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Periodic solutions for nonautonomous second order Hamiltonian systems with sublinear nonlinearity

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Abstract

Some existence and multiplicity of periodic solutions are obtained for nonautonomous second order Hamiltonian systems with sublinear nonlinearity by using the least action principle and minimax methods in critical point theory.

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1 Introduction and main results

Consider the second order systems

$$\begin{cases} \ddot{u}(t) = \nabla F(t, u(t)) \quad \text{a.e. } t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases} \tag{1.1}$$

where $T > 0$ and $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following assumption:

(A) $F(t, x)$ is measurable in t for every $x \in \mathbb{R}^N$ and continuously differentiable in x for a.e. $t \in [0, T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1(0, T; \mathbb{R}^+)$ such that

$$|F(t,x)| \le a(|x|)b(t), \quad |\nabla F(t,x)| \le a(|x|)b(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$.

The existence of periodic solutions for problem (1.1) has been studied extensively, a lot of existence and multiplicity results have been obtained, we refer the readers to [1-13] and the reference therein. In particular, under the assumptions that the nonlinearity $\nabla F(t, x)$ is bounded, that is, there exists $p(t) \in L^1(0, T; \mathbb{R}^+)$ such that

$$|\nabla F(t,x)| \le p(t) \tag{1.2}$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$, and that

$$\int_0^T F(t,x)\mathrm{d}t \to \pm\infty \quad \text{as } |x| \to +\infty, \tag{1.3}$$

Mawhin and Willem in [3] have proved that problem (1.1) admitted a periodic solution. After that, when the nonlinearity $\nabla F(t, x)$ is sublinear, that is, there exists $f(t)$, $g(t) \in L^1(0, T; \mathbb{R}^+)$ and $\alpha \in [0, 1)$ such that

$$|\nabla F(t,x)| \le f(t)|x|^{\alpha} + g(t) \tag{1.4}$$

for all $x \in \mathbb{R}^{\mathbb{N}}$ and a.e. $t \in [0, T]$, Tang in [7] have generalized the above results under the hypotheses

$$\frac{1}{|x|^{2\alpha}} \int_0^T F(t,x)\mathrm{d}t \to \pm\infty \quad \text{as } |x| \to +\infty. \tag{1.5}$$

Subsequently, Meng and Tang in [13] further improved condition (1.5) with $\alpha \in (0, 1)$ by using the following assumptions

$$\liminf_{|x|\to+\infty} \frac{1}{|x|^{2\alpha}} \int_0^T F(t,x)\mathrm{d}t > \frac{T}{24}\left(\int_0^T f(t)\mathrm{d}t\right)^2, \tag{1.6}$$

$$\limsup_{|x|\to+\infty} \frac{1}{|x|^{2\alpha}} \int_0^T F(t,x)\mathrm{d}t < -\frac{T}{8}\left(\int_0^T f(t)\mathrm{d}t\right)^2. \tag{1.7}$$

Recently, authors in [14] investigated the existence of periodic solutions for the second order nonautonomous Hamiltonian systems with p-Laplacian, here $p > 1$, it is assumed that the nonlinearity $\nabla F(t, x)$ may grow slightly slower than $|x|^{p-1}$, a typical example with $p = 2$ is

$$\nabla F(t,x) = \frac{t|x|}{\ln(100 + |x|^2)}, \tag{1.8}$$

solutions are found as saddle points to the corresponding action functional. Furthermore, authors in [12] have extended the ideas of [14], replacing in assumptions (1.4) and (1.5) the term $|x|$ with a more general function $h(|x|)$, which generalized the results of [3,7,10,11]. Concretely speaking, it is assumed that there exist $f(t), g(t) \in L^1(0, T; \mathbb{R}^+)$ and a nonnegative function $h \in C([0, +\infty), [0, +\infty))$ such that

$$|\nabla F(t,x)| \le f(t)h(|x|) + g(t)$$

for all $x \in \mathbb{R}^{\mathbb{N}}$ and a.e. $t \in [0, T]$, and that

$$\frac{1}{h^2(|x|)} \int_0^T F(t,x)\mathrm{d}t \to \pm\infty \quad \text{as —}x| \to +\infty,$$

where h be a control function with the properties:

$$\begin{array}{ll} (a)\ h(s) \le h(t) & \forall s \le t,\ s,t \in [0,+\infty), \\ (b)\ h(s+t) \le C^*(h(s)+h(t)) & \forall s,t \in [0,+\infty), \\ (c)\ 0 \le h(t) \le K_1 t^{\alpha} + K_2 & \forall t \in [0,+\infty), \\ (d)\ h(t) \to +\infty & \text{as } t \to +\infty, \end{array}$$

if $\alpha = 0$, $h(t)$ only need to satisfy conditions (a)-(c), here C^*, K_1 and K_2 are positive constants. Moreover, $\alpha \in [0, 1)$ is posed. Under these assumptions, periodic solutions of problem (1.1) are obtained. In addition, if the nonlinearity $\nabla F(t, x)$ grows more faster at infinity with the rate like $\frac{|x|}{\ln(100+|x|^2)}$, $f(t)$ satisfies some certain restrictions and

α is required in a more wider range, say, $\alpha \in [0,1]$, periodic solutions have also been established in [12] by minimax methods.

