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Positive solutions of fractional differential equations at resonance on the half-line

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Abstract

This article deals with the differential equations of fractional order on the half-line. By the recent Leggett-Williams norm-type theorem due to O'Regan and Zima, we present some new results on the existence of positive solutions for the fractional boundary value problems at resonance on unbounded domains.

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1 Introduction

In this article, we are concerned with the fractional differential equation

$$\begin{cases} D_{0+}^{\alpha} u(t) = f(t, u(t)), & t \in [0, +\infty), \\ u(0) = u'(0) = u''(0) = 0, & D_{0+}^{\alpha-1} u(0) = \lim_{t \to +\infty} D_{0+}^{\alpha-1} u(t), \end{cases} \tag{1.1}$$

where D_{0+}^{α} is the Riemann-Liouville fractional derivative, $3 < \alpha < 4$, and $f : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ satisfies the following condition:

(H) $f : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous and for each $l > 0$, there exists $\phi_l \in C[0, +\infty) \cap L^1[0, +\infty)$ satisfying $\sup_{t \geq 0} |\phi_l(t)| < +\infty$ and $\phi_l(t) > 0$, $t > 0$ such that

$$|u| < l \text{ implies } \left| f\left(t, \left(1 + t^{\alpha-1}\right)u\right) \right| \leq \phi_l(t), \quad \text{a.e. } t \geq 0.$$

The problem (1.1) happens to be at resonance in the sense that the kernel of the linear operator D_{0+}^{α} is not less than one-dimensional under the boundary value conditions.

Fractional calculus is a generalization of the ordinary differentiation and integration. It has played a significant role in science, engineering, economy, and other fields. Some books on fractional calculus and fractional differential equations have appeared recently (see [1–3]); furthermore, today there is a large number of articles dealing with the fractional differential equations (see [4–15]) due to their various applications.

In [8], the researchers dealt with the existence of solutions for boundary value problems of fractional order of the form

$$^C D_{0+}^{\alpha} y(t) = f(t, y(t)), \quad t \in [0, +\infty),$$

$$y(0) = y_0, \quad y \text{ is bounded in } [0, +\infty),$$

where $1 < \alpha \le 2$ and $f : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous. The results are based on the fixed point theorem of Schauder combined with the diagonalization method.

In [9], Su and Zhang studied the following fractional differential equations on the half-line using Schauder's fixed point theorem

$$D_{0+}^{\alpha} u(t) = f\big(t, u(t), D_{0+}^{\alpha-1} u(t)\big), \quad t \in (0, +\infty), 1 < \alpha \le 2,$$

$$u(0) = 0, \qquad \lim_{t \to \infty} D_{0+}^{\alpha-1} u(t) = u_{\infty}.$$

Employing the Leray-Schauder alternative theorem, in [12], Zhao and Ge considered the fractional boundary value problem

$$D_{0+}^{\alpha} u(t) + f\big(t, u(t)\big) = 0, \quad t \in (0, +\infty), 1 < \alpha < 2,$$

$$u(0) = 0, \qquad \lim_{t \to \infty} D_{0+}^{\alpha-1} u(t) = \beta u(\xi).$$

However, the articles on the existence of solutions of fractional differential equations on the half-line are still few, and most of them deal with the problems under nonresonance conditions. And as far as we know, recent articles, such as [4, 6, 7], investigating resonant problems are on the finite interval.

Motivated by the articles [16–20], in this article we study the differential equations (1.1) under resonance conditions on the unbounded domains. Moreover, we have successfully established the existence theorem by the recent Leggett-Williams norm-type theorem due to O'Regan and Zima. To our best knowledge, there is no article dealing with the resonant problems of fractional order on unbounded domains by the theorem.

The rest of the article is organized as follows. In Section 2, we give the definitions of the fractional integral and fractional derivative, some results about fractional differential equations, and the abstract existence theorem. In Section 3, we obtain the existence result of the solution for the problem (1.1) by the recent Leggett-Williams norm-type theorem. Then, an example is given in Section 4 to demonstrate the application of our result.

2 Preliminaries

First of all, we present some fundamental facts on the fractional calculus theory which we will use in the next section.

Definition 2.1 ([1–3]) The Riemann-Liouville fractional integral of order $v > 0$ of a function $h : (0, \infty) \to \mathbb{R}$ is given by

$$I_{0+}^{v} h(t) = D_{0+}^{-v} h(t) = \frac{1}{\Gamma(v)} \int_{0}^{t} (t - s)^{v-1} h(s) \, ds, \tag{2.1}$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.2 ([1–3]) The Riemann-Liouville fractional derivative of order $v > 0$ of a continuous function $h : (0, \infty) \to \mathbb{R}$ is given by

$$D_{0+}^{v} h(t) = \frac{1}{\Gamma(n - v)} \left(\frac{d}{dt} \right)^{n} \int_{0}^{t} (t - s)^{n-v-1} h(s) \, ds, \tag{2.2}$$

where $n = [v] + 1$, provided that the right-hand side is pointwise defined on $(0, \infty)$.

