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Multi-point boundary value problem for first order impulsive integro-differential equations with multi-point jump conditions

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Abstract

In this article we introduce a new definition of impulsive conditions for boundary value problems of first order impulsive integro-differential equations with multi-point boundary conditions. By using the method of lower and upper solutions in reversed order coupled with the monotone iterative technique, we obtain the extremal solutions of the boundary value problem. An example is also discussed to illustrate our results.

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1 Introduction

Impulsive differential equations describe processes which have a sudden change of their state at certain moments. Impulse effects are important in many real world applications, such as physics, medicine, biology, control theory, population dynamics, etc. (see, for example [1-3]). In this article, we consider the following boundary value problem for first order impulsive integro-differential equations (BVP):

$$\begin{cases} x'(t) = f(t, x(t), (Fx)(t), (Sx)(t)), & t \in J = [0, T], \quad t \neq t_k, \\ \Delta x(t_k) = I_k\left(\sum_{l=1}^{c_k} \rho_l^k x\left(\eta_l^k\right)\right), & k = 1, 2, \ldots m, \\ x(0) + \mu \sum_{k=1}^{m} \sum_{l=1}^{c_k} \tau_l^k x\left(\eta_l^k\right) = x(T), \end{cases} \tag{1.1}$$

where $f \in C(J \times R^3, R)$, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$,

$$(Fx)(t) = \int_0^t k(t,s)\,x(s)\,ds, \quad (Sx)(t) = \int_0^T h(t,s)\,x(s)\,ds,$$

$k \in C(D, R^+)$, $D = \{(t, s) \in J \times J: t \geq s\}$, $h \in C(J \times J, R^+)$. $I_k \in C(R, R)$, $\Delta x(t_k) = x\left(t_k^+\right) - x\left(t_k^-\right)$, $t_{k-1} < \eta_1^k < \eta_2^k < \ldots < \eta_{c_k}^k \leq t_k$, $\tau_l^k, \rho_l^k \geq 0$, $l = 1, 2, \ldots, c_k$, $c_k \in N = \{1, 2, \ldots\}$, $k = 1, 2, \ldots, m$, $\mu \geq 0$.

The monotone iterative technique coupled with the method of lower and upper solutions is a powerful method used to approximate solutions of several nonlinear problems (see [4-14]). Boundary value problems for first order impulsive functional differential equations with lower and upper solutions in reversed order have been widely discussed in recent years (see [15-20]). However, the discussion of multi-point boundary value problems for first order impulsive functional differential equations is very limited (see [21]). In all articles concerned with applications of the monotone iterative technique to impulsive problems, the authors have assumed that $\Delta x(t_k) = I_k\,(x(t_k))$, that is a short-term rapid change of the state at impulse point t_k depends on the left side of the limit of $x(t_k)$.

Recently, Tariboon [22] and Liu et al. [23] studied some types of impulsive boundary value problems with the impulsive integral conditions

$$\Delta x\,(t_k) = I_k\left(\int\limits_{t_k-\tau_k}^{t_k} x\,(s)\,ds - \int\limits_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}} x\,(s)\,ds\right), \quad k = 1, 2, \ldots, m. \tag{1.2}$$

It should be noticed that the terms $\int_{t_k-\tau_k}^{t_k} x\,(s)\,ds$ and $\int_{t_{k-1}}^{t_{k-1}+\sigma_{k-1}} x\,(s)\,ds$ of impulsive condition (1.2) illustrate the past memory state on $[t_k - \tau_k, t_k]$ before impulse points t_k and the history effects after the past impulse points t_{k-1} on $(t_{k-1}, t_{k-1} + \sigma_{k-1}]$, respectively.

The aim of the present article is to discuss the new impulsive multi-point condition

$$\Delta x\,(t_k) = I_k\left(\sum_{l=1}^{c_k} \rho_l^k x\left(\eta_l^k\right)\right) = I_k\left(\rho_1^k x\left(\eta_1^k\right) + \cdots + \rho_l^k x\left(\eta_l^k\right) + \cdots + \rho_{c_k}^k x\left(\eta_{c_k}^k\right)\right), \tag{1.3}$$

for $t_{k-1} < \eta_1^k < \eta_2^k < \cdots < \eta_{c_k}^k \le t_k$, $k = 1, 2, \ldots, m$. The new jump conditions mean that a sudden change of the state at impulse point t_k depends on the multi-point $\eta_l^k\ (l = 1, 2, \ldots, c_k)$ of past states on $(t_{k-1}, t_k]$. We note that if $c_k = 1$, $\eta_{c_k}^k = t_k$ and $\rho_{c_k}^k = 1$, then the impulsive condition (1.3) is reduced to the simple impulsive condition $\Delta x(t_k) = I_k\,(x(t_k))$.

Firstly, we introduce the definitions of lower and upper solutions and formulate some lemmas which are used in our discussion. In the main results, we obtain the existence of extreme solutions for BVP (1.1) by using the method of lower and upper solutions in reversed order and the monotone iterative technique. Finally, we give an example to illustrate the obtained results.

2 Preliminaries

Let $J^- = J \setminus \{t_1, t_2, ..., t_m\}$. $PC(J, R) = \{x\colon J \to R;\ x(t)$ is continuous everywhere except for some t_k at which $x\left(t_k^-\right)$ and $x\left(t_k^+\right)$ exist and $x\left(t_k^-\right) = x\,(t_k)$, $k = 1, 2, ..., m\}$, $PC^1(J, R) = \{x \in PC(J, R);\ x'(t)$ is continuous everywhere except for some t_k at which $x'\left(t_k^+\right)$ and $x'\left(t_k^-\right)$ exist and $x'\left(t_k^-\right) = x'\,(t_k)\ \}$. Let $E = PC(J, R)$ and $\mathcal{F} = PC^1\,(J, R)$, then E and $\mathcal{F}$ are Banach spaces with the nomes $||x||_E = \sup_{t\in J} |x(t)|$ and $\|x\|_{\mathcal{F}} = \max\left\{\|x\|_E,\ \left\|x'\right\|_E\right\}$, respectively. A function $x \in \mathcal{F}$ is called a solution of BVP (1.1) if it satisfies (1.1).