An interesting question naturally arises: Is it possible to handle both the case such as (1.8) and some cases like (1.4), (1.5), in which only $f(t) \in L^1(0, T; \mathbb{R}^+)$ and $\alpha \in [0, 1)$? In this paper, we will focus on this problem.

We now state our main results.

Theorem 1.1. *Suppose that F satisfies assumption* (*A*) *and the following conditions:*

(S_1) *There exist constants* $C \geq 0$, $C^* > 0$ *and a positive function* $h \in C(\mathbb{R}^+, \mathbb{R}^+)$ *with the properties:*

$$\begin{array}{ll} \text{(i) } h(s) \leq h(t) + C & \forall s \leq t,\ s, t \in \mathbb{R}^+, \\ \text{(ii) } h(s+t) \leq C^*(h(s) + h(t)) & \forall s, t \in \mathbb{R}^+, \\ \text{(iii) } th(t) - 2H(t) \to -\infty & \text{as } t \to +\infty, \\ \text{(iv) } \dfrac{H(t)}{t^2} \to 0 & \text{as } t \to +\infty, \end{array}$$

where $H(t) := \int_0^t h(s)\,ds$. *Moreover, there exist* $f \in L^1(0, T; \mathbb{R}^+)$ *and* $g \in L^1(0, T; \mathbb{R}^+)$ *such that*

$$|\nabla F(t, x)| \leq f(t)h(|x|) + g(t)$$

for all $x \in \mathbb{R}^{\mathbb{N}}$ *and a.e.* $t \in [0, T]$;

(S_2) *There exists a positive function* $h \in C(\mathbb{R}^+, \mathbb{R}^+)$ *which satisfies the conditions* (i)-(iv) *and*

$$\liminf_{|x| \to +\infty} \frac{1}{H(|x|)} \int_0^T F(t, x)\mathrm{d}t > 0.$$

Then, problem (1.1) *has at least one solution which minimizes the functional* ϕ *given by*

$$\varphi(u) := \frac{1}{2}\int_0^T |\dot{u}(t)|^2 \mathrm{d}t + \int_0^T [F(t, u(t)) - F(t, 0)]\,\mathrm{d}t$$

on the Hilbert space H_T^1*defined by*

$$H_T^1 := \left\{ u : [0, T] \to \mathbb{R}^{\mathbb{N}} \mid u \text{ is absolutely continuous}, u(0) = u(T), \dot{u} \in L^2(0, T; \mathbb{R}^{\mathbb{N}}) \right\}$$

with the norm

$$||u|| := \left(\int_0^T |u(t)|^2 \mathrm{d}t + \int_0^T |\dot{u}(t)|^2 \mathrm{d}t \right)^{1/2}.$$

Theorem 1.2. *Suppose that* (S_1) *and assumption* (*A*) *hold. Assume that*

$$(S_3) \quad \limsup_{|x| \to +\infty} \frac{1}{H(|x|)} \int_0^T F(t, x)\mathrm{d}t < 0.$$

Then, problem (1.1) *has at least one solution in* H_T^1.

Theorem 1.3. *Suppose that* (S_1), (S_3) *and assumption* (*A*) *hold. Assume that there exist* $\delta > 0$, $\varepsilon > 0$ *and an integer* $k > 0$ *such that*

$$-\frac{1}{2}(k+1)^2\omega^2|x|^2 \le F(t,x)-F(t,0) \tag{1.9}$$

for all $x \in \mathbb{R}^{\mathbb{N}}$ *and a.e.* $t \in [0, T]$, *and*

$$F(t,x)-F(t,0) \le -\frac{1}{2}k^2\omega^2(1+\varepsilon)|x|^2 \tag{1.10}$$

for all $|x| \le \delta$ *and a.e.* $t \in [0, T]$, *where* $\omega = \frac{2\pi}{T}$. *Then, problem* (1.1) *has at least two distinct solutions in* H_T^1.

Theorem 1.4. *Suppose that* (S_1), (S_2) *and assumption* (*A*) *hold. Assume that there exist* $\delta > 0$, $\varepsilon > 0$ *and an integer* $k \ge 0$ *such that*

$$-\frac{1}{2}(k+1)^2\omega^2|x|^2 \le F(t,x)-F(t,0) \le -\frac{1}{2}k^2\omega^2|x|^2 \tag{1.11}$$

for all $|x| \le \delta$ *and a.e.* $t \in [0, T]$. *Then, problem* (1.1) *has at least three distinct solutions in* H_T^1.

Remark 1.1.

(i) Let $\alpha \in [0, 1)$, in Theorems 1.1-1.4, $\nabla F(t, x)$ does not need to be controlled by $|x|^{2\alpha}$ at infinity; in particular, we can not only deal with the case in which $\nabla F(t, x)$ grows slightly faster than $|x|^{2\alpha}$ at infinity, such as the example (1.8), but also we can treat the cases like (1.4), (1.5).

(ii) Compared with [12], we remove the restriction on the function $f(t)$ as well as the restriction on the range of $\alpha \in [0, 1]$ when we are concerned with the cases like (1.8).

(iii) Here, we point out that introducing the control function $h(t)$ has also been used in [12,14], however, these control functions are different from ours because of the distinct characters of $h(t)$.

Remark 1.2. From (i) of (S_1), we see that, nonincreasing control functions $h(t)$ can be permitted. With respect to the detailed example on this assertion, one can see Example 4.3 of Section 4.