Lemma 2.1 ([1, 9]) *Assume that $h(t) \in L^1(0, +\infty)$. If $v_1, v_2, v > 0$, then*

$$I_{0+}^{v_1} I_{0+}^{v_2} h(t) = I_{0+}^{v_1+v_2} h(t), \qquad D_{0+}^{v} I_{0+}^{v} h(t) = h(t). \tag{2.3}$$

Lemma 2.2 ([9]) *Assume that $D_{0+}^{v} h(t) \in L^1(0, +\infty)$, $v > 0$. Then we have*

$$I_{0+}^{v} D_{0+}^{v} h(t) = h(t) + C_1 t^{v-1} + C_2 t^{v-2} + \cdots + C_N t^{v-N}, \quad t > 0, \tag{2.4}$$

for some $C_i \in \mathbb{R}$, $i = 1, 2, \ldots, N$, where N is the smallest integer greater than or equal to v.

Now, let us recall some standard facts and the fixed point theorem due to O'Regan and Zima, and these can be found in [16, 17, 21–23].

Let X, Z be real Banach spaces. Consider an operation equation

$$Lu = Nu,$$

where $L : \operatorname{dom} L \subset X \to Z$ is a linear operator, $N : X \to Z$ is a nonlinear operator. If $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in Z, then L is called a Fredholm mapping of index zero. And if L is a Fredholm mapping of index zero, there exist linear continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $\operatorname{Ker} L = \operatorname{Im} P$, $\operatorname{Im} L = \operatorname{Ker} Q$ and $X = \operatorname{Ker} L \oplus \operatorname{Ker} P$, $Z = \operatorname{Im} L \oplus \operatorname{Im} Q$. Then it follows that $L_P = L|_{\operatorname{dom} L \cap \operatorname{Ker} P} : \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$ is invertible. We denote the inverse of this map by K_P. For $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$.

It is known that the coincidence equation $Lu = Nu$ is equivalent to

$$u = (P + JQN)u + K_P(I - Q)Nu.$$

A nonempty convex closed set $C \subset X$ is called a cone if
(i) $\kappa x \in C$ for all $x \in C$ and $\kappa \geq 0$;
(ii) $x, -x \in C$ implies $x = 0$.
Note that C induces a partial order $\preceq$ in X by

$$x \preceq y \quad \text{if and only if} \quad y - x \in C.$$

The following lemma is valid for every cone in a Banach space.

Lemma 2.3 ([17, 23]) *Let C be a cone in the Banach space X. Then for every $u \in C \setminus \{0\}$, there exists a positive number $\sigma(u)$ such that*

$$\|x + u\| \geq \sigma(u)\|x\|,$$

for all $x \in C$.

Let $\gamma : X \to C$ be a retraction, i.e., a continuous mapping such that $\gamma(x) = x$ for all $x \in C$. Denote

$$\Psi := P + JQN + K_P(I - Q)N,$$

and

$$\Psi_\gamma := \Psi \circ \gamma.$$

Theorem 2.1 ([16, 17]) *Let C be a cone in X and let Ω_1, Ω_2 be open bounded subsets of X with $\overline{\Omega}_1 \subset \Omega_2$ and $C \cap (\overline{\Omega}_2 \setminus \Omega_1) \neq \emptyset$. Assume that:*

1° *L is a Fredholm operator of index zero;*
2° *$QN : X \to Z$ is continuous and bounded and $K_P(I - Q)N : X \to X$ is compact on every bounded subset of X;*
3° *$Lu \neq \lambda Nu$ for all $u \in C \cap \partial\Omega_2 \cap \operatorname{dom} L$ and $\lambda \in (0,1)$;*
4° *γ maps subsets of $\overline{\Omega}_2$ into bounded subsets of C;*
5° *$d_B([I - (P + JQN)\gamma]|_{\operatorname{Ker}L}, \operatorname{Ker} L \cap \Omega_2, 0) \neq 0$, where d_B stands for the Brouwer degree;*
6° *there exists $u_0 \in C \setminus \{0\}$ such that $\|u\| \leq \sigma(u_0)\|\Psi u\|$ for $u \in C(u_0) \cap \partial\Omega_1$, where $C(u_0) = \{u \in C : \mu u_0 \preceq u \text{ for some } \mu > 0\}$ and $\sigma(u_0)$ is such that $\|u + u_0\| \geq \sigma(u_0)\|u\|$ for every $u \in C$;*
7° *$(P + JQN)\gamma(\partial\Omega_2) \subset C$;*
8° *$\Psi_\gamma(\overline{\Omega}_2 \setminus \Omega_1) \subset C$.*

Then the equation $Lx = Nx$ has a solution in the set $C \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Let

$$X = \left\{ x \Big| x \in C[0, +\infty), \lim_{t \to +\infty} \frac{x(t)}{1 + t^{\alpha-1}} \text{ exists} \right\}$$

with the norm

$$\|x\|_X = \sup_{t \geq 0} \frac{|x(t)|}{1 + t^{\alpha-1}},$$

and

$$Z = \left\{ z \Big| z \in C[0, +\infty) \cap L^1[0, +\infty), \sup_{t \geq 0} |z(t)| < +\infty \right\},$$

equipped with the norm

$$\|z\|_Z = \sup_{t \geq 0} |z(t)| + \int_0^{+\infty} |z(t)| \, dt.$$

Remark 2.1 It is easy for us to prove that $(X, \|\cdot\|_X)$ and $(Z, \|\cdot\|_Z)$ are Banach spaces.

