)$ *provided that* $f : I \times \mathbb{R} \longrightarrow \mathbb{R}$ *satisfies (H1), (C3), and*

(H2) There exist admissible discontinuity curves $\gamma_n : I_n = [a_n, b_n] \longrightarrow \mathbb{R}$ *(*$n \in \mathbb{N}$*) such that for a.a.* $t \in I$ *the function* $x \mapsto f(t,x)$ *is continuous on* $\mathbb{R} \setminus \bigcup_{\{n: t\in I_n\}}\{\gamma_n(t)\}$.

Proof We start (exactly as in the proof of Theorem 2.2) considering $K \subset X = \mathcal{C}^1(I)$ and $T : K \longrightarrow K$ as in (2.2). Operator T is well defined and maps K into itself by (H1) and (C3).

The proof will be over once we have checked that condition (3.1) in Theorem 3.1 is satisfied. To do so, we fix an arbitrary function $x \in K$ and we consider three different cases (remember Remark 3.1). For simplicity, we use the notation $\mathbb{T}x$ as introduced in (3.2).

Case 1 - $m(\{t \in I_n : x(t) = \gamma_n(t)\}) = 0$ *for all* $n \in \mathbb{N}$. Let us prove that then T is continuous at x.

The assumption implies that for a.a. $t \in I$ the mapping $f(t,\cdot)$ is continuous at $x(t)$. Hence, if $x_k \to x$ in K, then

$$f\big(t, x_k(t)\big) \to f\big(t, x(t)\big) \quad \text{for a.a. } t \in I,$$

which, along with (C3), yield $Tx_k \to Tx$ in $\mathcal{C}^1(I)$.

Case 2 - $m(\{t \in I_n : x(t) = \gamma_n(t)\}) > 0$ *for some* $n \in \mathbb{N}$ *such that* γ_n *is inviable.* In this case we can prove that $x \notin \mathbb{T}x$.

First, we fix some notation. Let us assume that for some $n \in \mathbb{N}$ we have $m(\{t \in I_n : x(t) = \gamma_n(t)\}) > 0$ and there exist $\varepsilon > 0$ and $\psi \in L^1(I_n)$, $\psi(t) > 0$ for a.a. $t \in I_n$, such that (4.2) holds with γ replaced by γ_n. (The proof is similar if we assume (4.1) instead of (4.2), so we omit it.)

We denote $J = \{t \in I_n : x(t) = \gamma_n(t)\}$, and we deduce from Lemma 4.1 that there is a measurable set $J_0 \subset J$ with $m(J_0) = m(J) > 0$ such that for all $\tau_0 \in J_0$ we have

$$\lim_{t\to\tau_0^+} \frac{2\int_{[\tau_0,t]\setminus J} M(s)\,ds}{(1/4)\int_{\tau_0}^{t} \psi(s)\,ds} = 0 = \lim_{t\to\tau_0^-} \frac{2\int_{[t,\tau_0]\setminus J} M(s)\,ds}{(1/4)\int_{t}^{\tau_0} \psi(s)\,ds}. \tag{4.7}$$

By Corollary 4.2, there exists $J_1 \subset J_0$ with $m(J_0 \setminus J_1) = 0$ such that for all $\tau_0 \in J_1$ we have

$$\lim_{t\to\tau_0^+} \frac{\int_{[\tau_0,t]\cap J_0} \psi(s)\,ds}{\int_{\tau_0}^{t} \psi(s)\,ds} = 1 = \lim_{t\to\tau_0^-} \frac{\int_{[t,\tau_0]\cap J_0} \psi(s)\,ds}{\int_{t}^{\tau_0} \psi(s)\,ds}. \tag{4.8}$$

Let us now fix a point $\tau_0 \in J_1$. From (4.7) and (4.8), we deduce that there exist $t_- < \tau_0$ and $t_+ > \tau_0$, $t_\pm$ sufficiently close to τ_0 so that the following inequalities are satisfied:

$$2\int_{[\tau_0,t_+]\setminus J} M(s)\,ds < \frac{1}{4}\int_{\tau_0}^{t_+} \psi(s)\,ds, \tag{4.9}$$

$$\int_{[\tau_0,t_+]\cap J} \psi(s)\,ds \ge \int_{[\tau_0,t_+]\cap J_0} \psi(s)\,ds > \frac{1}{2}\int_{\tau_0}^{t_+} \psi(s)\,ds, \tag{4.10}$$

$$2\int_{[t_-,\tau_0]\setminus J} M(s)\,ds < \frac{1}{4}\int_{t_-}^{\tau_0} \psi(s)\,ds, \tag{4.11}$$

$$\int_{[t_-,\tau_0]\cap J} \psi(s)\,ds > \frac{1}{2}\int_{t_-}^{\tau_0} \psi(s)\,ds. \tag{4.12}$$

Finally, we define a positive number

$$\rho = \min\left\{\frac{1}{4}\int_{t_-}^{\tau_0} \psi(s)\,ds, \frac{1}{4}\int_{\tau_0}^{t_+} \psi(s)\,ds\right\}, \tag{4.13}$$

and we are now in a position to prove that $x \notin \mathbb{T}x$. By virtue of Proposition 3.2, it suffices to prove the following claim:

Claim - Let $\varepsilon > 0$ be given by our assumptions over γ_n and let ρ be as in (4.13). For every finite family $x_i \in B_\varepsilon(x) \cap K$ and $\lambda_i \in [0,1]$ $(i = 1,2,\dots,m)$, with $\sum \lambda_i = 1$, we have $\|x - \sum \lambda_i Tx_i\|_{\mathcal{C}^1} \ge \rho$.