Remark 1.3. There are functions $F(t, x)$ satisfying our theorems and not satisfying the results in [1-14]. For example, consider function

$$F(t,x) = f(t)\frac{|x|^2}{\ln(100+|x|^2)},$$

where $f(t) \in L^1(0, T; \mathbb{R}^+)$ and $f(t) > 0$ for a.e. $t \in [0, T]$. It is apparent that

$$|\nabla F(t,x)| \le 4f(t)\frac{|x|}{\ln(100+|x|^2)} \tag{1.12}$$

for all $x \in \mathbb{R}^{\mathbb{N}}$ and $t \in [0, T]$. (1.12) shows that (1.4) does not hold for any $\alpha \in [0, 1)$, moreover, note $f(t)$ only belongs to $L^1(0, T; \mathbb{R}^+)$ and no further requirements on the upper bound of $\int_0^T f(t)\mathrm{d}t$ are posed, then the approach of [12] cannot be repeated. This example cannot be solved by earlier results, such as [1-13].

On the other hand, take $h(t) = \frac{t}{\ln(100+t^2)}$, $H(t) = \int_0^t \frac{s}{\ln(100+s^2)}\,\mathrm{d}s$, $C = 0$, $C^* = 1$, then by simple computation, one has

$$\begin{aligned}
&\text{(i) } h(s) \le h(t) && \forall s \le t,\ s, t \in \mathbb{R}^+,\\
&\text{(ii) } h(s+t) = \frac{s+t}{\ln\left(100 + (s+t)^2\right)} \le h(s) + h(t) && \forall s, t \in \mathbb{R}^+,\\
&\text{(iii) } th(t) - 2H(t) = \frac{t^2}{\ln(100+t^2)} - 2\int_0^t \frac{1}{\ln(100+s^2)}\mathrm{d}\left(\frac{1}{2}s^2\right) &&\\
&\qquad = -\int_0^t \frac{2s^3}{(100+s^2)\ln^2(100+s^2)}\mathrm{d}s &&\\
&\qquad \to -\infty && \text{as } t \to +\infty,\\
&\text{(iv) } \frac{H(t)}{t^2} = \frac{\int_0^t \frac{s}{\ln(100+s^2)}\mathrm{d}s}{t^2} \to 0 && \text{as } t \to +\infty,
\end{aligned}$$

and

$$\frac{1}{H(|x|)}\int_0^T F(t,x)\mathrm{d}t = 2\int_0^T f(t)\mathrm{d}t > 0 \quad \text{as } |x| \to +\infty.$$

Hence, (S_1) and (S_2) are hold, by Theorem 1.1, problem (1.1) has at least one solution which minimizes the functional ϕ in H_T^1.

What's more, Theorem 1.1 can also deal with some cases which satisfy the conditions (1.4) and (1.5). For instance, consider function

$$F(t,x) = (0.6T - t)|x|^{\frac{3}{2}} + (q(t), x),$$

where $q(t) \in L^1(0, T; \mathbb{R}^{\mathbb{N}})$. It is not difficult to see that

$$|\nabla F(t,x)| \le \frac{3}{2}|0.6T - t||x|^{\frac{1}{2}} + |q(t)|$$

for all $x \in \mathbb{R}^{\mathbb{N}}$ and a.e. $t \in [0, T]$. Choose $h(t) = t^{\frac{1}{2}}$, $H(t) = \frac{2}{3}t^{\frac{3}{2}}$, $C = 0$, $C^* = 1$, $f(t) = \frac{3}{2}|0.6T - t|$ and $g(t) = |q(t)|$, then (S_1) and (S_2) hold, by Theorem 1.1, problem (1.1) has at least one solution which minimizes the functional ϕ in H_T^1. However, we can find that the results of [14] cannot cover this case. More examples are drawn in Section 4.

Our paper is organized as follows. In Section 2, we collect some notations and give a result regrading properties of control function $h(t)$. In Section 3, we are devote to the proofs of main theorems. Finally, we will give some examples to illustrate our results in Section 4.

2 Preliminaries

For $u \in H_T^1$, let $\bar{u} := \frac{1}{T}\int_0^T u(t)\mathrm{d}t$ and $\tilde{u}(t) := u(t) - \bar{u}$, then one has

$$\|\tilde{u}\|_\infty^2 \le \frac{T}{12}\int_0^T |\dot{u}(t)|^2\mathrm{d}t \quad \text{(Sobolev's inequality)},$$

and

$$\int_0^T |\tilde{u}(t)|^2\mathrm{d}t \le \frac{T^2}{4\pi^2}\int_0^T |\dot{u}(t)|^2\mathrm{d}t \quad \text{(Wirtinger's inequality)},$$

where $\|\tilde{u}\|_\infty := \max\limits_{0 \le t \le T}|\tilde{u}(t)|$.

It follows from assumption (A) that the corresponding function ϕ on H_T^1 given by

$$\varphi(u) := \frac{1}{2}\int_0^T |\dot{u}(t)|^2 \mathrm{d}t + \int_0^T [F(t, u(t)) - F(t, 0)]\, \mathrm{d}t$$

is continuously differentiable and weakly lower semi-continuous on H_T^1 (see[2]). Moreover, one has

$$(\varphi'(u), v) = \int_0^T (\dot{u}(t), \dot{v}(t))\mathrm{d}t + \int_0^T (\nabla F(t, u(t)), v(t))\mathrm{d}t$$

for all $u, v \in H_T^1$. It is well known that the solutions of problem (1.1) correspond to the critical point of ϕ.

