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Existence and uniqueness of a positive solution to singular fractional differential equations

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Abstract

In this paper, we discuss the existence and uniqueness of a positive solution to the following singular fractional differential equation with nonlocal boundary value conditions:

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = 0, \quad D_{0+}^{\beta} u(1) = \sum_{i=1}^{m-2} \eta_i D_{0+}^{\beta} u(\xi_i), \end{cases}$$

where $1 < \alpha \leq 2$, $0 < \beta < \alpha - 1$, $0 < \xi_1 < \dots < \xi_{m-2} < 1$ with $\sum_{i=1}^{m-2} \eta_i \xi_i^{\alpha-\beta-1} < 1$, D_{0+}^{α} is the standard Riemann-Liouville derivative, f may be singular at $t = 0$, $t = 1$, and $u = 0$.

MSC: 34B10; 34B15

Keywords: fractional differential equation; positive solution; iterative scheme; singular boundary value problem

1 Introduction

In this paper, we consider the following fractional differential equation:

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = 0, \quad D_{0+}^{\beta} u(1) = \sum_{i=1}^{m-2} \eta_i D_{0+}^{\beta} u(\xi_i), \end{cases} \quad (1.1)$$

where $1 < \alpha \leq 2$, $0 < \beta < \alpha - 1$, $0 < \xi_1 < \dots < \xi_{m-2} < 1$ with $\sum_{i=1}^{m-2} \eta_i \xi_i^{\alpha-\beta-1} < 1$, D_{0+}^{α} is the standard Riemann-Liouville derivative, $f \in C((0, 1) \times (0, +\infty) \rightarrow [0, +\infty))$ may be singular at $t = 0$, $t = 1$, and $u = 0$. In this paper, by a positive solution to (1.1), we mean a function $u \in C[0, 1]$ which satisfies $D_{0+}^{\alpha} u \in L(0, 1)$, positive on $(0, 1]$ and satisfies (1.1).

Recently, many results were obtained dealing with the existence of solutions for nonlinear fractional differential equations by using the techniques of nonlinear analysis; see [1–23] and references therein. The multi-point boundary value problems (BVP for short) have provoked a great deal of attention, for example [13–19]. In [10], the authors discussed some positive properties of the Green function for Dirichlet-type BVP of nonlinear frac-

tional differential equation

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = 0, & u(1) = 0, \end{cases} \quad (1.2)$$

where $1 < \alpha < 2$, D_{0+}^{α} is the standard Riemann-Liouville derivative, $f \in C([0, 1] \times [0, +\infty) \rightarrow [0, +\infty))$. By using the Krasnosel'skii fixed point theorem, the existence of positive solutions were obtained under suitable conditions on f .

In [14], the authors investigated the existence and multiplicity of positive solutions by using some fixed point theorems for the fractional differential equation

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = 0, & D_{0+}^{\beta} u(1) = aD_{0+}^{\beta} u(\xi), \end{cases} \quad (1.3)$$

where $1 < \alpha \leq 2$, $0 \leq \beta \leq 1$, $0 < \xi < 1$, $0 \leq a \leq 1$ with $a\xi^{\alpha-\beta-2} < 1 - \beta$, $0 \leq \alpha - \beta - 1$, $f : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$ satisfied Carathéodory type conditions.

In [20, 21], the authors considered the fractional differential equation given by

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t, u, u', \dots, u^{(n-2)}) = 0, & 0 < t < 1, n - 1 < \alpha \leq n, n \geq 2, \\ u(0) = u'(0) = \dots = u^{(n-2)}(0) = 0, & u^{(n-2)}(1) = 0. \end{cases} \quad (1.4)$$

In order to obtain the existence of positive solutions of (1.4), they considered the following fractional differential equation:

$$\begin{cases} D_{0+}^{\alpha-n+2} v(t) + f(t, I_{0+}^{n-2} v(t), I_{0+}^{n-3} v(t), \dots, I_{0+}^1 v(t), v(t)) = 0, & 0 < t < 1, \\ v(0) = v(1) = 0. \end{cases} \quad (1.5)$$

In [20], $f = q(t)(g + h)$, and g, h have different monotone properties. By using the fixed point theorem for the mixed monotone operator, Zhang obtained (1.4) and had a unique positive solution $u(t) = I_{0+}^{n-2} v(t)$ with $v \in Q =: \{x(t) : \frac{1}{M} t^{\alpha-n+1} \leq x(t) \leq M t^{\alpha-n+1}\}$. But the results are not true since $v(t)$ is a positive solution of (1.5), and $v(1) = 0$. What causes it lies in the unsuitable using of properties of the Green function.