Set

$$\operatorname{dom} L = \Big\{ u \in X \Big| D_{0+}^\alpha u(t) \in C[0, +\infty) \cap L^1[0, +\infty), u(0) = u'(0) = u''(0) = 0,$$

$$D_{0+}^{\alpha-1} u(0) = \lim_{t \to +\infty} D_{0+}^{\alpha-1} u(t) \Big\}.$$

Define

$$L : \operatorname{dom} L \to Z, \qquad u \to D_{0+}^\alpha u(t), \tag{2.5}$$

and

$$N : X \to Z, \qquad u \to f\big(t, u(t)\big). \tag{2.6}$$

Then the multi-point boundary value problem (1.1) can be written by

$$Lu = Nu, \qquad u \in \mathrm{dom}\, L.$$

Definition 2.3 $u \in X$ is called a solution of the problem (1.1) if $u \in \mathrm{dom}\, L$ and u satisfied Equation (1.1).

Next, similar to the compactness criterion in [12, 24], we establish the following criterion, and it can be proved in a similar way.

Lemma 2.4 $\mathcal{U}$ *is a relatively compact set in* X *if and only if the following conditions are satisfied:*

(a) $\mathcal{U}$ *is uniformly bounded, that is, there exists a constant* $R > 0$ *such that for each* $u \in \mathcal{U}$, $\|u\|_X \le R$.

(b) *The functions from* $\mathcal{U}$ *are equicontinuous on any compact subinterval of* $[0, +\infty)$, *that is, let* J *be a compact subinterval of* $[0, +\infty)$, *then* $\forall \varepsilon > 0$, *there exists* $\delta = \delta(\varepsilon) > 0$ *such that for* $t_1, t_2 \in J$, $|t_1 - t_2| < \delta$,

$$\left| \frac{u(t_1)}{1 + t_1^{\alpha-1}} - \frac{u(t_2)}{1 + t_2^{\alpha-1}} \right| < \varepsilon, \quad \forall u \in \mathcal{U}.$$

(c) *The functions from* $\mathcal{U}$ *are equiconvergent, that is, given* $\varepsilon > 0$, *there exists* $T = T(\varepsilon) > 0$ *such that*

$$\left| \frac{u(s_1)}{1 + s_1^{\alpha-1}} - \frac{u(s_2)}{1 + s_2^{\alpha-1}} \right| < \varepsilon,$$

for $s_1, s_2 > T$, $\forall u \in \mathcal{U}$.

3 Main results

In this section, we will present the existence theorem for the fractional differential equation on the half-line. In order to prove our main result, we need the following lemmas.

Lemma 3.1 *Let* $g \in Z$. *Then* $u \in X$ *is the solution of the following fractional differential equation:*

$$\begin{cases} D_{0+}^{\alpha} u(t) = g(t), & t \in [0, +\infty), \\ u(0) = u'(0) = u''(0) = 0, & D_{0+}^{\alpha-1} u(0) = \lim_{t \to +\infty} D_{0+}^{\alpha-1} u(t), \end{cases}$$

if and only if

$$u(t) = c t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} g(s)\, ds, \quad c \in \mathbb{R},$$

and

$$\int_0^{+\infty} g(t)\,dt = 0.$$

Proof In view of Lemmas 2.1 and 2.2, we can certify the conclusion easily, so we omit the details here. □

Lemma 3.2 *The operator L is a Fredholm mapping of index zero. Moreover,*

$$\operatorname{Ker} L = \{u | u = ct^{\alpha-1}, t \geq 0, c \in \mathbb{R}\} \subset X, \tag{3.1}$$

and

$$\operatorname{Im} L = \left\{ g \in Z \Big| \int_0^{+\infty} g(t)\,dt = 0 \right\} \subset Z. \tag{3.2}$$

Proof It is obvious that Lemma 3.1 implies (3.1) and (3.2). Now, let us focus our minds on proving that L is a Fredholm mapping of index zero.