In order to prove our main theorems, we prepare the following auxiliary result, which will be used frequently later on.

Lemma 2.1. *Suppose that there exists a positive function h which satisfies the conditions* (i), (iii), (iv) *of* (S_1), *then we have the following estimates:*

$$\begin{array}{ll} (1)\ 0 < h(t) \le \varepsilon t + C_0 & \textit{for any } \varepsilon > 0, C_0 > 0, t \in \mathbb{R}^+, \\ (2)\ \frac{h^2(t)}{H(t)} \to 0 & \textit{as } t \to +\infty, \\ (3)\ H(t) \to +\infty & \textit{as } t \to +\infty. \end{array}$$

Proof. It follows from (iv) of (S_1) that, for any ε >0, there exists M_1 >0 such that

$$H(t) \le \varepsilon t^2 \qquad \forall t \ge M_1. \tag{2.1}$$

By (iii) of (S_1), there exists M_2 >0 such that

$$th(t) - 2H(t) \le 0 \qquad \forall t \ge M_2, \tag{2.2}$$

which implies that

$$h(t) \le \frac{2H(t)}{t} \le \varepsilon t \qquad \forall t \ge M, \tag{2.3}$$

where $M := \max\{M_1, M_2\}$. Hence, we obtain

$$h(t) \le \varepsilon t + h(M) + C \tag{2.4}$$

for all t >0 by (i) of (S_1). Obviously, $h(t)$ satisfies (1) due to the definition of $h(t)$ and (2.4).

Next, we come to check condition (2). Recalling the property (iv) of (S_1) and (2.2), we get

$$0 < \frac{h^2(t)}{H(t)} = \frac{h^2(t)}{H^2(t)} \cdot H(t) \le \left(\frac{2}{t}\right)^2 \cdot H(t) = 4 \cdot \frac{H(t)}{t^2} \to 0 \quad \textit{as } t \to +\infty.$$

Therefore, condition (2) holds.

Finally, we show that (3) is also true. By (iii) of (S_1), one arrives at, for every β >0, there exists M_3 >0 such that

$$th(t) - 2H(t) \le -2\beta \qquad \forall t \ge M_3. \tag{2.5}$$

Let $\theta \ge 1$, using (2.5) and integrating the relation

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\frac{H(\theta t)}{\theta^2}\right] = \frac{\theta t \cdot h(\theta t) - 2H(\theta t)}{\theta^3} \le \frac{-2\beta}{\theta^3}$$

over an interval $[1, S] \subset [1, +\infty)$, we obtain

$$\frac{H(St)}{S^2} - H(t) \le \beta \left[\frac{1}{S^2} - 1 \right].$$

Thus, since $\lim\limits_{S \to +\infty} \frac{H(St)}{S^2} = 0$ by (iv) of (S_1), one has

$$H(t) \ge \beta$$

for all $t \ge M_3$. That is,

$$H(t) \to +\infty \qquad \text{as } t \to +\infty,$$

which completes the proof. □

3 Proof of main results

For the sake of convenience, we will denote various positive constants as C_i, $i = 1, 2, 3, \ldots$. Now, we are ready to proof our main results.

Proof of Theorem 1.1. For $u \in H_T^1$, it follows from (S_1), Lemma 2.1 and Sobolev's inequality that

$$\begin{aligned}
&\left| \int_0^T [F(t, u(t)) - F(t, \bar{u})] \mathrm{d}t \right| \\
&= \left| \int_0^T \int_0^1 (\nabla F(t, \bar{u} + s\tilde{u}(t)), \tilde{u}(t)) \mathrm{d}s \mathrm{d}t \right| \\
&\le \int_0^T \int_0^1 f(t) h(|\bar{u} + s\tilde{u}(t)|) |\tilde{u}(t)| \mathrm{d}s \mathrm{d}t + \int_0^T \int_0^1 g(t) |\tilde{u}(t)| \mathrm{d}s \mathrm{d}t \\
&\le \int_0^T \int_0^1 f(t) \left[h(|\bar{u}| + |\tilde{u}(t)|) + C \right] |\tilde{u}(t)| \mathrm{d}s \mathrm{d}t + ||\tilde{u}||_\infty \int_0^T g(t) \mathrm{d}t \\
&\le \int_0^T \int_0^1 f(t) \left[C^* \left(h(|\bar{u}|) + h(|\tilde{u}(t)|) \right) + C \right] |\tilde{u}(t)| \mathrm{d}s \mathrm{d}t + ||\tilde{u}||_\infty \int_0^T g(t) \mathrm{d}t \\
&\le C^* [h(|\bar{u}|) + h(|\tilde{u}(t)|)] ||\tilde{u}||_\infty \int_0^T f(t) \mathrm{d}t + C ||\tilde{u}||_\infty \int_0^T f(t) \mathrm{d}t + ||\tilde{u}||_\infty \int_0^T g(t) \mathrm{d}t \qquad (3.1) \\
&\le C^* \left[\frac{3}{C^* T} ||\tilde{u}||_\infty^2 + \frac{C^* T}{3} h^2(|\bar{u}|) \left(\int_0^T f(t) \mathrm{d}t \right)^2 \right] + ||\tilde{u}||_\infty \int_0^T g(t) \mathrm{d}t \\
&\quad + C^* [h(||\tilde{u}||_\infty) + C] ||\tilde{u}||_\infty \int_0^T f(t) \mathrm{d}t + C ||\tilde{u}||_\infty \int_0^T f(t) \mathrm{d}t \\
&\le \frac{1}{4} \int_0^T |\dot{u}(t)|^2 \mathrm{d}t + C_1 h^2(|\bar{u}|) + C^* \left[\varepsilon ||\tilde{u}||_\infty + C_0 + C \right] ||\tilde{u}||_\infty \int_0^T f(t) \mathrm{d}t \\
&\quad + C ||\tilde{u}||_\infty \int_0^T f(t) \mathrm{d}t + ||\tilde{u}||_\infty \int_0^T g(t) \mathrm{d}t \\
&\le \left(\frac{1}{4} + \varepsilon C_2 \right) \int_0^T |\dot{u}(t)|^2 \mathrm{d}t + C_1 h^2(|\bar{u}|) + C_3 \left(\int_0^T |\dot{u}(t)|^2 \mathrm{d}t \right)^{1/2},
\end{aligned}$$