In [21], $f \in C([0, 1] \times [0, +\infty) \times R^{n-2} \rightarrow [0, +\infty))$, $f(t, y_1, y_2, \dots, y_{n-1})$ is increasing for $y_i \geq 0$, $i = 1, 2, \dots, n - 1$. By using the positive properties of the Green function obtained in [10] and fixed point theory for the u_0 concave operator, the authors obtained the uniqueness of a positive solution for the BVP (1.4).

Motivated by the works mentioned above, in this paper we aim to establish the existence and uniqueness of a positive solution to the BVP (1.1). Our work presented in this paper has the following features. Firstly, the BVP (1.1) possesses singularity, that is, f may be singular at $t = 0$, $t = 1$, and $u = 0$. Secondly, we impose weaker positivity conditions on the nonlocal boundary term, that is, some of the coefficients η_i can be negative. Thirdly, the unique positive solution can be approximated by an iterative scheme.

The rest of the paper is organized as follows. In Section 2, we present some preliminaries and lemmas that will be used to prove our main results. We also develop some new positive properties of the Green function. In Section 3, we discuss the existence and uniqueness of

a positive solution of the BVP (1.1), we also give an example to demonstrate the application of our theoretical results.

2 Preliminaries

For the convenience of the reader, we present here the necessary definitions from fractional calculus theory. These definitions can be found in recent literature.

Definition 2.1 The fractional integral of order $\alpha > 0$ of a function $u : (0, +\infty) \rightarrow R$ is given by

$$I_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds$$

provided the right-hand side is defined pointwise on $(0, +\infty)$.

Definition 2.2 The fractional derivative of order $\alpha > 0$ of a continuous function $u : (0, +\infty) \rightarrow R$ is given by

$$D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} u(s) ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integral part of the number α , provided the right-hand side is pointwisely defined on $(0, +\infty)$.

Definition 2.3 By $u \in L(0,1)$, we mean $\int_0^1 |u(t)| dt < \infty$.

Lemma 2.1 ([3]) Let $\alpha > 0$. Then the following equality holds for $u \in L(0,1)$, $D_{0+}^{\alpha} u \in L(0,1)$,

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \dots + c_n t^{\alpha-n},$$

where $c_i \in R$, $i = 1, 2, \dots, n$, $n-1 < \alpha \leq n$.

Set

$$G_0(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-\beta-1}, & 0 \leq t \leq s \leq 1, \\ t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \end{cases} \tag{2.1}$$

$$p(s) = 1 - \sum_{s \leq \xi_i} \eta_i \left(\frac{\xi_i - s}{1-s}\right)^{\alpha-\beta-1}, \tag{2.2}$$

$$G(t,s) = G_0(t,s) + q(s)t^{\alpha-1}, \tag{2.3}$$

where

$$q(s) = \frac{p(s) - p(0)}{\Gamma(\alpha)p(0)} (1-s)^{\alpha-\beta-1}, \quad p(0) = 1 - \sum_{i=1}^{m-2} \eta_i \xi_i^{\alpha-\beta-1}. \tag{2.4}$$

For the convenience in presentation, we here list the assumption to be used throughout the paper.

(H₁) $p(0) > 0$, $q(s) \geq 0$ on $[0,1]$.

Remark 2.1 If $\eta_i = 0$ ($i = 1, \dots, m - 2$), we have $p(0) = 1$ and $q(s) \equiv 0$. If $\eta_i \geq 0$ ($i = 1, \dots, m - 2$) and $\sum_{i=1}^{m-2} \eta_i \xi_i^{\alpha-\beta-1} < 1$, we have $p(0) > 0$ and $q(s) \geq 0$ on $[0, 1]$.