Define $Q : Z \to Z$

$$(Qg)(t) = e^{-t} \int_0^{+\infty} g(s)\,ds, \quad t \geq 0, \tag{3.3}$$

where $g \in Z$. Evidently, $\operatorname{Ker} Q = \operatorname{Im} L$, $\operatorname{Im} Q = \{g | g = ce^{-t}, t \geq 0, c \in \mathbb{R}\}$, and $Q : Z \to Z$ is a continuous linear projector. In fact, for an arbitrary $g \in Z$, we have

$$Q^2 g = Q(Qg) = Q\left(e^{-t}\int_0^{+\infty} g(s)\,ds\right) = Q(e^{-t})\int_0^{+\infty} g(s)\,ds = e^{-t}\int_0^{+\infty} g(s)\,ds = Qg,$$

that is to say, $Q : Z \to Z$ is idempotent.

Let $g = g - Qg + Qg = (I - Q)g + Qg$, where $g \in Z$ is an arbitrary element. Since $Qg \in \operatorname{Im} Q$ and $(I - Q)g \in \operatorname{Ker} Q$, we obtain that $Z = \operatorname{Im} Q + \operatorname{Ker} Q$. Take $z_0 \in \operatorname{Im} Q \cap \operatorname{Ker} Q$, then z_0 can be written as $z_0 = ce^{-t}$, $c \in \mathbb{R}$, for $z_0 \in \operatorname{Im} Q$. Since $z_0 \in \operatorname{Ker} Q = \operatorname{Im} L$, by (3.2), we get that $Q(z_0) = Q(ce^{-t}) = cQ(e^{-t}) = ce^{-t} = 0$, which implies that $c = 0$, and then $z_0 = 0$. Therefore, $\operatorname{Im} Q \cap \operatorname{Ker} Q = \{0\}$, thus, $Z = \operatorname{Im} Q \oplus \operatorname{Ker} Q = \operatorname{Im} Q \oplus \operatorname{Im} L$.

Now, $\dim \operatorname{Ker} L = 1 = \dim \operatorname{Im} Q = \operatorname{codim} \operatorname{Ker} Q = \operatorname{codim} \operatorname{Im} L < +\infty$, and observing that $\operatorname{Im} L$ is closed in Z, so L is a Fredholm mapping of index zero. □

Let $P : X \to X$ be defined by

$$(Pu)(t) = \left(\frac{1}{\Gamma(\alpha)} \int_0^{+\infty} e^{-s} u(s)\,ds\right) t^{\alpha-1}, \quad t \geq 0, u \in X. \tag{3.4}$$

It is clear that $P : X \to X$ is a linear continuous projector and

$$\operatorname{Im} P = \{u | u = ct^{\alpha-1}, t \geq 0, c \in \mathbb{R}\} = \operatorname{Ker} L.$$

Also, proceeding with the proof of Lemma 3.2, we can show that $X = \operatorname{Im} P \oplus \operatorname{Ker} P = \operatorname{Ker} L \oplus \operatorname{Ker} P$.

Consider the mapping $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$

$$(K_P g)(t) = -\left(\frac{1}{\Gamma(\alpha)} \int_0^{+\infty} e^{-s} g(s)\, ds\right) t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\, ds, \quad g \in \operatorname{Im} L.$$

Note that

$$(K_P L)u = K_P(Lu) = u, \quad \forall u \in \operatorname{dom} L \cap \operatorname{Ker} P,$$

and

$$(LK_P)g = L(K_P g) = g, \quad \forall g \in \operatorname{Im} L.$$

Thus, $K_P = (L_P)^{-1}$, where $L_P = L|_{\operatorname{dom} L \cap \operatorname{Ker} P} : \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} P$.

Define the linear isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$ as

$$J\left(ce^{-t}\right) = ct^{\alpha-1}, \quad t \geq 0, c \in \mathbb{R}.$$

Thus, $JQN + K_P(I - Q)N : X \to X$ is given by

$$\left[JQN + K_P(I - Q)N\right]u(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^{+\infty} G(t,s) f\left(s, u(s)\right) ds, \quad t \geq 0, \tag{3.5}$$

where

$$G(t,s) = \begin{cases} 0, & t = 0; \\ \Gamma(\alpha) + \dfrac{1}{2} - e^{-s} - \displaystyle\int_0^t \frac{(t-\tau)^{\alpha-1}}{t^{\alpha-1}} e^{-\tau}\, d\tau + \frac{(t-s)^{\alpha-1}}{t^{\alpha-1}}, \\ \quad t \neq 0 \text{ and } 0 \leq s \leq t < +\infty; \\ \Gamma(\alpha) + \dfrac{1}{2} - e^{-s} - \displaystyle\int_0^t \frac{(t-\tau)^{\alpha-1}}{t^{\alpha-1}} e^{-\tau}\, d\tau, & 0 < t \leq s < +\infty. \end{cases}$$

Then, it is easy to verify that

$$0 < \Gamma(\alpha) - \frac{1}{2} \leq G(t,s) \leq \Gamma(\alpha) + \frac{3}{2}. \tag{3.6}$$

Now, we state the main result on the existence of the positive solutions to the problem (1.1) in the following.