which implies that

$$\begin{aligned}\varphi(u) &= \frac{1}{2}\int_0^T |\dot{u}(t)|^2\mathrm{d}t + \int_0^T [F(t,u(t)) - F(t,\bar{u})]\mathrm{d}t + \int_0^T F(t,\bar{u})\mathrm{d}t - \int_0^T F(t,0)\mathrm{d}t \\ &\geq \left(\frac{1}{4} - \varepsilon C_2\right)\int_0^T |\dot{u}(t)|^2\mathrm{d}t - C_3\left(\int_0^T |\dot{u}(t)|^2\mathrm{d}t\right)^{1/2} - \int_0^T F(t,0)\mathrm{d}t \\ &\quad + H(|\bar{u}|)\left[\frac{1}{H(|\bar{u}|)}\int_0^T F(t,\bar{u})\mathrm{d}t - C_1\frac{h^2(|\bar{u}|)}{H(|\bar{u}|)}\right].\end{aligned} \tag{3.2}$$

Taking into account Lemma 2.1 and (S_2), one has

$$H(|\bar{u}|)\left[\frac{1}{H(|\bar{u}|)}\int_0^T F(t,\bar{u})\mathrm{d}t - C_1\frac{h^2(|\bar{u}|)}{H(|\bar{u}|)}\right] \to +\infty \qquad \text{as —}\bar{u}| \to +\infty. \tag{3.3}$$

As $||u|| \to +\infty$ if and only if $\left(|\bar{u}|^2 + \int_0^T |\dot{u}(t)|^2\mathrm{d}t\right)^{1/2} \to +\infty$, for ε small enough, (3.2) and (3.3) deduce that

$$\varphi(u) \to +\infty \quad \text{as } ||u|| \to +\infty.$$

Hence, by the least action principle, problem (1.1) has at least one solution which minimizes the function ϕ in H_T^1. □

Proof of Theorem 1.2. First, we prove that ϕ satisfies the (PS) condition. Suppose that $\{u_n\} \subset H_T^1$ is a (PS) sequence of ϕ, that is, $\phi'(u_n) \to 0$ as $n \to +\infty$ and $\{\phi(u_n)\}$ is bounded. In a way similar to the proof of Theorem 1.1, we have

$$\begin{aligned}&\left|\int_0^T (\nabla F(t,u_n(t)), \tilde{u}_n(t))\mathrm{d}t\right| \\ &\leq \left(\frac{1}{4} + \varepsilon C_2\right)\int_0^T |\dot{u}_n(t)|^2\mathrm{d}t + C_1 h^2(|\bar{u}_n|) + C_3\left(\int_0^T |\dot{u}_n(t)|^2\mathrm{d}t\right)^{1/2}\end{aligned}$$

for all n. Hence, we get

$$\begin{aligned}||\tilde{u}_n|| &\geq (\varphi'(u_n), \tilde{u}_n) = \int_0^T |\dot{u}_n(t)|^2\mathrm{d}t + \int_0^T (\nabla F(t,u_n(t)), \tilde{u}_n(t))\mathrm{d}t \\ &\geq \left(\frac{3}{4} - \varepsilon C_2\right)\int_0^T |\dot{u}_n(t)|^2\mathrm{d}t - C_1 h^2(|\bar{u}_n|) - C_3\left(\int_0^T |\dot{u}_n(t)|^2\mathrm{d}t\right)^{1/2}\end{aligned} \tag{3.4}$$

for large n. On the other hand, it follows from Wirtinger's inequality that

$$||\tilde{u}_n|| \leq \left(\frac{T^2}{4\pi^2} + 1\right)^{1/2}\left(\int_0^T |\dot{u}_n(t)|^2\mathrm{d}t\right)^{1/2} \tag{3.5}$$

for all n. Combining (3.4) with (3.5), we obtain

$$C_4 h(|\bar{u}_n|) \geq \left(\int_0^T |\dot{u}_n(t)|^2 \mathrm{d}t\right)^{1/2} - C_5 \tag{3.6}$$