Definition 2.1. A function $\alpha_0 \in \mathcal{F}$ is called a lower solution of BVP (1.1) if:

$$\begin{cases} \alpha'_0\,(t) \le f\,(t, \alpha_0\,(t), (F\alpha_0)\,(t), (S\alpha_0)\,(t)), & t \in J^-, \\ \Delta\alpha_0\,(t_k) \le I_k\left(\sum\limits_{l=1}^{c_k} \rho_l^k \alpha_0\left(\eta_l^k\right)\right), & k = 1, 2, \ldots, m, \\ \alpha_0\,(0) + \mu \sum\limits_{k=1}^{m}\sum\limits_{l=1}^{c_k} \tau_l^k \alpha_0\left(\eta_l^k\right) \le \alpha_0\,(T). \end{cases}$$

Analogously, a function $\beta_0 \in \mathcal{F}$ is called an upper solution of BVP (1.1) if:

$$\begin{cases} \beta'_0(t) \geq f(t, \beta_0(t), (F\beta_0)(t), (S\beta_0)(t)), & t \in J^-, \\ \Delta\beta_0(t_k) \geq I_k\left(\sum_{l=1}^{c_k} \rho_l^k \beta_0\left(\eta_l^k\right)\right), & k = 1, 2, \ldots, m, \\ \beta_0(0) + \mu \sum_{k=1}^{m} \sum_{l=1}^{c_k} \tau_l^k \beta_0\left(\eta_l^k\right) \geq \beta_0(T), \end{cases}$$

where $t_{k-1} < \eta_l^k \leq t_k, \rho_l^k, \tau_l^k \geq 0$, $l = 1, 2, ..., c_k$, $c_k \in N = \{1, 2, ...\}$, $k = 1, 2, ..., m$ and $\mu \geq 0$.

Let us consider the following boundary value problem of a linear impulsive integro-differential equation (BVP):

$$\begin{cases} x'(t) - Mx(t) = H(Fx)(t) + K(Sx)(t) + v(t), & t \in J^-, \\ \Delta x(t_k) = L_k \sum_{l=1}^{c_k} \rho_l^k x\left(\eta_l^k\right) + I_k\left(\sum_{l=1}^{c_k} \rho_l^k \sigma\left(\eta_l^k\right)\right) & \\ \qquad -L_k \sum_{l=1}^{c_k} \rho_l^k \sigma\left(\eta_l^k\right), & k = 1, 2, \ldots, m, \\ x(0) + \mu \sum_{k=1}^{m} \sum_{l=1}^{c_k} \tau_l^k \sigma\left(\eta_l^k\right) = x(T), \end{cases} \tag{2.1}$$

where $M > 0$, $H, K \geq 0$, $L_k \geq 0$, $t_{k-1} < \eta_l^k \leq t_k, \tau_l^k, \rho_l^k \geq 0$, $l = 1, 2, ..., c_k$, $c_k \in N = \{1, 2, ...\}$, $k = 1, 2, ..., m$ are constants and $v(t), \sigma(t) \in E$.

Lemma 2.1. *$x \in \mathcal{F}$ is a solution of* (2.1) *if and only if $x \in E$ is a solution of the impulsive integral equation*

$$\begin{aligned} x(t) = {} & \frac{\mu e^{Mt}}{e^{MT} - 1} \sum_{k=1}^{m} \sum_{l=1}^{c_k} \tau_l^k \sigma\left(\eta_l^k\right) - \int_0^T G(t, s) P(s)\, ds - \sum_{k=1}^{m} G(t, t_k) \left[L_k \sum_{l=1}^{c_k} \rho_l^k x\left(\eta_l^k\right) \right. \\ & \left. + I_k\left(\sum_{l=1}^{c_k} \rho_l^k \sigma\left(\eta_l^k\right)\right) - L_k \sum_{l=1}^{c_k} \rho_l^k \sigma\left(\eta_l^k\right) \right], \qquad t \in J, \end{aligned} \tag{2.2}$$

where $P(t) = H(Fx)(t) + K(Sx)(t) + v(t)$ *and*

$$G(t, s) = \begin{cases} \dfrac{e^{M(t-s)}}{e^{MT} - 1}, & 0 \leq s < t \leq T, \\ \dfrac{e^{M(T+t-s)}}{e^{MT} - 1}, & 0 \leq t \leq s \leq T. \end{cases}$$

Proof. Assume that $x(t)$ is a solution of BVP (2.1). By using the variation of parameters formula, we get

$$\begin{aligned} x(t) = {} & x(0) e^{Mt} + \int_0^t e^{M(t-s)} P(s)\, ds + \sum_{0 < t_k < t} e^{M(t-t_k)} \left[L_k \sum_{l=1}^{c_k} \rho_l^k x\left(\eta_l^k\right) + I_k\left(\sum_{l=1}^{c_k} \rho_l^k \sigma\left(\eta_l^k\right)\right) \right. \\ & \left. -L_k \sum_{l=1}^{c_k} \rho_l^k \sigma\left(\eta_l^k\right) \right]. \end{aligned} \tag{2.3}$$

Putting $t = T$ in (2.3), we have

$$\begin{aligned} x(T) = {} & x(0) e^{MT} + \int_0^T e^{M(T-s)} P(s)\, ds + \sum_{k=1}^{m} e^{M(T-t_k)} \left[L_k \sum_{l=1}^{c_k} \rho_l^k x\left(\eta_l^k\right) + I_k\left(\sum_{l=1}^{c_k} \rho_l^k \sigma\left(\eta_l^k\right)\right) \right. \\ & \left. -L_k \sum_{l=1}^{c_k} \rho_l^k \sigma\left(\eta_l^k\right) \right]. \end{aligned} \tag{2.4}$$

From $x(0)+\mu\sum_{k=1}^{m}\sum_{l=1}^{c_k}\tau_l^k\sigma\left(\eta_l^k\right)=x(T)$, we obtain

$$x(0)=\frac{-1}{e^{MT}-1}\left\{-\mu\sum_{k=1}^{m}\sum_{l=1}^{c_k}\tau_l^k\sigma\left(\eta_l^k\right)+\int_0^T e^{M(T-s)}P(s)\,ds\right.$$
$$\left.+\sum_{k=1}^{m}e^{M(T-t_k)}\left[L_k\sum_{l=1}^{c_k}\rho_l^k x\left(\eta_l^k\right)+I_k\left(\sum_{l=1}^{c_k}\rho_l^k\sigma\left(\eta_l^k\right)\right)-L_k\sum_{l=1}^{c_k}\rho_l^k\sigma\left(\eta_l^k\right)\right]\right\}. \tag{2.5}$$

Substituting (2.5) into (2.3), we see that $x\in E$ satisfies (2.2). Hence, $x(t)$ is also the solution of (2.2).