Lemma 2.2 ([14]) *Assume that $g \in L(0, 1)$ and $\alpha > 1 \geq \beta \geq 0$. Then*

$$D_{0+}^{\beta} \int_0^t (t-s)^{\alpha-1} g(s) ds = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} g(s) ds.$$

Lemma 2.3 *Assume (H_1) holds, and $y \in L(0, 1)$. Then the unique solution of the problem*

$$\begin{cases} D_{0+}^{\alpha} u(t) + y(t) = 0, & 0 < t < 1, \\ u(0) = 0, & D_{0+}^{\beta} u(1) = \sum_{i=1}^{m-2} \eta_i D_{0+}^{\beta} u(\xi_i), \end{cases} \quad (2.5)$$

is

$$u(t) = \int_0^1 G(t, s) y(s) ds,$$

where $G(t, s)$ is called the Green function of BVP (2.5).

Proof From Lemma 2.1, we have the solution of (2.5) given by

$$u(t) = -I_{0+}^{\alpha} y(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}.$$

Consequently,

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}.$$

From $u(0) = 0$, we have $c_2 = 0$.

By Lemma 2.2, we have

$$D_{0+}^{\beta} u(t) = -\frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} y(s) ds + \frac{c_1 \Gamma(\alpha)}{\Gamma(\alpha-\beta)} t^{\alpha-\beta-1}.$$

Therefore,

$$D_{0+}^{\beta} u(1) = -\frac{1}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} y(s) ds + \frac{c_1 \Gamma(\alpha)}{\Gamma(\alpha-\beta)},$$

and

$$D_{0+}^{\beta} u(\xi_i) = -\frac{1}{\Gamma(\alpha-\beta)} \int_0^{\xi_i} (\xi_i-s)^{\alpha-\beta-1} y(s) ds + \frac{c_1 \Gamma(\alpha)}{\Gamma(\alpha-\beta)} \xi_i^{\alpha-\beta-1}.$$

By $D_{0+}^{\beta} u(1) = \sum_{i=1}^{m-2} \eta_i D_{0+}^{\beta} u(\xi_i)$, we have

$$c_1 = \frac{\int_0^1 (1-s)^{\alpha-1} y(s) ds - \sum_{i=1}^{m-2} \eta_i \int_0^{\xi_i} (\xi_i-s)^{\alpha-\beta-1} y(s) ds}{\Gamma(\alpha)p(0)} = \frac{\int_0^1 (1-s)^{\alpha-\beta-1} p(s) y(s) ds}{\Gamma(\alpha)p(0)}.$$

Therefore, the solution of (2.5) is

$$u(t) = c_1 t^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds = \int_0^1 G(t,s) y(s) ds. \quad \square$$

Lemma 2.4 *The function $G_0(t,s)$ has the following properties:*

- (1) $G_0(t,s) > 0$, for $t,s \in (0,1)$;
- (2) $\Gamma(\alpha)G_0(t,s) \leq t^{\alpha-1}$, for $t,s \in [0,1]$;
- (3) $\beta t^{\alpha-1}h(s) \leq \Gamma(\alpha)G_0(t,s) \leq h(s)t^{\alpha-2}$, for $t,s \in (0,1)$, where

$$h(s) = s(1-s)^{\alpha-\beta-1}. \quad (2.6)$$

Proof It is obvious that (1), (2) hold. In the following, we will prove (3).

(i) When $0 < s \leq t < 1$, noticing $0 < \beta < \alpha - 1 \leq 1$, we have

$$\begin{aligned} & \frac{\partial}{\partial \beta} \{ t^{\alpha-2} s(1-s)^{\alpha-\beta-1} - t^{\alpha-1} (1-s)^{\alpha-\beta-1} \} \\ & = t^{\alpha-2} (1-s)^{\alpha-\beta-1} (t-s) \ln(1-s) \leq 0. \end{aligned} \quad (2.7)$$

Therefore,

$$\begin{aligned} & t^{\alpha-2} s(1-s)^{\alpha-\beta-1} - (t^{\alpha-1} (1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}) \\ & \geq t^{\alpha-2} s - t^{\alpha-1} + (t-s)^{\alpha-1} \\ & = -t^{\alpha-2} (t-s) + (t-s)^{\alpha-1} \geq 0, \end{aligned}$$

which implies

$$\Gamma(\alpha)G_0(t,s) \leq h(s)t^{\alpha-2}. \quad (2.8)$$

On the other hand, we have

$$\frac{\partial}{\partial s} \{ \beta s + (1-s)^\beta \} \leq 0, \quad s \in [0,1].$$