for all large n. By (3.1), (3.6), Lemma 2.1 and (S_3), one has

$$\begin{aligned}
\varphi(u_n) &= \frac{1}{2}\int_0^T |\dot{u}_n(t)|^2 \mathrm{d}t + \int_0^T [F(t, u_n(t)) - F(t, \bar{u}_n)]\ \mathrm{d}t \\
&\quad + \int_0^T F(t, \bar{u}_n)\mathrm{d}t - \int_0^T F(t, 0)\mathrm{d}t \\
&\leq \left(\frac{3}{4} + \varepsilon C_2\right)\int_0^T |\dot{u}_n(t)|^2 \mathrm{d}t + C_1 h^2(|\bar{u}_n|) + C_3\left(\int_0^T |\dot{u}(t)|^2 \mathrm{d}t\right)^{1/2} \\
&\quad + \int_0^T F(t, \bar{u}_n)\mathrm{d}t - \int_0^T F(t, 0)\mathrm{d}t \\
&\leq C_6[C_4 h(|\bar{u}_n|) + C_5]^2 + C_1 h^2(|\bar{u}_n|) + C_3[C_4 h(|\bar{u}_n|) + C_5] \\
&\quad + \int_0^T F(t, \bar{u}_n)\mathrm{d}t - \int_0^T F(t, 0)\mathrm{d}t \\
&\leq C_7 h^2(|\bar{u}_n|) + C_8 h(|\bar{u}_n|) + C_9 + \int_0^T F(t, \bar{u}_n)\mathrm{d}t - \int_0^T F(t, 0)\mathrm{d}t \\
&\leq H(|\bar{u}_n|)\left[C_7 \frac{h^2(|\bar{u}_n|)}{H(|\bar{u}_n|)} + C_8 \frac{h(|\bar{u}_n|)}{H(|\bar{u}_n|)} + \frac{1}{H(|\bar{u}_n|)}\int_0^T F(t, \bar{u}_n)\mathrm{d}t\right] \\
&\quad + C_9 - \int_0^T F(t, 0)\mathrm{d}t \to -\infty \quad \text{as } |\bar{u}_n| \to +\infty.
\end{aligned}$$

This contradicts the boundedness of $\{\phi(u_n)\}$. So, $\{\bar{u}_n\}$ is bounded. Notice (3.6) and (1) of Lemma 2.1, hence $\{u_n\}$ is bounded. Arguing then as in Proposition 4.1 in [3], we conclude that the (PS) condition is satisfied.

In order to apply the saddle point theorem in [2,3], we only need to verify the following conditions:

(ϕ1) $\phi(u) \to +\infty$ as $||u|| \to +\infty$in $\tilde{H}_T^1$, where $\tilde{H}_T^1 := \left\{u \in H_T^1 |\ \bar{u} = 0\right\}$,
(ϕ2) $\phi(u) \to -\infty$ as $|u(t)| \to +\infty$.

In fact, for all $u \in \tilde{H}_T^1$, by (S_1), Sobolev's inequality and Lemma 2.1, we have

$$\begin{aligned}\left|\int_0^T [F(t,u(t))-F(t,0)]\,\mathrm{d}t\right| &= \left|\int_0^T\int_0^1 (\nabla F(t,\bar{u}+su(t)),u(t))\,\mathrm{d}s\mathrm{d}t\right|\\ &\le \int_0^T f(t)h(|su(t)|)|u(t)|\mathrm{d}t+\int_0^T g(t)|u(t)|\mathrm{d}t\\ &\le \int_0^T f(t)[h(|u(t)|)+C]|u(t)|\mathrm{d}t\ +||u||_\infty\int_0^T g(t)\mathrm{d}t\\ &\le \varepsilon||u||_\infty^2\int_0^T f(t)\mathrm{d}t+(C_0+C)||u||_\infty\int_0^T f(t)\mathrm{d}t\ +||u||_\infty\int_0^T g(t)\mathrm{d}t\\ &\le \varepsilon C_{10}\int_0^T|\dot{u}(t)|^2\mathrm{d}t+C_{11}\left(\int_0^T|\dot{u}(t)|^2\mathrm{d}t\right)^{1/2},\end{aligned}$$

which implies that

$$\begin{aligned}\varphi(u) &= \frac{1}{2}\int_0^T|\dot{u}(t)|^2\mathrm{d}t+\int_0^T[F(t,u(t))-F(t,0)]\ \mathrm{d}t\\ &\ge \left(\frac{1}{2}-\varepsilon C_{10}\right)\int_0^T|\dot{u}(t)|^2\mathrm{d}t-C_{11}\left(\int_0^T|\dot{u}(t)|^2\mathrm{d}t\right)^{1/2}.\end{aligned}\tag{3.7}$$

By Wirtinger's inequality, one has

$$||u||\ \to +\infty \Leftrightarrow \left(\int_0^T|\dot{u}(t)|^2\mathrm{d}t\right)^{1/2}\ \to +\infty\qquad \text{on } \tilde{H}_T^1.$$

Hence, for ε small enough, (ϕ_1) follows from (3.7).