Theorem 3.1 *Let $f : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ satisfy the condition (H). Assume that there exist six nonnegative functions $\alpha_i(t)$ ($i = 1, 2, 3$), $\beta_j(t)$ ($j = 1, 2$) and $\mu(t)$ such that*

$$f(t, u) \leq -\alpha_1(t)\left|f(t, u)\right| + \alpha_2(t)\frac{u}{1 + t^{\alpha-1}} + \alpha_3(t), \quad t \geq 0, \tag{3.7}$$

and

$$-\mu(t)\frac{u}{1 + t^{\alpha-1}} \leq f(t, u) \leq -\beta_1(t)\frac{u}{1 + t^{\alpha-1}} + \beta_2(t), \quad t \geq 0, \tag{3.8}$$

where $0 \le \frac{u}{1+t^{\alpha-1}} \le R$, $R > R_0$, *and* R_0 *is defined by* (3.12), $\alpha_1(t)$ *is bounded on* $[0, +\infty)$, $\beta_1(t) > 0$, $t \ge 0$, $\alpha_2(t), \alpha_3(t), \beta_1(t), \beta_2(t) \in L^1[0, +\infty)$,

$$\alpha_0 := \inf_{t \ge 0} \alpha_1(t) > 0, \qquad \int_0^{+\infty} \alpha_3(t)\,dt > 0, \qquad \int_0^{+\infty} \beta_2(t)\,dt > 0, \tag{3.9}$$

$$\Gamma_0 := \frac{2}{\alpha_0} \sup_{t \ge 0} \frac{\alpha_2(t) + \alpha_0 e^{-t}(1 + t^{\alpha-1})/2}{\beta_1(t)} < +\infty, \tag{3.10}$$

and

$$\int_0^{+\infty} \mu(t)\,dt < \frac{\Gamma(\alpha) + 1}{2(\Gamma(\alpha) + 1/2)(\Gamma(\alpha) + 3/2)}, \qquad e^t \mu(t) < \frac{1 + t^{\alpha-1}}{\Gamma(\alpha) + 3/2}. \tag{3.11}$$

Then the problem (1.1) *has at least one positive solution in* $\operatorname{dom} L$.

Proof For the simplicity of notation, we denote

$$\varepsilon_1 := \frac{\Gamma(\alpha)}{\Gamma(\alpha) + 1/2} + \frac{\Gamma(\alpha) + 3/2}{\Gamma(\alpha) + 1} \int_0^{+\infty} \mu(s)\,ds < 1, \qquad \beta_0 := \int_0^{+\infty} \frac{s^{\alpha-1}\beta_1(s)}{1 + s^{\alpha-1}}\,ds,$$

and

$$R_0 := \max\left\{ \frac{\Gamma_0}{\Gamma(\alpha)} \int_0^{+\infty} \beta_2(s)\,ds + \frac{2}{\alpha_0\Gamma(\alpha)} \int_0^{+\infty} \alpha_3(s)\,ds, \frac{1}{\beta_0} \int_0^{+\infty} \beta_2(s)\,ds \right\}. \tag{3.12}$$

Consider the cone

$$C = \{u \mid u \in X, u(t) \ge 0, t \ge 0\}.$$

Set

$$\Omega_1 = \left\{ u \in X \,\middle|\, \varepsilon_0 \|u\|_X < \frac{u(t)}{1 + t^{\alpha-1}} < R_1, t \ge 0 \right\}, \qquad \Omega_2 = \left\{ u \in X \mid \|u\|_X < R_2, t \ge 0 \right\},$$

where $R_2 \in (R_0, R), R_1 \in (0, R_2), \varepsilon_0 \in (\varepsilon_1, 1)$. Clearly, Ω_1 and Ω_2 are an open bounded set of X.

Step 1: In view of Lemma 3.2, the condition 1° of Theorem 2.1 is fulfilled.

Step 2: By virtue of Lemma 2.4, we can get that $QN : X \to Z$ is continuous and bounded and $K_P(I - Q)N : X \to X$ is compact on every bounded subset of X, which ensures that the assumption 2° of Theorem 2.1 holds.

Step 3: Suppose that there exist $u^* \in C \cap \partial\Omega_2 \cap \operatorname{dom} L$ and $\lambda^* \in (0, 1)$ such that $Lu^* = \lambda^* Nu^*$.

Since

$$u^* = (I - P)u^* + Pu^* = K_P L(I - P)u^* + Pu^* = K_P Lu^* + Pu^*,$$

we have

$$\frac{u^*(t)}{1 + t^{\alpha-1}} = -\frac{1}{\Gamma(\alpha)} \int_0^{+\infty} e^{-s} D_{0+}^\alpha u^*(s)\,ds \cdot \frac{t^{\alpha-1}}{1 + t^{\alpha-1}} + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{1 + t^{\alpha-1}} D_{0+}^\alpha u^*(s)\,ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} e^{-s} u^*(s) \, ds \cdot \frac{t^{\alpha-1}}{1 + t^{\alpha-1}} \tag{3.13}$$

$$< \frac{2}{\Gamma(\alpha)} \int_0^{+\infty} \left| D_{0+}^\alpha u^*(s) \right| ds + \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} e^{-s} u^*(s) \, ds.$$