Therefore, $\beta s + (1-s)^\beta \leq 1$, which implies

$$[1 - (1-s)^\beta] \geq \beta s.$$

Then

$$\begin{aligned} \Gamma(\alpha)G_0(t,s) & = t^{\alpha-1} (1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1} \\ & \geq t^{\alpha-1} (1-s)^{\alpha-\beta-1} - (t-s)^\beta (t-t) s^{\alpha-\beta-1} \\ & = \left[1 - \left(1 - \frac{s}{t} \right)^\beta \right] t^{\alpha-1} (1-s)^{\alpha-\beta-1} \\ & \geq [1 - (1-s)^\beta] t^{\alpha-1} (1-s)^{\alpha-\beta-1} \geq \beta t^{\alpha-1} h(s). \end{aligned} \quad (2.9)$$

(ii) When $0 < t \leq s < 1$, we have

$$\begin{aligned} \Gamma(\alpha)G_0(t,s) &= t^{\alpha-1}(1-s)^{\alpha-\beta-1} = t^{\alpha-2}t(1-s)^{\alpha-\beta-1} \\ &\leq t^{\alpha-2}s(1-s)^{\alpha-\beta-1} = h(s)t^{\alpha-2}. \end{aligned} \tag{2.10}$$

On the other hand, clearly we have

$$\Gamma(\alpha)G_0(t,s) = t^{\alpha-1}(1-s)^{\alpha-\beta-1} \geq \beta t^{\alpha-1}h(s). \tag{2.11}$$

(2.8)-(2.11) implies (3) holds. □

By Lemma 2.4 we have the following results.

Lemma 2.5 *Assume (H_1) holds, then the Green function defined by (2.3) satisfies:*

- (1) $G(t,s) > 0, \forall t,s \in (0,1)$;
- (2) $G(t,s) \leq t^{\alpha-1}(\frac{1}{\Gamma(\alpha)} + q(s)), \forall t,s \in [0,1]$;
- (3) $\beta t^{\alpha-1}\Phi(s) \leq G(t,s) \leq t^{\alpha-2}\Phi(s), \forall t,s \in (0,1)$, where

$$\Phi(s) = \frac{h(s)}{\Gamma(\alpha)} + q(s).$$

Lemma 2.6 *Assume (H_1) holds, then the function $G^*(t,s) =: t^{2-\alpha}G(t,s)$ satisfies:*

- (1) $G^*(t,s) > 0, \forall t,s \in (0,1)$;
- (2) $G^*(t,s) \leq t(\frac{1}{\Gamma(\alpha)} + q(s)), \forall t,s \in [0,1]$;
- (3) $\beta t\Phi(s) \leq G^*(t,s) \leq \Phi(s), \forall t,s \in [0,1]$.

For convenience, we list here two more assumptions to be used later:

(H₂) $f(t,u) = g(t,u,u)$, here $g \in C((0,1) \times [0,+\infty) \times (0,+\infty) \rightarrow [0,+\infty))$, $g(t,u,v)$ is non-decreasing on u , nonincreasing on v , and there exists $\mu \in (0,1)$ such that

$$g\left(t, ru, \frac{v}{r}\right) \geq r^\mu g(t,u,v), \quad \forall u,v > 0, r \in (0,1). \tag{2.12}$$

(H₃)

$$0 < \int_0^1 g(s, s^{\alpha-1}, s^{\alpha-1}) ds < +\infty. \tag{2.13}$$

Remark 2.2 Inequality (2.12) is equivalent to

$$g\left(t, \frac{u}{r}, rv\right) \leq r^{-\mu} g(t,u,v), \quad \forall u,v > 0, r \in (0,1). \tag{2.14}$$

Let $E = C[0,1]$ be endowed with the maximum norm $\|u\| = \max_{0 \leq t \leq 1} |u(t)|$. Define a cone P by

$$P = \{u \in E : \text{there exists } l_u > 0 \text{ such that } \beta \|u\|t \leq u(t) \leq l_u t\}.$$

Let

$$A(u, v)(t) = \int_0^1 G^*(t, s)g(s, s^{\alpha-2}u(s), s^{\alpha-2}v(s)) ds. \tag{2.15}$$

Set $Q = P \setminus \{\theta\}$, where θ is the zero element of E . We have the following lemma.