On the other hand, by (S_3) and Lemma 2.1, we get

$$\int_0^T F(t,u(t))\mathrm{d}t\to-\infty\qquad \text{as } |u(t)|\to+\infty\quad \text{in } \mathbb{R}^{\mathbb{N}},$$

which implies that

$$\varphi(u)=\int_0^T F(t,u(t))\mathrm{d}t-\int_0^T F(t,0)\mathrm{d}t\to-\infty\qquad \text{as } |u(t)|\to+\infty\quad \text{in } \mathbb{R}^{\mathbb{N}}.$$

Thus, (ϕ_2) is verified. The proof of Theorem 1.2 is completed. □

Proof of Theorem 1.3. Let $E=H_T^1$,

$$H_k:=\left\{\sum_{j=1}^k(a_j\cos j\omega t+b_j\sin\omega t)|a_j,b_j\in\mathbb{R}^{\mathbb{N}},j=1,2,\ldots,k\right\}$$

and ψ = -ϕ. Then, $\psi\in C^1(E,\mathbb{R})$ satisfies the (PS) condition by the proof of Theorem 1.2. In view of Theorem 5.29 and Example 5.26 in [2], we only need to prove that

(ψ_1) $\liminf ||u||^{-2}\psi(u) > 0$ as $u \to 0$ in H_k,
(ψ_2) $\psi(u) \le 0$ for all u in $H_k^{\perp}$, and
(ψ_3) $\psi(u) \to -\infty$ as $||u|| \to \infty$ in $H_{k-1}^{\perp}$.

We see that

$$F(t,x) - F(t,0) = \int_0^1 (\nabla F(t,sx), x)\mathrm{d}s$$

for all $x \in \mathbb{R}^{\mathbb{N}}$ and a.e. $t \in [0, T]$. By (S_1) and Lemma 2.1, one has

$$\begin{aligned} F(t,x) - F(t,0) &\le \int_0^1 (f(t)h(|sx|) + g(t), x)\mathrm{d}s \\ &\le f(t)[h(|x|) + C]|x| + g(t)|x| \\ &\le f(t)[\varepsilon|x| + C_0 + C]|x| + g(t)|x| \\ &= \varepsilon f(t)|x|^2 + [f(t)(C_0 + C) + g(t)]|x| \ \le Q(t)|x|^3 \end{aligned}$$

for all $|x| \ge \delta$, a.e. $t \in [0, T]$ and some $Q(t) \in L^1(0, T; \mathbb{R}^+)$ given by

$$Q(t) := \varepsilon f(t)\delta^{-1} + [f(t)(C_0 + C) + g(t)]\delta^{-2}.$$

Now, it follows from (1.10) that

$$F(t,x) - F(t,0) \le -\frac{1}{2}k^2\omega^2(1+\varepsilon)|x|^2 + Q(t)|x|^3$$

for all $x \in \mathbb{R}^{\mathbb{N}}$ and a.e. $t \in [0, T]$. Hence, we obtain

$$\begin{aligned} \psi(u) &= -\frac{1}{2}\int_0^T |\dot{u}(t)|^2\mathrm{d}t - \int_0^T [F(t,u(t)) - F(t,0)]\,\mathrm{d}t \\ &\ge -\frac{1}{2}\int_0^T |\dot{u}(t)|^2\mathrm{d}t + \frac{1}{2}k^2\omega^2(1+\varepsilon)\int_0^T |u(t)|^2\mathrm{d}t - \int_0^T Q(t)|u(t)|^3\mathrm{d}t \\ &\ge \frac{1}{2}\varepsilon\int_0^T |\dot{u}(t)|^2\mathrm{d}t + \frac{1}{2}k^2\omega^2(1+\varepsilon)|\bar{u}|^2T - ||u||_\infty^3\int_0^T Q(t)\mathrm{d}t \\ &\ge C_{12}||u||^2 - C_{13}||u||^3 \end{aligned}$$

for all $u \in H_k$. Then, (ψ_1) follows from the above inequality.

For $u \in H_k^{\perp}$, by (1.9), one has

$$\psi(u) \le -\frac{1}{2}\int_0^T |\dot{u}(t)|^2\mathrm{d}t + \frac{1}{2}(k+1)^2\omega^2\int_0^T |u(t)|^2\mathrm{d}t \le 0.$$

So, (ψ_2) is obtained. At last, (ψ_3) follows from (ϕ_1) which are appeared in the proof of Theorem 1.2. Then the proof of Theorem 1.3 is completed. □

Proof of Theorem 1.4. From the proof of Theorem 1.1, we know that ϕ is coercive which implies that ϕ satisfies the (PS) condition. With the similar manner to [4,7], we can get the multiplicity results, here we omit the details. □

4 Examples

In this section, we give some examples to illustrate our results.

Example 4.1. Consider the function

$$F(t,x)=\left(\frac{1}{3}T-t\right)\frac{|x|^2}{\ln\left(100\ +|x|^2\right)}+(d(t),x),$$

where $d(t)\in\ L^1(0,\ T;\ \mathbb{R}^{\mathbb{N}})$. Let $h(t)=\frac{t}{\ln\left(100+t^2\right)}$, then $H(t)=\int_0^t\frac{s}{\ln\left(100+s^2\right)}\,\mathrm{d}s$, by a direct computation, (S_1) and (S_3) hold. Then, by Theorem 1.2, we conclude that problem (1.1) has one solution in H_T^1. However, as the reason of Remark 1.3, the results in [1-13] cannot be applied.