From (3.7) and (3.8), we get that

$$D_{0+}^\alpha u^*(t) = \lambda^* f(t, u^*(t)) \le -\lambda^* \alpha_1(t) \left| f(t, u^*(t)) \right| + \lambda^* \alpha_2(t) \frac{u^*(t)}{1 + t^{\alpha-1}} + \lambda^* \alpha_3(t)$$

$$\le -\alpha_1(t) \left| D_{0+}^\alpha u^*(t) \right| + \alpha_2(t) \frac{u^*(t)}{1 + t^{\alpha-1}} + \alpha_3(t), \tag{3.14}$$

and

$$D_{0+}^\alpha u^*(t) = \lambda^* f(t, u^*(t)) \le -\lambda^* \beta_1(t) \frac{u^*(t)}{1 + t^{\alpha-1}} + \lambda^* \beta_2(t). \tag{3.15}$$

On account of the fact that

$$\int_0^{+\infty} D_{0+}^\alpha u^*(s) \, ds = \int_0^{+\infty} D\left(D_{0+}^{\alpha-1} u^*(s) \right) ds = \lim_{t \to +\infty} D_{0+}^{\alpha-1} u^*(t) - D_{0+}^{\alpha-1} u^*(0) = 0,$$

and considering (3.14) and (3.15), we have

$$0 = \int_0^{+\infty} D_{0+}^\alpha u^*(s) \, ds$$

$$\le - \int_0^{+\infty} \alpha_1(s) \left| D_{0+}^\alpha u^*(s) \right| ds + \int_0^{+\infty} \alpha_2(s) \frac{u^*(s)}{1 + s^{\alpha-1}} \, ds + \int_0^{+\infty} \alpha_3(s) \, ds,$$

and

$$0 = \int_0^{+\infty} D_{0+}^\alpha u^*(s) \, ds \le - \int_0^{+\infty} \lambda^* \beta_1(s) \frac{u^*(s)}{1 + s^{\alpha-1}} \, ds + \int_0^{+\infty} \lambda^* \beta_2(s) \, ds.$$

Thus,

$$\int_0^{+\infty} \left| D_{0+}^\alpha u^*(s) \right| ds \le \frac{1}{\alpha_0} \int_0^{+\infty} \alpha_2(s) \frac{u^*(s)}{1 + s^{\alpha-1}} \, ds + \frac{1}{\alpha_0} \int_0^{+\infty} \alpha_3(s) \, ds,$$

and

$$\int_0^{+\infty} \beta_1(s) \frac{u^*(s)}{1 + s^{\alpha-1}} \, ds \le \int_0^{+\infty} \beta_2(s) \, ds.$$

By (3.9), (3.10) and (3.13), we obtain that

$$\frac{u^*(t)}{1 + t^{\alpha-1}}$$

$$< \frac{2}{\Gamma(\alpha)} \int_0^{+\infty} \left| D_{0+}^\alpha u^*(s) \right| ds + \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} e^{-s} u^*(s) \, ds$$

$$\le \frac{2}{\alpha_0 \Gamma(\alpha)} \int_0^{+\infty} \alpha_2(s) \frac{u^*(s)}{1 + s^{\alpha-1}} \, ds + \frac{2}{\alpha_0 \Gamma(\alpha)} \int_0^{+\infty} \alpha_3(s) \, ds + \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} e^{-s} u^*(s) \, ds$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} \left(\frac{2}{\alpha_0} \frac{\alpha_2(s) + \alpha_0 e^{-s}(1 + s^{\alpha-1})/2}{\beta_1(s)} \right) \beta_1(s) \frac{u^*(s)}{1 + s^{\alpha-1}} \, ds + \frac{2}{\alpha_0 \Gamma(\alpha)} \int_0^{+\infty} \alpha_3(s) \, ds$$

$$\le \frac{\Gamma_0}{\Gamma(\alpha)} \int_0^{+\infty} \beta_1(s) \frac{u^*(s)}{1 + s^{\alpha-1}} \, ds + \frac{2}{\alpha_0 \Gamma(\alpha)} \int_0^{+\infty} \alpha_3(s) \, ds$$

$$\le \frac{\Gamma_0}{\Gamma(\alpha)} \int_0^{+\infty} \beta_2(s) \, ds + \frac{2}{\alpha_0 \Gamma(\alpha)} \int_0^{+\infty} \alpha_3(s) \, ds \le R_0 < R_2,$$

which is a contradiction to $u^* \in C \cap \partial\Omega_2 \cap \operatorname{dom} L$. Therefore, $3°$ is satisfied.

Step 4: Let $(\gamma u)(t) = |u(t)|$, then we can verify that $\gamma : X \to C$ is a retraction and $4°$ holds.