Lemma 2.7 *Suppose that (H_1) - (H_3) hold. Then $A : Q \times Q \rightarrow Q$.*

Proof For any $u, v \in Q$, there exists $l_1, l_2 > 0$, such that

$$\beta \|u\|t \leq u(t) \leq l_1 t, \quad \beta \|v\|t \leq v(t) \leq l_2 t.$$

By (H_2) , (H_3) and (2) of Lemma 2.6, we get

$$\begin{aligned} A(u, v)(t) &= \int_0^1 G^*(t, s)g(s, s^{\alpha-2}u(s), s^{\alpha-2}v(s)) ds \\ &\leq t \int_0^1 \left(\frac{1}{\Gamma(\alpha)} + q(s) \right) g(s, s^{\alpha-2}u(s), s^{\alpha-2}v(s)) ds \\ &\leq t \int_0^1 \left(\frac{1}{\Gamma(\alpha)} + q(s) \right) g(s, l_1 s^{\alpha-1}, \beta \|v\| s^{\alpha-1}) ds \\ &\leq t \int_0^1 \left(\frac{1}{\Gamma(\alpha)} + q(s) \right) g\left(s, (1 + l_1) s^{\alpha-1}, \frac{\beta \|v\|}{(1 + \beta)(1 + \|v\|)} s^{\alpha-1}\right) ds \\ &\leq L_1 t \int_0^1 \left(\frac{1}{\Gamma(\alpha)} + q(s) \right) g(s, s^{\alpha-1}, s^{\alpha-1}) ds < +\infty, \end{aligned} \tag{2.16}$$

where $L_1 = \max\{(1 + l_1)^\mu, (\frac{(1+\beta)(1+\|v\|)}{\beta\|v\|})^\mu\}$. This implies that A is well defined in $Q \times Q$.

On the other hand, by (3) of Lemma 2.6, we have

$$\begin{aligned} A(u, v)(t) &= \int_0^1 G^*(t, s)g(s, s^{\alpha-2}u(s), s^{\alpha-2}v(s)) ds \\ &\geq \beta t \int_0^1 \Phi(s)g(s, s^{\alpha-2}u(s), s^{\alpha-2}v(s)) ds, \\ A(u, v)(t) &= \int_0^1 G^*(t, s)g(s, s^{\alpha-2}u(s), s^{\alpha-2}v(s)) ds \\ &\leq \int_0^1 \Phi(s)g(s, s^{\alpha-2}u(s), s^{\alpha-2}v(s)) ds. \end{aligned}$$

Therefore, $A(u, v)(t) \geq \beta \|A(u, v)\|t$. Combining with (2.16), we have $A : Q \times Q \rightarrow Q$. □

Remark 2.3 By (H_2) and (2.15), A is a mixed monotone operator.

3 Main results

Theorem 3.1 *Suppose that (H_1) - (H_3) hold. Then the BVP (1.1) has a unique positive solution.*

Proof For any $r \in (0, 1)$, by Remark 2.2, we have

$$A\left(\frac{u}{r}, rv\right) \leq r^{-\mu} A(u, v), \quad \forall u, v \in Q.$$

For any $w \in Q$, noticing $A(w, w) \in Q$, we can choose $r_0 \in (0, 1)$ small enough such that

$$r_0^{1-\mu} w \leq A(w, w) \leq r_0^{-(1-\mu)} w. \tag{3.1}$$

Set

$$u_0 = r_0 w, \quad v_0 = r_0^{-1} w. \tag{3.2}$$

Clearly,

$$u_0, v_0 \in Q, \quad u_0 \leq v_0.$$

Let

$$u_n = A(u_{n-1}, v_{n-1}), \quad v_n = A(v_{n-1}, u_{n-1}), \quad n = 1, 2, \dots \tag{3.3}$$

It is easy to see that

$$u_0 \leq u_1 \leq \dots \leq u_n \leq \dots \leq v_n \leq \dots \leq v_1 \leq v_0. \tag{3.4}$$