Conversely, we assume that $x(t)$ is a solution of (2.2). By computing directly, we have

$$G_t'(t,s)=\begin{cases}\dfrac{Me^{M(t-s)}}{e^{MT}-1}, & 0\le s<t\le T,\\ \dfrac{Me^{M(T+t-s)}}{e^{MT}-1}, & 0\le t\le s\le T,\end{cases}=MG(t,s).$$

Differentiating (2.2) for $t\neq t_k$, we obtain

$$x'(t)=Mx(t)+H(Fx)(t)+K(Sx)(t)+v(t).$$

It is easy to see that

$$\Delta x(t_k)=L_k\sum_{l=1}^{c_k}\rho_l^k x\left(\eta_l^k\right)+I_k\left(\sum_{l=1}^{c_k}\rho_l^k\sigma\left(\eta_l^k\right)\right)-L_k\sum_{l=1}^{c_k}\rho_l^k\sigma\left(\eta_l^k\right).$$

Since $G(0,s)=G(T,s)$, then $x(0)+\mu\sum_{k=1}^{m}\sum_{l=1}^{c_k}\tau_l^k\sigma\left(\eta_l^k\right)=x(T)$. This completes the proof. □

Lemma 2.2. *Assume that* $M>0$, $H, K\geq 0$, $L_k\geq 0, t_{k-1}<\eta_l^k\leq t_k, \rho_l^k\geq 0$, $l=1,2,\ldots,c_k$, $c_k\in N=\{1,2,\ldots\}$, $k=1,2,\ldots,m$, *and the following inequality holds:*

$$\frac{e^{MT}}{e^{MT}-1}\int_0^T\left[H\int_0^s k(s,r)\,dr+K\int_0^T h(s,r)\,dr\right]ds+\frac{e^{MT}}{e^{MT}-1}\sum_{k=1}^{m}L_k\left(\sum_{l=1}^{c_k}\rho_l^k\right)<1. \tag{2.6}$$

Then BVP (2.1) *has a unique solution.*

Proof. For any $x\in E$, we define an operator A by

$$(Ax)(t)=\frac{\mu e^{Mt}}{e^{MT}-1}\sum_{k=1}^{m}\sum_{l=1}^{c_k}\tau_l^k\sigma\left(\eta_l^k\right)-\int_0^T G(t,s)\left[H(Fx)(s)+K(Sx)(s)+v(s)\right]ds$$
$$-\sum_{k=1}^{m}G(t,t_k)\left[L_k\sum_{l=1}^{c_k}\rho_l^k x\left(\eta_l^k\right)+I_k\left(\sum_{l=1}^{c_k}\rho_l^k\sigma\left(\eta_l^k\right)\right)-L_k\sum_{l=1}^{c_k}\rho_l^k\sigma\left(\eta_l^k\right)\right],\quad t\in J, \tag{2.7}$$

where $G(t,s)$ is defined as in Lemma 2.1. Since $\max_{t\in[0,T]}\{G(t,s)\}=\dfrac{e^{MT}}{e^{MT}-1}$, we have for any $x, y\in E$, that

$$\begin{aligned}\left\|Ax - Ay\right\|_E &= \Bigg| -\int_0^T G(t,s)\left[H\int_0^s k(s,r)\big(x(r)-y(r)\big)\,dr\right.\\ &\quad \left. + K\int_0^T h(s,r)\big(x(r)-y(r)\big)\,dr\right]ds\\ &\quad - \sum_{k=1}^{m} G(t,t_k)\,L_k\sum_{l=1}^{c_k}\rho_l^k\left(x\left(\eta_l^k\right)-y\left(\eta_l^k\right)\right)\Bigg|\\ &\le \left[\frac{e^{MT}}{e^{MT}-1}\int_0^T\left[H\int_0^s k(s,r)\,dr + K\int_0^T h(s,r)\,dr\right]ds\right.\\ &\quad \left. + \frac{e^{MT}}{e^{MT}-1}\sum_{k=1}^{m}L_k\left(\sum_{l=1}^{c_k}\rho_l^k\right)\right]\left\|x-y\right\|_E.\end{aligned}$$

From (2.6) and the Banach fixed point theorem, A has a unique fixed point $\bar{x} \in E$. By Lemma 2.1, $\bar{x}$ is also the unique solution of (2.1). □

Lemma 2.3. *Assume that* $x \in \mathcal{F}$ *satisfies*

$$\begin{cases} x'(t) \ge Mx(t) + H(Fx)(t) + K(Sx)(t), & t \in J^-,\\ \Delta x(t_k) \ge L_k\sum\limits_{l=1}^{c_k}\rho_l^k x\left(\eta_l^k\right), & k = 1,2,\ldots,m,\\ x(0) \ge x(T), \end{cases} \tag{2.8}$$

where $M > 0$, $H, K \ge 0$, $L_k \ge 0, t_{k-1} < \eta_l^k \le t_k, \rho_l^k \ge 0$, $l = 1, 2, \ldots, c_k$, $c_k \in N = \{1, 2, \ldots\}$, $k = 1, 2, \ldots, m$. *In addition assume that*

$$e^{MT}\left[\int_0^T q(s)\,ds + \sum_{k=1}^{m}L_k\sum_{l=1}^{c_k}\rho_l^k e^{-M\left(t_k-\eta_l^k\right)}\right] \le 1, \tag{2.9}$$

where $q(t) = H\int_0^t k(t,s)\,e^{-M(t-s)}ds + K\int_0^T h(t,s)\,e^{-M(t-s)}ds$. *Then,* $x(t) \le 0$ *for all* $t \in J$.

Proof. Set $u(t) = x(t)e^{-Mt}$ for $t \in J$, then we have

$$\begin{cases} u'(t) \ge H\int_0^t k(t,s)\,e^{-M(t-s)}u(s)\,ds + K\int_0^T h(t,s)\,e^{-M(t-s)}u(s)\,ds, & t \in J^-,\\ \Delta u(t_k) \ge L_k\sum_{l=1}^{c_k}\rho_l^k e^{-M\left(t_k-\eta_l^k\right)}u\left(\eta_l^k\right), & k = 1,2,\ldots,m,\\ u(0) \ge e^{MT}u(T). \end{cases} \tag{2.10}$$

Obviously, the function $u(t)$ and $x(t)$ have the same sign. Suppose, to the contrary, that $u(t) > 0$ for some $t \in J$. Then, there are two cases:

(i) There exists a $t^* \in J$, such that $u(t^*) > 0$ and $u(t) \ge 0$ for all $t \in J$.

(ii) There exists $t^*, t_* \in J$, such that $u(t^*) > 0$ and $u(t_*) < 0$.

Case (i): Equation (2.10) implies that $u'(t) \ge 0$ for $t \in J^-$ and $\Delta u(t_k) \ge 0$ for $k = 1, 2, \ldots, m$. This means that $u(t)$ is nondecreasing in J. Therefore, $u(T) \ge u(t^*) > 0$ and $u(T) \ge u(0) \ge u(T)e^{MT}$, which is a contradiction.