Step 5: Let $u \in \operatorname{Ker} L \cap \overline{\Omega}_2$, then $u(t) = ct^{\alpha-1}$, $t \ge 0$, $c \in \mathbb{R}$. Inspired by Aijun and Wang [5], we set

$$H(ct^{\alpha-1}, \rho) = [I - \rho(P + JQN)\gamma](ct^{\alpha-1}) = \left(c - \rho|c| - \rho \int_0^{+\infty} f(s, |c|s^{\alpha-1}) \, ds \right) t^{\alpha-1},$$

where $c \in [-R_2, R_2]$ and $\rho \in [0, 1]$.

Define homeomorphism $J_1 : \operatorname{Ker} L \cap \overline{\Omega}_2 \to \mathbb{R}$ by $J_1(ct^{\alpha-1}) = c$, then

$$d_B(H(ct^{\alpha-1}, \rho), \operatorname{Ker} L \cap \Omega_2, 0) = d_B(J_1 H(J_1^{-1}c, \rho), J_1(\operatorname{Ker} L \cap \Omega_2), J_1(0))$$

$$= d_B(J_1 H(J_1^{-1}c, \rho), J_1(\operatorname{Ker} L \cap \Omega_2), 0).$$

It is obvious that $J_1 H(J_1^{-1}c, \rho) = 0$ implies that $c \ge 0$ by (3.8) and (3.11).

Take $c_0 \in J_1(\operatorname{Ker} L \cap \partial\Omega_2)$, then $|c_0| = R_2$. Suppose that $J_1 H(J_1^{-1}c, \rho) = 0$, $\rho \in (0, 1]$, then we have that $c_0 = R_2$. Also, in view of (3.8),

$$R_2 = \rho \left(R_2 - \int_0^{+\infty} f(s, R_2 s^{\alpha-1}) \, ds \right)$$

$$\le \rho \left(R_2 + R_2 \int_0^{+\infty} \beta_1(s) \frac{s^{\alpha-1}}{1 + s^{\alpha-1}} \, ds + \int_0^{+\infty} \beta_2(s) \, ds \right)$$

$$< \rho R_2 \le R_2.$$

It is a contradiction. Besides, if $\rho = 0$, then $R_2 = 0$, which is impossible. Hence, for $c \in J_1(\operatorname{Ker} L \cap \partial\Omega_2)$, $J_1 H(J_1^{-1}c, \rho) \ne 0$, $\rho \in [0, 1]$.

Therefore,

$$d_B([I - (P + JQN)\gamma]|_{\operatorname{Ker} L}, \operatorname{Ker} L \cap \Omega_2, 0) = d_B(H(\cdot, 1), \operatorname{Ker} L \cap \Omega_2, 0)$$

$$= d_B(J_1 H(J_1^{-1}c, 1), J_1(\operatorname{Ker} L \cap \Omega_2), 0)$$

$$= d_B(J_1 H(J_1^{-1}c, 0), J_1(\operatorname{Ker} L \cap \Omega_2), 0)$$

$$= d_B(I, J_1(\operatorname{Ker} L \cap \Omega_2), 0) = 1 \ne 0,$$

which shows that $5°$ is true.

Step 6: Let $u_0 = 1 + t^{\alpha-1} \in C \setminus \{0\}$, then we have

$$C(u_0) = \left\{ u \in C \,\bigg|\, \inf_{t \ge 0} \frac{u(t)}{1 + t^{\alpha-1}} > 0 \right\}.$$

And we can take $\sigma(u_0) = 1$.

Let $t_0 > 0$ such that

$$\frac{t_0^{\alpha-1}}{1 + t_0^{\alpha-1}} > \frac{\Gamma(\alpha) + 1/2}{\Gamma(\alpha) + 1}.$$

For $u \in C(u_0) \cap \partial\Omega_1$, we have that

$$\|u\|_X \le R_1 < R_2, \qquad \frac{u(t)}{1 + t^{\alpha-1}} \ge \varepsilon_0 \|u\|_X.$$

Therefore, combining (3.6), (3.8) and (3.11), we get that

$$
\begin{aligned}
&\frac{(\Psi u)(t_0)}{1 + t_0^{\alpha-1}} \\
&\quad = \frac{1}{1 + t_0^{\alpha-1}} \big(Pu(t_0) + [JQN + K_P(I - Q)N]u(t_0) \big) \\
&\quad = \frac{t_0^{\alpha-1}}{1 + t_0^{\alpha-1}} \left(\frac{1}{\Gamma(\alpha)} \int_0^{+\infty} e^{-s} u(s)\, ds + \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} G(t_0, s) f(s, u(s))\, ds \right) \\
&\quad > \frac{\Gamma(\alpha) + 1/2}{\Gamma(\alpha) + 1} \left(\varepsilon_0 \|u\|_X \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} e^{-s}\big(1 + s^{\alpha-1}\big)\, ds - \frac{\|u\|_X}{\Gamma(\alpha)} \int_0^{+\infty} G(t_0, s)\mu(s)\, ds \right) \\
&\quad \ge \|u\|_X \frac{\Gamma(\alpha) + 1/2}{\Gamma(\alpha) + 1} \left(\varepsilon_0 \frac{\Gamma(\alpha) + 1}{\Gamma(\alpha)} - \frac{\Gamma(\alpha) + 3/2}{\Gamma(\alpha)} \int_0^{+\infty} \mu(s)\, ds \right) \\
&\quad = \|u\|_X.
\end{aligned}
$$

Thus, $\|u\|_X \le \sigma(u_0)\|\Psi u\|_X$ for all $u \in C(u_0) \cap \partial\Omega_1$. So, $6°$ holds.