Noticing

$$\begin{aligned} u_1 &= A(r_0 w, r_0^{-1} w) \geq r_0^\mu A(w, w), \\ v_1 &= A(r_0^{-1} w, r_0 w) \leq r_0^{-\mu} A(w, w), \end{aligned}$$

therefore,

$$u_1 \geq r_0^{2\mu} v_1.$$

Suppose that $u_n \geq r_0^{2\mu^n} v_n$, then $v_n \leq r_0^{-2\mu^n} u_n$, and

$$u_{n+1} = A(u_n, v_n) \geq A(r_0^{2\mu^n} v_n, r_0^{-2\mu^n} u_n) \geq r_0^{2\mu^{n+1}} A(v_n, u_n).$$

By induction, we can get

$$u_n \geq r_0^{2\mu^n} v_n, \quad n = 1, 2, \dots \tag{3.5}$$

By (3.4), (3.5), we have

$$0 \leq u_{n+m} - u_n \leq v_n - u_n \leq (1 - r_0^{2\mu^n}) v_n \leq (1 - r_0^{2\mu^n}) v_0,$$

which implies $\{u_n\}$ is a Cauchy sequence. Similarly, $\{v_n\}$ is a Cauchy sequence. Noticing (3.4), there exist $u^*, v^* \in Q$, such that $\{u_n\}$ converges to u^* and $\{v_n\}$ converges to v^* . Moreover,

$$u_n \leq u^* \leq v^* \leq v_n, \quad n = 1, 2, \dots \tag{3.6}$$

(3.5) and (3.6) imply that

$$\|v^* - u^*\| \leq \|v_n - u_n\| \leq (1 - r_0^\mu) \|v_0\|, \quad n = 1, 2, \dots \tag{3.7}$$

This implies that $u^* = v^*$.

By the mixed monotone property of A and (3.6), we have

$$A(u^*, v^*) \geq A(u_n, v_n) = u_{n+1}, \quad A(v^*, u^*) \leq A(v_n, u_n) = v_{n+1}, \quad n = 1, 2, \dots$$

Let $n \rightarrow +\infty$, we get

$$u^* \leq A(u^*, v^*) = A(v^*, u^*) \leq v^*.$$

Since $u^* = v^*$, we have u^* is a positive fixed point of A .

In the following, we will prove the positive fixed point of A is unique.

Suppose $u \neq u^*$ is a positive fixed point of A . By Lemma 2.6, we can get $u \in Q$. Let

$$r_1 = \sup\{r \in (0, 1) : ru^* \leq u \leq r^{-1}u^*\}.$$

Then $0 < r_1 < 1$, and $r_1u^* \leq u \leq r_1^{-1}u^*$. Therefore

$$\begin{aligned} u &= A(u, u) \geq A(r_1u^*, r_1^{-1}u^*) \geq r_1^\mu A(u^*, u^*) = r_1^\mu u^*, \\ u &= A(u, u) \leq A(r_1^{-1}u^*, r_1u^*) \leq r_1^{-\mu} A(u^*, u^*) = r_1^{-\mu} u^*. \end{aligned}$$

Thus, $r_1^\mu u^* \leq u \leq r_1^{-\mu} u^*$, which contradicts the definition of r_1 . Consequently, the positive fixed point of A is unique.

It is clear that $y(t) = t^{\alpha-2}u^*(t)$ satisfies

$$y(t) = \int_0^1 G(t, s)g(s, y(s), y(s)) \, ds = \int_0^1 G(t, s)f(s, y(s)) \, ds, \quad t \in (0, 1].$$

On the other hand, since $u^* \in Q$, we have $\beta \|u^*\| t \leq u^*(t) \leq l_{u^*} t$. Then, $\beta \|u^*\| t^{\alpha-1} \leq y(t) \leq l_{u^*} t^{\alpha-1}$. By Lemma 2.5 and $(H_2), (H_3)$, we can get $g(t, y(t), y(t)) \in L(0, 1)$. Moreover,

$$\begin{aligned} \lim_{t \rightarrow 0} y(t) &= \lim_{t \rightarrow 0} \int_0^1 G(t, s)g(s, y(s), y(s)) \, ds \\ &\leq t^{\alpha-1} \int_0^1 \left(\frac{1}{\Gamma(\alpha)} + q(s) \right) g(s, y(s), y(s)) \, ds \\ &= 0. \end{aligned}$$

Lemma 2.3 implies $y(t) = t^{\alpha-2}u^*(t)$ is a positive solution of (1.1).