Let x_i and λ_i be as in the claim and, for simplicity, denote $y = \sum \lambda_i Tx_i$. For a.a. $t \in J = \{t \in I_n : x(t) = \gamma_n(t)\}$, we have

$$y''(t) = \sum_{i=1}^{m} \lambda_i (Tx_i)''(t) = \sum_{i=1}^{m} \lambda_i f\bigl(t, x_i(t)\bigr). \tag{4.14}$$

On the other hand, for every $i \in \{1,2,\dots,m\}$ and every $t \in J$ we have

$$\bigl|x_i(t) - \gamma_n(t)\bigr| = \bigl|x_i(t) - x(t)\bigr| < \varepsilon,$$

and then the assumptions on γ_n ensure that for a.a. $t \in J$ we have

$$y''(t) = \sum_{i=1}^{m} \lambda_i f\bigl(t, x_i(t)\bigr) < \sum_{i=1}^{m} \lambda_i \bigl(\gamma_n''(t) - \psi(t)\bigr) = x''(t) - \psi(t). \tag{4.15}$$

Now we compute

$$\begin{aligned} y'(\tau_0) - y'(t_-) &= \int_{t_-}^{\tau_0} y''(s)\,ds = \int_{[t_-,\tau_0]\cap J} y''(s)\,ds + \int_{[t_-,\tau_0]\setminus J} y''(s)\,ds \\ &< \int_{[t_-,\tau_0]\cap J} x''(s)\,ds - \int_{[t_-,\tau_0]\cap J} \psi(s)\,ds \\ &\quad + \int_{[t_-,\tau_0]\setminus J} M(s)\,ds \quad \bigl(\text{by (4.15), (4.14) and (C3)}\bigr) \\ &= x'(\tau_0) - x'(t_-) - \int_{[t_-,\tau_0]\setminus J} x''(s)\,ds - \int_{[t_-,\tau_0]\cap J} \psi(s)\,ds \\ &\quad + \int_{[t_-,\tau_0]\setminus J} M(s)\,ds \\ &\le x'(\tau_0) - x'(t_-) - \int_{[t_-,\tau_0]\cap J} \psi(s)\,ds + 2\int_{[t_-,\tau_0]\setminus J} M(s)\,ds \\ &< x'(\tau_0) - x'(t_-) - \frac{1}{4}\int_{t_-}^{\tau_0} \psi(s)\,ds \quad \bigl(\text{by (4.11) and (4.12)}\bigr), \end{aligned}$$

hence $\|x - y\|_{\mathcal{C}^1} \ge y'(t_-) - x'(t_-) \ge \rho$ provided that $y'(\tau_0) \ge x'(\tau_0)$.

Similar computations with t_+ instead of t_- show that if $y'(\tau_0) \le x'(\tau_0)$ then we also have $\|x - y\|_{\mathcal{C}^1} \ge \rho$. The claim is proven.

Case 3 - $m(\{t \in I_n : x(t) = \gamma_n(t)\}) > 0$ only for some of those $n \in \mathbb{N}$ such that γ_n is viable. Let us prove that in this case the relation $x \in \mathbb{T}x$ implies $x = Tx$.

Let us consider the subsequence of all viable admissible discontinuity curves in the conditions of Case 3, which we denote again by $\{\gamma_n\}_{n\in\mathbb{N}}$ to avoid overloading notation. We have $m(J_n) > 0$ for all $n \in \mathbb{N}$, where

$$J_n = \bigl\{t \in I_n : x(t) = \gamma_n(t)\bigr\}.$$

For each $n \in \mathbb{N}$ and for a.a. $t \in J_n$, we have

$$x''(t) = \gamma_n''(t) = f\bigl(t, \gamma_n(t)\bigr) = f\bigl(t, x(t)\bigr),$$

and, therefore, $x''(t) = f(t, x(t))$ a.e. in $J = \bigcup_{n\in\mathbb{N}} J_n$.

Now we assume that $x \in \mathbb{T}x$ and we prove that it implies that $x''(t) = f(t, x(t))$ a.e. in $I \setminus J$, thus showing that $x = Tx$.