Step 7: For $u \in \partial\Omega_2$, from (3.8) and (3.11), we have

$$
\begin{aligned}
(P + JQN)(\gamma u)(t) &= \left(\frac{1}{\Gamma(\alpha)} \int_0^{+\infty} e^{-s} |u(s)|\, ds + \int_0^{+\infty} f(s, |u(s)|)\, ds \right) t^{\alpha-1} \\
&\ge \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^{+\infty} \big[e^{-s}\big(1 + s^{\alpha-1}\big) - \mu(s) \big] \frac{|u(s)|}{1 + s^{\alpha-1}}\, ds \ge 0,
\end{aligned}
$$

which implies that $(P + JQN)\gamma(\partial\Omega_2) \subset C$. Hence, $7°$ holds.

Step 8: For $u \in \overline{\Omega}_2 \setminus \Omega_1$, by (3.6), (3.8) and (3.11), we obtain that

$$
\begin{aligned}
\Psi_\gamma u(t) &= \big[P + JQN + K_P(I - Q)N \big] |u(t)| \\
&= \left(\frac{1}{\Gamma(\alpha)} \int_0^{+\infty} e^{-s} |u(s)|\, ds + \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} G(t, s) f(s, |u(s)|)\, ds \right) t^{\alpha-1} \\
&\ge \frac{t^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^{+\infty} \big[e^{-s}\big(1 + s^{\alpha-1}\big) - G(t, s)\mu(s) \big] \frac{|u(s)|}{1 + s^{\alpha-1}}\, ds \right) \\
&\ge \frac{t^{\alpha-1}}{\Gamma(\alpha)} \left(\int_0^{+\infty} \left[e^{-s}\big(1 + s^{\alpha-1}\big) - \left(\Gamma(\alpha) + \frac{3}{2} \right) \mu(s) \right] \frac{|u(s)|}{1 + s^{\alpha-1}}\, ds \right) \\
&\ge 0.
\end{aligned}
$$

Thus, $\Psi_\gamma(\overline{\Omega}_2 \setminus \Omega_1) \subset C$, that is, $8°$ is satisfied.

Hence, applying Theorem 2.1, the problem (1.1) has a positive solution in the set $C \cap (\overline{\Omega}_2 \setminus \Omega_1)$. $\qquad\square$

4 Examples

To illustrate our main result, we will present an example.

Example 4.1

$$
\begin{cases}
D_{0+}^{\alpha} u(t) = f\big(t, u(t)\big), & t \in [0, +\infty), \\
u(0) = u'(0) = u''(0) = 0, & D_{0+}^{\alpha-1} u(0) = \lim_{t \to +\infty} D_{0+}^{\alpha-1} u(t),
\end{cases}
\tag{4.1}
$$

where $\alpha = 3.5$, and for $(t, u) \in \mathbb{R}^2$,

$$
f(t, u) = -\beta_1(t) \frac{u}{1 + t^{\alpha-1}} + \beta_2(t),
$$

and

$$
\beta_1(t) = \frac{1}{40} e^{-t} \big(1 + t^{\alpha-1}\big), \qquad \beta_2(t) = \frac{1}{1 + t^2}.
$$

It is easy for us to certify that f satisfies the condition (H).

Noting that

$$
f(t, u) \leq -\alpha_1(t) \big| f(t, u) \big| + \alpha_2(t) \frac{u}{1 + t^{\alpha-1}} + \alpha_3(t), \quad t \geq 0,
$$

and

$$
-\mu(t) \frac{u}{1 + t^{\alpha-1}} \leq f(t, u) \leq -\beta_1(t) \frac{u}{1 + t^{\alpha-1}} + \beta_2(t), \quad t \geq 0,
$$

for $u \geq 0$, where

$$
\alpha_1(t) = 2, \qquad \alpha_2(t) = \beta_1(t), \qquad \alpha_3(t) = 3\beta_2(t), \qquad \mu(t) = \beta_1(t).
$$

Evidently, $\mu(t)$ satisfies (3.11).

Meanwhile, by simple computation we can get that

$$
\alpha_0 = 2, \qquad \int_0^{+\infty} \alpha_3(t)\, dt = \frac{3\pi}{2}, \qquad \int_0^{+\infty} \beta_2(t)\, dt = \frac{\pi}{2}, \qquad \Gamma_0 = 41.
$$

Thus, to sum up the points which we have just indicated, by Theorem 3.1, we can conclude that the problem (4.1) has at least one positive solution.

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