Case (ii): Let $t_* \in (t_i, t_{i+1}]$, $i \in \{0, 1, \ldots, m\}$, such that $u(t_*) = \inf\{u(t): t \in J\} < 0$ and $t^* \in (t_j, t_{j+1}]$, $j \in \{0, 1, \ldots, m\}$, such that $u(t^*) > 0$. We first claim that $u(0) \le 0$. Otherwise, if $u(0) > 0$, then by (2.10), we have

$$
\begin{aligned}
u(t_*) - u(0) &\geq H \int_0^{t_*} \int_0^{s} k(s,r)\, e^{-M(s-r)} u(r)\, drds \\
&\quad + K \int_0^{t_*} \int_0^{T} h(s,r)\, e^{-M(s-r)} u(r)\, drds + \sum_{k=1}^{i} \Delta u(t_k) \\
&\geq u(t_*) \left[\int_0^{t_*} q(s)\, ds + \sum_{k=1}^{i} L_k \sum_{l=1}^{c_k} \rho_l^k e^{-M\left(t_k - \eta_l^k\right)} \right] \\
&\geq u(t_*)\,,
\end{aligned} \tag{2.11}
$$

a contradiction, and so $u(0) \leq 0$.

If $t^* < t_*$, then $j \leq i$. Integrating the differential inequality in (2.10) from t^* to t_*, we obtain

$$
\begin{aligned}
u(t_*) - u\left(t^*\right) &\geq H \int_{t^*}^{t_*} \int_0^{s} k(s,r)\, e^{-M(s-r)} u(r)\, drds \\
&\quad + K \int_{t^*}^{t_*} \int_0^{T} h(s,r)\, e^{-M(s-r)} u(r)\, drds + \sum_{k=j+1}^{i} \Delta u(t_k) \\
&\geq u(t_*) \int_{t^*}^{t_*} q(s)\, ds + \sum_{k=j+1}^{i} \Delta u(t_k) \\
&\geq u(t_*) \int_{t^*}^{t_*} q(s)\, ds + \sum_{k=j+1}^{i} L_k \sum_{l=1}^{c_k} \rho_l^k e^{-M\left(t_k - \eta_l^k\right)} u\left(\eta_l^k\right) \\
&\geq u(t_*) \left[\int_0^{T} q(s)\, ds + \sum_{k=1}^{m} L_k \sum_{l=1}^{c_k} \rho_l^k e^{-M\left(t_k - \eta_l^k\right)} \right] \\
&\geq u(t_*)\,,
\end{aligned}
$$

which is a contradiction to $u(t^*) > 0$.

Now, assume that $t_* < t^*$. Since $0 \geq u(0) \geq e^{MT} u(T)$, then $u(T) \leq 0$. From (2.10), we have

$$
\begin{aligned}
u(T) - u\left(t^*\right) &\geq H \int_{t^*}^{T} \int_0^{s} k(s,r)\, e^{-M(s-r)} u(r)\, drds \\
&\quad + K \int_{t^*}^{T} \int_0^{T} h(s,r)\, e^{-M(s-r)} u(r)\, drds + \sum_{k=j+1}^{m} \Delta u(t_k) \\
&\geq u(t_*) \left[\int_{t^*}^{T} q(s)\, ds + \sum_{k=j+1}^{m} L_k \sum_{l=1}^{c_k} \rho_l^k e^{-M\left(t_k - \eta_l^k\right)} \right],
\end{aligned}
$$

and $u(0) \geq e^{MT}\, u(T)$. In consequence,

$$
u(0) \geq e^{MT} u(T) \geq e^{MT} u(t^*) + u(t_*)\, e^{MT} \left[\int_{t^*}^{T} q(s)\, ds + \sum_{k=j+1}^{m} L_k \sum_{l=1}^{c_k} \rho_l^k e^{-M\left(t_k - \eta_l^k\right)} \right] \tag{2.12}
$$

can be obtained.

If $t_* = 0$, then

$$\begin{aligned} u(t_*) &\geq e^{MT} u\left(t^*\right) + u(t_*) e^{MT} \left[\int_{t^*}^{T} q(s)\, ds + \sum_{k=j+1}^{m} L_k \sum_{l=1}^{c_k} \rho_l^k e^{-M\left(t_k - \eta_l^k\right)} \right] \\ &\geq e^{MT} u\left(t^*\right) + u(t_*) . \end{aligned}$$

This contradicts the fact that $u(t^*) > 0$.

If $t_* > 0$, we obtain from (2.11),

$$u(t_*) - u(t_*) \left[\int_0^{t_*} q(s)\, ds + \sum_{k=1}^{i} L_k \sum_{l=1}^{c_k} \rho_l^k e^{-M\left(t_k - \eta_l^k\right)} \right] \geq u(0) .$$

This joint to (2.12) yields

$$\begin{aligned} & u(t_*) - u(t_*) \left[\int_0^{t_*} q(s)\, ds + \sum_{k=1}^{i} L_k \sum_{l=1}^{c_k} \rho_l^k e^{-M\left(t_k - \eta_l^k\right)} \right] \\ & \qquad \geq e^{MT} u\left(t^*\right) + u(t_*) e^{MT} \left[\int_{t^*}^{T} q(s)\, ds + \sum_{k=j+1}^{m} L_k \sum_{l=1}^{c_k} \rho_l^k e^{-M\left(t_k - \eta_l^k\right)} \right] . \end{aligned}$$

Therefore,

$$\begin{aligned} u(t_*) - e^{MT} u\left(t^*\right) &\geq u(t_*) e^{MT} \left[\int_{t^*}^{T} q(s)\, ds + \sum_{k=j+1}^{m} L_k \sum_{l=1}^{c_k} \rho_l^k e^{-M\left(t_k - \eta_l^k\right)} \right] \\ &\quad + u(t_*) \left[\int_0^{t_*} q(s)\, ds + \sum_{k=1}^{i} L_k \sum_{l=1}^{c_k} \rho_l^k e^{-M\left(t_k - \eta_l^k\right)} \right] \\ &\geq u(t_*) e^{MT} \left[\int_{t^*}^{T} q(s)\, ds + \sum_{k=j+1}^{m} L_k \sum_{l=1}^{c_k} \rho_l^k e^{-M\left(t_k - \eta_l^k\right)} \right] \\ &\quad + u(t_*) e^{MT} \left[\int_0^{t_*} q(s)\, ds + \sum_{k=1}^{i} L_k \sum_{l=1}^{c_k} \rho_l^k e^{-M\left(t_k - \eta_l^k\right)} \right] \\ &\geq u(t_*) e^{MT} \left[\int_0^{T} q(s)\, ds + \sum_{k=1}^{m} L_k \sum_{l=1}^{c_k} \rho_l^k e^{-M\left(t_k - \eta_l^k\right)} \right] \\ &\geq u(t_*) . \end{aligned}$$

This is a contradiction and so $u(t) \leq 0$ for all $t \in J$. The proof is complete. □

3 Main results

In this section, we are in a position to prove our main results concerning the existence criteria for solutions of BVP (1.1).