Example 4.2. Consider the function

$$F(t,x)=\begin{cases}\left(\frac{2}{3}T-t\right)\frac{|x|^2}{\ln\left(100+|x|^2\right)}+A(t)|x|+B(t), & |x|\ >1,\\ -\frac{1}{4}\omega^2|x|^2+\left(\frac{1}{2}\omega^2+\frac{3}{2}T-\frac{9}{4}t\right)|x|^4-\left(\frac{1}{4}\omega^2+\frac{5}{6}T-\frac{5}{4}t\right)|x|^6, & |x|\ \le 1,\end{cases}$$

where $A(t)$, $B(t)$ are suitable functions which insure assumption (A) hold. Also, put $H(t)=\int_0^t\frac{s}{\ln\left(100+s^2\right)}\,\mathrm{d}s$, $H(t)=\int_0^t\frac{s}{\ln\left(100+s^2\right)}\,\mathrm{d}s$, we see that (S_1), (S_2) and (1.11) hold. By virtue of Theorem 1.4, problem (1.1) has at least three distinct solutions in H_T^1.

Example 4.3. Consider the function

$$F(t,x)=\left(\frac{2}{3}T-t\right)\ln(100\ +|x|^2).$$

We observe that

$$|\nabla F(t,x)|\ \le\left|\frac{2}{3}T-t\right|\frac{2|x|}{100+|x|^2}\le 2\left|\frac{2}{3}T-t\right|,$$

which means $\nabla F(t,\ x)$ is bounded, moreover, one has

$$\int\limits_0^T F(t,x)\mathrm{d}t\to+\infty\quad\text{as }|x|\to+\infty.$$

Then, by the results in [3,7,12], problem (1.1) has one solution which minimizes the functional ϕ in H_T^1.

In fact, our Theorem 1.1 can also handle this case. In this situation, let $h(t)=\frac{t}{100+t^2}$, $H(t)=\displaystyle\int_0^t\frac{s}{100+s^2}\,\mathrm{d}s$, and choose $C=2$, $C^*=1$ $f(t)=2\left|\frac{2}{3}T-t\right|$, $g(t)\equiv 0$, we infer

$$\begin{aligned}
&\text{(i)}\ \ h(s)\le h(t)+C && \forall s\le t,\ \ s,t\in\mathbb{R}^+,\\
&\text{(ii)}\ \ h(s+t)=\frac{s+t}{100+(s+t)^2}\le h(s)+h(t) && \forall s,t\in\mathbb{R}^+,\\
&\text{(iii)}\ \ th(t)-2H(t)=\frac{t^2}{100+t^2}-2\left[\frac{1}{2}\ln(100+t^2)-\frac{1}{2}\ln 100\right] && \text{as } t\to+\infty,\\
&\qquad\qquad\qquad\to-\infty &&\\
&\text{(iv)}\ \ \frac{H(t)}{t^2}=\frac{\int\limits_0^t\frac{s}{100+s^2}\mathrm{d}s}{t^2}\to 0 && \text{as } t\to+\infty.
\end{aligned}$$

and

$$\frac{1}{H(|x|)}\int_0^T F(t,x)\mathrm{d}x > 0 \quad \text{as —}x| \to +\infty.$$

So, by Theorem 1.1, problem (1.1) has one solution which minimizes the functional ϕ in H_T^1.

Remark 4.1. Unlike the control functions in [12], where $h(t)$ is nondecreasing, here control function $h(t) = \frac{t}{100+t^2}$ is bounded but not increasing.

Example 4.4. Consider the function

$$F(t,x) = \left(\frac{1}{3}T - t\right)|x|^{\frac{4}{3}} + (k(t),x),$$

where $k(t) \in L^1(0, T; \mathbb{R}^{\mathbb{N}})$. It is easy to check that

$$|\nabla F(t,x)| \le \frac{4}{3}\left|\frac{1}{3}T - t\right||x|^{\frac{1}{3}} + |k(t)|.$$

The above inequality leads to (1.4) hold with

$$f(t) = \frac{4}{3}\left|\frac{1}{3}T - t\right|, g(t) = |k(t)|.$$

Take $\alpha = \frac{1}{3}$, then

$$\frac{1}{|x|^{2\alpha}}\int_0^T F(t,x)\mathrm{d}t \to -\infty \quad \text{as } |x| \to +\infty.$$

So, by the theorems in [3,7,12,13], problem (1.1) has at least one solution in H_T^1.

Indeed, our Theorem 1.2 can also deal with this case. Let $h(t) = t^{\frac{1}{3}}$, $H(t) = \frac{3}{4}t^{\frac{4}{3}}$, and choose $C = 0$, $C^* = 1$, $f(t) = \frac{4}{3}\left|\frac{4}{3}T - t\right|$, $g(t) = |k(t)|$, we know

$$\begin{aligned}
&\text{(i)}\ \ h(s) \le h(t) && \forall s \le t,\ s,t \in \mathbb{R}^+,\\
&\text{(ii)}\ \ h(s+t) = (s+t)^{\frac{1}{3}} \le 8(h(s)+h(t)) && \forall s,t \in \mathbb{R}^+,\\
&\text{(iii)}\ \ th(t) - 2H(t) = -\frac{1}{2}t^{\frac{4}{3}} \to -\infty && \text{as } t \to +\infty,\\
&\text{(iv)}\ \ \frac{H(t)}{t^2} = \frac{3}{4t^{\frac{2}{3}}} \to 0 && \text{as } t \to +\infty.
\end{aligned}$$

Furthermore, one has

$$\frac{1}{H(|x|)}\int_0^T F(t,x)\mathrm{d}x < 0 \quad \text{as } |x| \to +\infty.$$

Hence, (S_1) and (S_3) are true, by Theorem 1.2, problem (1.1) has at least one solution in H_T^1. However, we can find that the results in [14] cannot deal with this case.

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