Since $x \in \mathbb{T}x$ then for each $k \in \mathbb{N}$, we can use Proposition 3.2 with $\varepsilon = \rho = 1/k$ to guarantee that we can find functions $x_{k,i} \in B_{1/k}(x) \cap K$ and coefficients $\lambda_{k,i} \in [0,1]$ ($i = 1, 2, \dots, m(k)$) such that $\sum \lambda_{k,i} = 1$ and

$$\left\| x - \sum_{i=1}^{m(k)} \lambda_{k,i} T x_{k,i} \right\|_{\mathcal{C}^1} < \frac{1}{k}.$$

Let us denote $y_k = \sum_{i=1}^{m(k)} \lambda_{k,i} T x_{k,i}$, and notice that $y_k' \to x'$ uniformly in I and $\|x_{k,i} - x\|_{\mathcal{C}^1} \le 1/k$ for all $k \in \mathbb{N}$ and all $i \in \{1, 2, \dots, m(k)\}$.

For a.a. $t \in I \setminus J$, we have that $f(t,\cdot)$ is continuous at $x(t)$, so for any $\varepsilon > 0$ there is some $k_0 = k_0(t) \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $k \geq k_0$, we have

$$\left|f\left(t, x_{k,i}(t)\right) - f\left(t, x(t)\right)\right| < \varepsilon \quad \text{for all } i \in \left\{1, 2, \ldots, m(k)\right\}$$

and, therefore,

$$\left|y_k''(t) - f\left(t, x(t)\right)\right| \leq \sum_{i=1}^{m(k)} \lambda_{k,i} \left|f\left(t, x_{k,i}(t)\right) - f\left(t, x(t)\right)\right| < \varepsilon.$$

Hence, $y_k''(t) \to f(t, x(t))$ for a.a. $t \in I \setminus J$, and then Corollary 4.3 guarantees that $x''(t) = f(t, x(t))$ for a.a. $t \in I \setminus J$. □

Finally, we go back to problem (2.3) for an illustrative example. In this case we can quickly prove the existence of solutions by redefining the nonlinear part over the discontinuity curves so that all of them become viable, and then we show that solutions of the modified problem are solutions of the former one (which is not true in general).

Proposition 4.5 *Problem (2.3) has at least one solution.*

Proof We can identify (2.3) with (2.1) for

$$f(t,x) = \phi^{1/3}\left(\frac{1}{t^2 - x}\right) \quad \text{for } t \in I,\ t > 0, \text{ and } x \leq 0,$$

and $f(t,x) = \phi^{1/3}(t^{-2})$ for $t > 0$ and $x > 0$ (this definition makes $f(t,\cdot)$ be continuous for all $x \in [0,\infty)$ and for a.a. $t \in I$, and, moreover, any possible solution of (2.1) is nonpositive and, therefore, it is a solution of (2.3)).

For a.a. $t \in I$, the function $f(t,\cdot)$ is continuous on $\mathbb{R} \setminus \bigcup_{\{k: t \in I_k\}} \{\gamma_k(t)\}$, where for each $k \in \mathbb{N}$

$$\gamma_k(t) = t^2 - n_k^{-1} \quad \text{for all } t \in I_k = \left[0, n_k^{-2}\right],$$

and $\{n_k\}_{k \in \mathbb{N}}$ is the sequence of all discontinuity points of ϕ.

Some of the γ_k's are inviable, some of them might be viable, but unluckily, some of them are not admissible discontinuity curves. To overcome this difficulty, we consider a modified problem (2.1) with f replaced by $\tilde{f}$, where for each $k \in \mathbb{N}$ we define

$$\tilde{f}\left(t, \gamma_k(t)\right) = 2\left(= \gamma_k''(t)\right) \quad \text{a.e. in } I_k,$$

and $\tilde{f}(t,x) = f(t,x)$ elsewhere.

Similar arguments to those in the proof of Proposition 1.1 show that $\tilde{f}$ satisfies (H1) and (C3) (take $M(t) = \max\{t^{-2/3}, 2\}$ for a.a. $t \in I$).

Plainly, for each $k \in \mathbb{N}$, γ_k is a viable discontinuity curve for $\tilde{f}$, and therefore Theorem 4.4 ensures that (2.1) with f replaced by $\tilde{f}$ has at least one solution $x \in W^{2,1}(I)$.

Since x is convex and $x(0) = x(1) = 0$, then x can only intersect each γ_k once, so x is also a solution to (2.1). □

Remark 4.2 Theorem 3.1 yields similar results for other types of problems, not only for (2.1). Indeed, we have successfully adapted the proof of Theorem 4.4 to readily get a nice analogous existence result for

$$x' = f(t,x), \quad t \in [0,1], \qquad x(0) = x_0 \in \mathbb{R},$$

but we have decided not to include it here because it was just a particular case to [8, Theorem 2.4] (although easier to prove).

Competing interests
The author declares that they have no competing interests.

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Endnote
[a] We recommend readers to visit Alberto Cabada's webpage where a very useful program for computing Green's functions can be downloaded [4].

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