For $\beta_0, \alpha_0 \in \mathcal{F}$, we denote

$$[\beta_0, \alpha_0] = \{x \in \mathcal{F} : \beta_0(t) \leq x(t) \leq \alpha_0(t), \quad t \in J\},$$

and we write $\beta_0 \leq \alpha_0$ if $\beta_0(t) \leq \alpha_0(t)$ for all $t \in J$.

Theorem 3.1. *Let the following conditions hold.*

(H_1) *The functions* α_0 *and* β_0 *are lower and upper solutions of BVP* (1.1), *respectively, such that* $\beta_0(t) \le \alpha_0(t)$ *on* J.

(H_2) *The function* $f \in C(J \times R^3, R)$ *satisfies*

$$f\left(t, x, y, z\right) - f\left(t, \bar{x}, \bar{y}, \bar{z}\right) \le M\left(x - \bar{x}\right) + H\left(y - \bar{y}\right) + K\left(z - \bar{z}\right),$$

for $\beta_0(t) \le \bar{x}(t) \le x(t) \le \alpha_0(t), (F\beta_0)(t) \le \bar{y}(t) \le y(t) \le (F\alpha_0)(t)\ (S\beta_0)(t) \le \bar{z}(t) \le z(t) \le (S\alpha_0)(t)\ t \in J$.

(H_3) *The function* $I_k \in C(R, R)$ *satisfies*

$$I_k\left(\sum_{l=1}^{c_k} \rho_l^k x\left(\eta_l^k\right)\right) - I_k\left(\sum_{l=1}^{c_k} \rho_l^k y\left(\eta_l^k\right)\right) \le L_k \sum_{l=1}^{c_k} \rho_l^k \left(x\left(\eta_l^k\right) - y\left(\eta_l^k\right)\right),$$

whenever $\beta_0\left(\eta_l^k\right) \le y\left(\eta_l^k\right) \le x\left(\eta_l^k\right) \le \alpha_0\left(\eta_l^k\right)$, $l = 1, 2, ..., c_k$, $c_k \in N = \{1, 2, ...\}$, $L_k \ge 0$, $k = 1, 2, ..., m$.

(H_4) *Inequalities* (2.6) *and* (2.9) *hold.*

Then there exist monotone sequences $\{\alpha_n\}, \{\beta_n\} \subset \mathcal{F}$ *such that* $\lim_{n\to\infty} \alpha_n(t) = x^*(t)$, $\lim_{n\to\infty} \beta_n(t) = x_*(t)$ *uniformly on* J *and* x^*, x_* *are maximal and minimal solutions of BVP* (1.1), *respectively, such that*

$$\beta_0 \le \beta_1 \le \beta_2 \le \cdots \le \beta_n \le x_* \le x \le x^* \le \alpha_n \le \cdots \le \alpha_2 \le \alpha_1 \le \alpha_0,$$

on J, *where* x *is any solution of BVP* (1.1) *such that* $\beta_0(t) \le x(t) \le \alpha_0(t)$ *on* J.

Proof. For any $\sigma \in [\beta_0, \alpha_0]$, we consider BVP (2.1) with

$$v(t) = f(t, \sigma(t), (F\sigma)(t), (S\sigma)(t)) - M\sigma(t) - H(F\sigma)(t) - K(S\sigma)(t).$$

By Lemma 2.2, BVP (2.1) has a unique solution $x(t)$ for $t \in J$. We define an operator A by $x = A\sigma$, then the operator A is an operator from $[\beta_0, \alpha_0]$ to $\mathcal{F}$ and A has the following properties.

(i) $\beta_0 \le A\beta_0$, $A\alpha_0 \le \alpha_0$;

(ii) For any $\sigma_1, \sigma_2 \in [\beta_0, \alpha_0]$, $\sigma_1 \le \sigma_2$ implies $A\sigma_1 \le A\sigma_2$.

To prove (i), set $\phi = \beta_0 - \beta_1$, where $\beta_1 = A\beta_0$. Then from (H_1) and (2.1) for $t \in J'$, we have

$$\begin{aligned}\varphi'(t) &= \beta'_0(t) - \beta'_1(t),\\ &\ge f(t, \beta_0(t), (F\beta_0)(t), (S\beta_0)(t)) - [M\beta_1(t) + H(F\beta_1)(t) + K(S\beta_1)(t)\\ &\quad + f(t, \beta_0(t), (F\beta_0)(t), (S\beta_0)(t)) - M\beta_0(t) - H(F\beta_0)(t) - K(S\beta_0)(t)]\\ &= M\varphi(t) + H(F\varphi)(t) + K(S\varphi)(t),\end{aligned}$$

$$\begin{aligned}\Delta\varphi(t_k) &= \Delta\beta_0(t_k) - \Delta\beta_1(t_k)\\ &\ge I_k\left(\sum_{l=1}^{c_k} \rho_l^k \beta_0\left(\eta_l^k\right)\right) - \left[L_k \sum_{l=1}^{c_k} \rho_l^k \beta_1\left(\eta_l^k\right) + I_k\left(\sum_{l=1}^{c_k} \rho_l^k \beta_0\left(\eta_l^k\right)\right)\right.\\ &\quad \left. - L_k \sum_{l=1}^{c_k} \rho_l^k \beta_0\left(\eta_l^k\right)\right]\\ &= L_k \sum_{l=1}^{c_k} \rho_l^k \varphi\left(\eta_l^k\right), \quad k = 1, 2, \ldots, m,\end{aligned}$$

and

$$\begin{aligned}\varphi(0) &= \beta_0(0) - \beta_1(0)\\ &\geq \beta_0(T) - \mu\sum_{k=1}^{m}\sum_{l=1}^{c_k}\tau_l^k\beta_0\left(\eta_l^k\right) - \beta_1(T) + \mu\sum_{k=1}^{m}\sum_{l=1}^{c_k}\tau_l^k\beta_0\left(\eta_l^k\right)\\ &= \varphi(T).\end{aligned}$$

By Lemma 2.3, we get that $\phi(t) \leq 0$ for all $t \in J$, i.e., $\beta_0 \leq A\beta_0$. Similarly, we can prove that $A\alpha_0 \leq \alpha_0$.