On the other hand, if $y(t)$ is a positive solution of (1.1), then

$$y(t) = \int_0^1 G(t,s)g(s,y(s),y(s)) ds.$$

By Lemma 2.5, we have there exists $l_1, l_2 > 0$ such that

$$l_2 t^{\alpha-1} \leq y(t) \leq l_1 t^{\alpha-1}.$$

Set $u(t) = t^{2-\alpha}y(t)$, we have

$$l_2 t \leq u(t) \leq l_1 t,$$

and

$$u(t) = \int_0^1 G^*(t,s)g(s,s^{\alpha-2}u(s),s^{\alpha-2}v(s)) ds,$$

which implies u is a positive fixed point of A .

Then $y(t) = t^{\alpha-2}u^*(t)$ is the unique positive solution of the BVP (1.1). □

Remark 3.1 The unique positive solution y of (1.1) can be approximated by the iterative schemes: for any $w \in Q$, let u_0, v_0 be defined as (3.2) and $u_n = A(u_{n-1}, v_{n-1}), v_n = A(v_{n-1}, u_{n-1}), n = 1, 2, \dots$, then $t^{\alpha-2}u_n \rightarrow y$.

Example 3.1 (A 4-point BVP with coefficients of both signs) Consider the following problem:

$$\begin{cases} D_{0+}^{\frac{7}{4}}u(t) + f(t, u(t)) = 0, & t \in (0, 1), \\ u(0) = 0, & D_{0+}^{\frac{1}{4}}u(1) = D_{0+}^{\frac{1}{4}}u(\frac{1}{4}) - \frac{1}{2}D_{0+}^{\frac{1}{4}}u(\frac{4}{9}) \end{cases} \tag{3.8}$$

with

$$f(t, x) = -x^{\frac{1}{2}} \ln t - x^{-\frac{1}{3}} \ln(1-t) + x^{-\frac{1}{3}} \ln(1+x^{\frac{1}{2}}).$$

Then

$$G_0(t,s) = \frac{1}{\Gamma(\frac{7}{4})} \begin{cases} t^{\frac{3}{4}}(1-s)^{\frac{1}{2}}, & 0 \leq t \leq s \leq 1, \\ t^{\frac{3}{4}}(1-s)^{\frac{1}{2}} - (t-s)^{\frac{3}{4}}, & 0 \leq s \leq t \leq 1 \end{cases}$$

and

$$p(s) = \begin{cases} 1 - (\frac{1-s}{1-s})^{\frac{1}{2}} - \frac{1}{2}(\frac{4-s}{1-s})^{\frac{1}{2}}, & 0 \leq s \leq \frac{1}{4}, \\ 1 - \frac{1}{2}(\frac{4-s}{1-s})^{\frac{1}{2}}, & \frac{1}{4} < s \leq \frac{4}{9}, \\ 1, & \frac{4}{9} < s \leq 1. \end{cases}$$

By direct calculations, we have $p(0) = \frac{5}{6}$ and $q(s) \geq 0$, which implies (H_1) holds.

Let

$$g(t, x, y) = -x^{\frac{1}{2}} \ln t - y^{-\frac{1}{3}} \ln(1-t) + y^{-\frac{1}{3}} \ln(1+x^{\frac{1}{2}}).$$

Obviously, $g \in C((0, 1) \times [0, +\infty) \times (0, +\infty) \rightarrow [0, +\infty))$, $g(t, x, y)$ is nondecreasing on x and nonincreasing on y . It is easy to see that

$$\ln(1+rx) \geq r \ln(1+x), \quad \forall x \geq 0, r \in (0, 1). \quad (3.9)$$

Then

$$g\left(t, rx, \frac{y}{r}\right) \geq r^{\frac{5}{6}} g(t, x, y), \quad \forall x, y > 0, r \in (0, 1).$$

Therefore (H_2) holds. It is easy to get that (H_3) holds. Therefore, the assumptions of Theorem 3.1 are satisfied. Thus Theorem 3.1 ensures that the BVP (3.8) has a unique positive solution.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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