To prove (ii), let $u_1 = A\sigma_1$, $u_2 = A\sigma_2$, where $\sigma_1 \leq \sigma_2$ on J and $\sigma_1, \sigma_2 \in [\beta_0, \alpha_0]$. Set $\phi = u_1 - u_2$. Then for $t \in J^{-}$ and by (H_2), we obtain

$$\begin{aligned}\varphi'(t) &= u'_1(t) - u'_2(t)\\ &= Mu_1(t) + H(Fu_1)(t) + K(Su_1)(t) + f(t, \sigma_1(t), (F\sigma_1)(t), (S\sigma_1)(t))\\ &\quad - M\sigma_1(t) - H(F\sigma_1)(t) - K(S\sigma_1)(t) - (Mu_2(t) + H(Fu_2)(t) + K(Su_2)(t)\\ &\quad + f(t, \sigma_2(t), (F\sigma_2)(t), (S\sigma_2)(t)) - M\sigma_2(t) - H(F\sigma_2)(t) - K(S\sigma_2)(t))\\ &\geq M(u_1(t) - u_2(t)) + H(F(u_1 - u_2))(t) + K(S(u_1 - u_2))(t),\\ &= M\varphi(t) + H(F\varphi)(t) + K(S\varphi)(t),\end{aligned}$$

and by $(H3)$;

$$\begin{aligned}\Delta\varphi(t_k) &= \Delta u_1(t_k) - \Delta u_2(t_k)\\ &= L_k\sum_{l=1}^{c_k}\rho_l^k u_1\left(\eta_l^k\right) + I_k\left(\sum_{l=1}^{c_k}\rho_l^k\sigma_1\left(\eta_l^k\right)\right) - L_k\sum_{l=1}^{c_k}\rho_l^k\sigma_1\left(\eta_l^k\right) - \left[L_k\sum_{l=1}^{c_k}\rho_l^k u_2\left(\eta_l^k\right)\right.\\ &\quad \left. + I_k\left(\sum_{l=1}^{c_k}\rho_l^k\sigma_2\left(\eta_l^k\right)\right) - L_k\sum_{l=1}^{c_k}\rho_l^k\sigma_2\left(\eta_l^k\right)\right]\\ &\geq L_k\sum_{l=1}^{c_k}\rho_l^k\left[u_1\left(\eta_l^k\right) - u_2\left(\eta_l^k\right)\right] = L_k\sum_{l=1}^{c_k}\rho_l^k\varphi\left(\eta_l^k\right), \quad k = 1, 2, \ldots, m.\end{aligned}$$

It is easy to see that

$$\begin{aligned}\varphi(0) &= u_1(0) - u_2(0)\\ &= u_1(T) - \mu\sum_{k=1}^{m}\sum_{l=1}^{c_k}\tau_l^k\sigma_1\left(\eta_l^k\right) - u_2(T) + \mu\sum_{k=1}^{m}\sum_{l=1}^{c_k}\tau_l^k\sigma_2\left(\eta_l^k\right)\\ &\geq \varphi(T).\end{aligned}$$

Then by using Lemma 2.3, we have $\phi(t) \leq 0$, which implies that $A\sigma_1 \leq A\sigma_2$.

Now, we define the sequences $\{\alpha_n\}$, $\{\beta_n\}$ such that $\alpha_{n+1} = A\alpha_n$ and $\beta_{n+1} = A\beta_n$. From (i) and (ii) the sequence $\{\alpha_n\}$, $\{\beta_n\}$ satisfy the inequality

$$\beta_0 \leq \beta_1 \leq \cdots \leq \beta_n \leq \cdots \leq \alpha_n \leq \cdots \leq \alpha_1 \leq \alpha_0,$$

for all $n \in N$. Obviously, each α_n, β_n $(n = 1, 2, \ldots)$ satisfy

$$\begin{cases}\alpha'_n(t) = M\alpha_n(t) + H(F\alpha_n)(t) + K(S\alpha_n)(t) + f(t, \alpha_{n-1}(t), (F\alpha_{n-1})(t), (S\alpha_{n-1})(t))\\ \qquad -M\alpha_{n-1}(t) - H(F\alpha_{n-1})(t) - K(S\alpha_{n-1})(t), \qquad t \in J^{-},\\ \Delta\alpha_n(t_k) = L_k\sum_{l=1}^{c_k}\rho_l^k\alpha_n\left(\eta_l^k\right) + I_k\left(\sum_{l=1}^{c_k}\rho_l^k\alpha_{n-1}\left(\eta_l^k\right)\right)\\ \qquad -L_k\sum_{l=1}^{c_k}\rho_l^k\alpha_{n-1}\left(\eta_l^k\right), \qquad k = 1, 2, \ldots, m,\\ \alpha_n(0) + \mu\sum_{k=1}^{m}\sum_{l=1}^{c_k}\tau_l^k\alpha_{n-1}\left(\eta_l^k\right) = \alpha_n(T),\end{cases}$$

and

$$\begin{cases} \beta'_n(t) = M\beta_n(t) + H(F\beta_n)(t) + K(S\beta_n)(t) + f(t, \beta_{n-1}(t), (F\beta_{n-1})(t), (S\beta_{n-1})(t)) \\ \qquad -M\beta_{n-1}(t) - H(F\beta_{n-1})(t) - K(S\beta_{n-1})(t), \qquad t \in J^-, \\ \Delta\beta_n(t_k) = L_k \sum_{l=1}^{c_k} \rho_l^k \beta_n(\eta_l^k) + I_k\left(\sum_{l=1}^{c_k} \rho_l^k \beta_{n-1}(\eta_l^k)\right) \\ \qquad -L_k \sum_{l=1}^{c_k} \rho_l^k \beta_{n-1}(\eta_l^k), \qquad k = 1, 2, \ldots, m, \\ \beta_n(0) + \mu \sum_{k=1}^{m} \sum_{l=1}^{c_k} \tau_l^k \beta_{n-1}(\eta_l^k) = \beta_n(T). \end{cases}$$

Therefore, there exist x_* and x^*, such that $\lim_{n\to\infty} \beta_n = x_*$ and $\lim_{n\to\infty} \alpha_n = x^*$ uniformly on J. Clearly, x_*, x^* are solutions of BVP (1.1).

Finally, we are going to prove that x_*, x^* are minimal and maximal solutions of BVP (1.1). Assume that $x(t)$ is any solution of BVP (1.1) such that $x \in [\beta_0, \alpha_0]$ and that there exists a positive integer n such that $\beta_n(t) \leq x(t) \leq \alpha_n(t)$ on J. Let $\phi = \beta_{n+1} - x$, then for $t \in J^-$,

$$\begin{aligned} \varphi'(t) &= \beta'_{n+1}(t) - x'(t) \\ &= M\beta_{n+1}(t) + H(F\beta_{n+1})(t) + K(S\beta_{n+1})(t) + f(t, \beta_n(t), (F\beta_n)(t), (S\beta_n)(t)) \\ &\quad - M\beta_n(t) - H(F\beta_n)(t) - K(S\beta_n)(t) - f(t, x(t), (Fx)(t), (Sx)(t)) \\ &\geq M\varphi(t) + H(F\varphi)(t) + K(S\varphi)(t), \end{aligned}$$

$$\begin{aligned} \Delta\varphi(t_k) &= \Delta\beta_{n+1}(t_k) - \Delta x(t_k) \\ &= L_k \sum_{l=1}^{c_k} \rho_l^k \beta_{n+1}(\eta_l^k) + I_k\left(\sum_{l=1}^{c_k} \rho_l^k \beta_n(\eta_l^k)\right) - L_k \sum_{l=1}^{c_k} \rho_l^k \beta_n(\eta_l^k) - I_k\left(\sum_{l=1}^{c_k} \rho_l^k x(\eta_l^k)\right) \\ &\geq L_k \sum_{l=1}^{c_k} \rho_l^k \left[\beta_{n+1}(\eta_l^k) - x(\eta_l^k)\right] = L_k \sum_{l=1}^{c_k} \rho_l^k \varphi(\eta_l^k), \qquad k = 1, 2, \ldots, m, \end{aligned}$$

and

$$\begin{aligned} \varphi(0) &= \beta_{n+1}(0) - x(0) \\ &= \beta_{n+1}(T) - \mu \sum_{k=1}^{m} \sum_{l=1}^{c_k} \tau_l^k \beta_n(\eta_l^k) - x(T) + \mu \sum_{k=1}^{m} \sum_{l=1}^{c_k} \tau_l^k x(\eta_l^k) \\ &\geq \varphi(T). \end{aligned}$$

Then by using Lemma 2.3, we have $\phi(t) \leq 0$, which implies that $\beta_{n+1} \leq x$ on J. Similarly we obtain $x \leq \alpha_{n+1}$ on J. Since $\beta_0 \leq x \leq \alpha_0$ on J, by induction we get $\beta_n \leq \times \leq \alpha_n$ on J for every n. Therefore, $x_*(t) \leq x(t) \leq x^*(t)$ on J by taking $n \to \infty$. The proof is complete. □

4 An example

In this section, in order to illustrate our results, we consider an example.

Example 4.1. Consider the BVP

$$\begin{cases} x'(t) = t^3(1 + x(t)) + \frac{1}{54} t\left[\int_0^t tsx(s)\,ds\right]^3 + \frac{1}{81} t^2\left[\int_0^1 tsx(s)\,ds\right]^3, \qquad t \in J = [0, 1], \quad t \neq \frac{1}{2}, \\ \Delta x\left(\frac{1}{2}\right) = \frac{1}{4}\left[\frac{1}{5}x\left(\frac{1}{10}\right) + \frac{3}{10}x\left(\frac{1}{5}\right) + \frac{1}{10}x\left(\frac{3}{10}\right) + \frac{1}{5}x\left(\frac{2}{5}\right) + \frac{1}{5}x\left(\frac{1}{2}\right)\right], \quad k = 1, \\ x(0) + \frac{1}{5}\left[\frac{1}{5}x\left(\frac{1}{5}\right) + \frac{2}{5}x\left(\frac{3}{10}\right) + \frac{2}{5}x\left(\frac{1}{2}\right)\right] = x(1), \end{cases} \tag{4.1}$$

where $k(t, s) = h(t, s) = ts$, $m = 1$, $t_1 = \frac{1}{2}, c_1 = 5, \rho_1^1 = \frac{1}{5}, \rho_2^1 = \frac{3}{10}, \rho_3^1 = \frac{1}{10}, \rho_4^1 = \frac{1}{5}, \rho_5^1 = \frac{1}{5}, \eta_1^1 = \frac{1}{10}, \eta_2^1 = \frac{1}{5}, \eta_3^1 = \frac{3}{10}, \eta_4^1 = \frac{2}{5}, \eta_5^1 = \frac{1}{2}, \tau_1^1 = 0, \tau_2^1 = \frac{1}{5}, \tau_3^1 = \frac{2}{5}, \tau_4^1 = 0, \tau_5^1 = \frac{2}{5}, \mu = \frac{1}{5}$.

Obviously, $\alpha_0 = 0,\ \beta_0 = \begin{cases} -5, & t \in \left[0, \frac{1}{2}\right] \\ -6, & t \in \left(\frac{1}{2}, 1\right] \end{cases}$ are lower and upper solutions for (4.1), respectively, and $\beta_0 \leq \alpha_0$.

Let

$$f\left(t, x, y, z\right) = t^3\left(1 + x\right) + \frac{1}{54}ty^3 + \frac{1}{81}t^2 z^3.$$

Then,

$$f\left(t, x, y, z\right) - f\left(t, \bar{x}, \bar{y}, \bar{z}\right) \leq \left(x - \bar{x}\right) + \frac{1}{2}\left(y - \bar{y}\right) + \frac{1}{3}\left(z - \bar{z}\right),$$

where $\beta_0(t) \leq \bar{x}(t) \leq x(t) \leq \alpha_0(t)$, $(F\beta_0)(t) \leq \bar{y}(t) \leq y(t) \leq (F\alpha_0)(t)$, $(S\beta_0)(t) \leq \bar{z}(t) \leq z(t) \leq (S\alpha_0)(t)$, $t \in J$. It is easy to see that

$$I_1\left(\sum_{l=1}^{5}\rho_l^1 x\left(\eta_l^1\right)\right) - I_1\left(\sum_{l=1}^{5}\rho_l^1 y\left(\eta_l^1\right)\right) = \frac{1}{4}\sum_{l=1}^{5}\rho_l^1\left(x\left(\eta_l^1\right) - y\left(\eta_l^1\right)\right),$$

whenever $\beta_0\left(\eta_l^1\right) \leq y\left(\eta_l^1\right) \leq x\left(\eta_l^1\right) \leq \alpha_0\left(\eta_l^1\right)$, $l = 1, \ldots, 5$.

Taking $L_1 = \frac{1}{4}, M = 1, H = \frac{1}{2}, K = \frac{1}{3}$, it follows that

$$\begin{aligned} &e^{MT}\left[\int_0^T H\int_0^s k\left(s, r\right)e^{-M(s-r)}dr + K\int_0^T h\left(s, r\right)e^{-M(s-r)}drds \right.\\ &\qquad \left. + \sum_{k=1}^{m} L_k e^{-Mt_k}\left(\sum_{l=1}^{c_k}\rho_l^k e^{M\eta_l^k}\right)\right] \\ &= e\left[\int_0^1 \frac{1}{2}\int_0^s sre^{-(s-r)}dr + \frac{1}{3}\int_0^1 sre^{-(s-r)}drds \right.\\ &\qquad \left. + \frac{1}{4}e^{-\frac{1}{2}}\left(\frac{1}{5}e^{\frac{1}{10}} + \frac{3}{10}e^{\frac{1}{5}} + \frac{1}{10}e^{\frac{3}{10}} + \frac{1}{5}e^{\frac{2}{5}} + \frac{1}{5}e^{\frac{1}{2}}\right)\right] \\ &\approx 0.9287149 \leq 1, \end{aligned}$$

and

$$\begin{aligned} &\frac{e^{MT}}{e^{MT} - 1}\int_0^T\left[H\int_0^s k\left(s, r\right)dr + K\int_0^T h\left(s, r\right)dr\right]ds + \frac{e^{MT}}{e^{MT} - 1}\sum_{k=1}^{m} L_k\left(\sum_{l=1}^{c_k}\rho_l^k\right) \\ &= \frac{e}{e - 1}\int_0^1\left[\frac{1}{2}\int_0^s \mathrm{srdr} + \frac{1}{3}\int_0^1 \mathrm{srdr}\right]ds + \frac{e}{e - 1}\left(\frac{1}{4}\right)\left(\frac{1}{5} + \frac{3}{10} + \frac{1}{10} + \frac{1}{5} + \frac{1}{5}\right) \\ &\approx 0.6261991 < 1. \end{aligned}$$

Therefore, (4.1) satisfies all conditions of Theorem 3.1. So, BVP (4.1) has minimal and maximal solutions in the segment $[\beta_0, \alpha_0]$.

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Authors' contributions
All authors contributed equally in this article. They read and approved the final manuscript.

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References
1. Lakshmikantham, V, Bainov, DD, Simeonov, PS: Theory of Impulsive Differential Equations. World Scientific, Singapore. (1989)
2. Bainov, DD, Simeonov, PS: Impulsive Differential Equations: Periodic Solutions and Applications. Longman Scientific & Technical, New York. (1993)
3. Samoilenko, AM, Perestyuk, NA: Impulsive Differential Equations. World Scientific, Singapore. (1995)
4. Ding, W, Mi, J, Han, M: Periodic boundary value problems for the first order impulsive functional differential equations. Appl Math Comput. **165**, 433–446 (2005). doi:10.1016/j.amc.2004.06.022
5. Zhang, F, Li, M, Yan, J: Nonhomogeneous boundary value problem for first-order impulsive differential equations with delay. Comput Math Appl. **51**, 927–936 (2006). doi:10.1016/j.camwa.2005.11.028
6. Chen, L, Sun, J: Nonlinear boundary problem of first order impulsive integro-differential equations. J Comput Appl Math. **202**, 392–401 (2007). doi:10.1016/j.cam.2005.10.041
7. Liang, R, Shen, J: Periodic boundary value problem for the first order impulsive functional differential equations. J Comput Appl Math. **202**, 498–510 (2007). doi:10.1016/j.cam.2006.03.017
8. Ding, W, Xing, Y, Han, M: Anti-periodic boundary value problems for first order impulsive functional differential equations. Appl Math Comput. **186**, 45–53 (2007). doi:10.1016/j.amc.2006.07.087
9. Yang, X, Shen, J: Nonlinear boundary value problems for first order impulsive functional differential equations. Appl Math Comput. **189**, 1943–1952 (2007). doi:10.1016/j.amc.2006.12.085
10. Luo, Z, Jing, Z: Periodic boundary value problem for first-order impulsive functional differential equations. Comput Math Appl. **55**, 2094–2107 (2008). doi:10.1016/j.camwa.2007.08.036
11. Wang, X, Zhang, J: Impulsive anti-periodic boundary value problem of first-order integro-differential equations. J Comput Appl Math. **234**, 3261–3267 (2010). doi:10.1016/j.cam.2010.04.024
12. Song, G, Zhao, Y, Sun, X: Integral boundary value problems for first order impulsive integro-differential equations of mixed type. J Comput Appl Math. **235**, 2928–2935 (2011). doi:10.1016/j.cam.2010.12.007
13. Nieto, JJ, Rodríguez-López, R: Existence and approximation of solutions for nonlinear functional differential equations with periodic boundary value conditions. Comput Appl Math. **40**, 433–442 (2000). doi:10.1016/S0898-1221(00)00171-1
14. Nieto, JJ, Rodríguez-López, R: Periodic boundary value problem for non-Lipschitzian impulsive functional differential equations. J Math Anal Appl. **318**, 593–610 (2006). doi:10.1016/j.jmaa.2005.06.014
15. He, Z, Yu, J: Periodic boundary value problem for first-order impulsive functional differential equations. J Comput Appl Math. **138**, 205–217 (2002). doi:10.1016/S0377-0427(01)00381-8
16. He, Z, Yu, J: Periodic boundary value problem for first-order impulsive ordinary differential equations. J Math Anal Appl. **272**, 67–78 (2002). doi:10.1016/S0022-247X(02)00133-6
17. Chen, L, Sun, J: Nonlinear boundary value problem of first order impulsive functional differential equations. J Math Anal Appl. **318**, 726–741 (2006). doi:10.1016/j.jmaa.2005.08.012
18. Chen, L, Sun, J: Nonlinear boundary value problem for first order impulsive integro-differential equations of mixed type. J Math Anal Appl. **325**, 830–842 (2007). doi:10.1016/j.jmaa.2006.01.084
19. Wang, G, Zhang, L, Song, G: Extremal solutions for the first order impulsive functional differential equations with upper and lower solutions in reversed order. J Comput Appl Math. **235**, 325–333 (2010). doi:10.1016/j.cam.2010.06.014
20. Zhang, L: Boundary value problem for first order impulsive functional integro-differential equations. J Comput Appl Math. **235**, 2442–2450 (2011). doi:10.1016/j.cam.2010.10.045
21. Zhang, Y, Zhang, F: Multi-point boundary value problem of first order impulsive functional differential equations. J Appl Math Comput. **31**, 267–278 (2009). doi:10.1007/s12190-008-0209-2
22. Tariboon, J: Boundary value problems for first order functional differential equations with impulsive integral conditions. J Comput Appl Math. **234**, 2411–2419 (2010). doi:10.1016/j.cam.2010.03.007
23. Liu, Z, Han, J, Fang, L: Integral boundary value problems for first order integro-differential equations with impulsive integral conditions. Comput Math Appl. **61**, 3035–3043 (2011). doi:10.1016/j.camwa.2011.03.094

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