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Existence of multiple positive solutions for singular boundary value problems of nonlinear fractional differential equations

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Abstract

In this paper, we consider the properties of the Green's function for the nonlinear fractional differential equation boundary value problem $D^q_{0+}u(t) = f(t, u(t))$, $t \in J := [0, 1]$, $u(0) = u'(1) = 0$, where $1 < q \leq 2$ is a real number, and D^q_{0+} is the standard Riemann-Liouville differentiation. As an application of the Green's function, we give some multiple positive solutions for singular boundary value problems, and we also give the uniqueness of solution for a singular problem by means of the Leray-Schauder nonlinear alternative, a fixed-point theorem on cones, and a mixed monotone method.

Keywords: boundary value problem; fractional differential equations; Riemann-Liouville fractional derivative; positive solution; fixed-point theorem

1 Introduction

This paper is mainly concerned with the existence and multiplicity of positive solutions of the nonlinear fractional differential equation boundary value problem (BVP for short)

$$D^q_{0^+}u(t) = f\left(t, u(t)\right), \quad 0 < t < 1, \tag{1.1}$$

$$u(0) = u'(1) = 0, \tag{1.2}$$

where $1 < q \leq 2$ is a real number and $D^q_{0^+}$ is the standard Riemann-Liouville differentiation, and f is a given function satisfying some assumptions that will be specified later, with $\lim_{u\to 0} f(\cdot, u) = +\infty$ (*i.e.*, f is singular at $u = 0$).

In the last few years, fractional differential equations (in short FDEs) have been studied extensively. The motivation for those works stems from both the development of the theory of fractional calculus itself and the applications of such constructions in various sciences such as physics, mechanics, chemistry, engineering, and so on. For an extensive collection of such results, we refer the readers to the monographs by Kilbas *et al.* [1], Miller and Ross [2], Oldham and Spanier [3], Podlubny [4] and Samko *et al.* [5].

Some basic theory for the initial value problems of FDE involving Riemann-Liouville differential operator has been discussed by Lakshmikantham [6–8], Babakhani and Daftardar-Gejji [9–11] and Bai [12], and so on. Also, there are some papers which deal with the existence and multiplicity of solutions (or positive solution) for nonlinear FDE of BVPs by using techniques of nonlinear analysis (fixed-point theorems, Leray-Shauder theory, topological degree theory, *etc.*); see [13–34] and the references therein.

Bai and Lü [15] studied the following two-point boundary value problem of FDEs:

$$D_{0+}^{q}u(t) + f\bigl(t, u(t)\bigr) = 0, \qquad u(0) = u(1) = 0, \quad 0 < t < 1, 1 < q \leq 2,$$

where D_{0+}^{q} is the standard Riemann-Liouville fractional derivative. They obtained the existence of positive solutions by means of the Guo-Krasnosel'skii fixed-point theorem and the Leggett-Williams fixed-point theorem.

Zhang [23] considered the existence and multiplicity of positive solutions for the nonlinear fractional boundary value problem

$$^{c}D_{0+}^{q}u(t) = f\bigl(t, u(t)\bigr), \quad 0 < t < 1, \qquad u(0) + u'(0) = 0, \qquad u(1) + u'(1) = 0, \tag{1.3}$$

where $1 < q \leq 2$ is a real number, $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ and $^{c}D_{0+}^{q}$ is the standard Caputo's fractional derivative. The author obtained the existence and multiplicity results of positive solutions by means of the Guo-Krasnosel'skii fixed-point theorem and the Leggett-Williams fixed-point theorem.

Qiu and Bai [33] considered the existence of positive solutions for the nonlinear fractional boundary value problem

$$^{c}D_{0+}^{q}u(t) + f\bigl(t, u(t)\bigr) = 0, \quad 0 < t < 1, \qquad u(0) = u'(1) = u''(0) = 0, \tag{1.4}$$

where $2 < q \leq 3$ is a real number, $f : (0,1] \times [0,+\infty) \to [0,+\infty)$ with $\lim_{t\to 0^{+}} f(t,\cdot) = \infty$ (*i.e.*, f is singular at $t = 0$), and $^{c}D_{0+}^{q}$ is the standard Caputo's fractional derivative. The authors proved the existence of one positive solution by using the Guo-Krasnosel'skii fixed-point theorem and the nonlinear alternative of Leray-Schauder type in a cone and assuming certain hypotheses on the function f.

Mena *et al.* [34] proved the existence and uniqueness of a positive and nondecreasing solution for the problem (1.4) by using a fixed-point theorem in partially ordered sets.

From the above works, we can see a fact, although the fractional boundary value problems have been investigated by some authors, singular boundary value problems are seldom considered, in particular, f is singular at $u = 0$. Motivated by all the works above, in this paper we discuss the boundary value problem (1.1)-(1.2). Using the Leray-Schauder nonlinear alternative theorem and the Guo-Krasnosel'skii fixed-point theorem, we give some new existence criteria for the singular boundary value problem (1.1)-(1.2). Finally, we obtain new uniqueness criteria for the singular boundary value problem (1.1)-(1.2) by a mixed monotone method.

The plan of this paper is as follows. In Section 2, we shall give some definitions and lemmas to prove our main results. In Section 3, we establish the existence of multiple positive solutions for the singular boundary value problem (1.1)-(1.2) by the Leray-Schauder nonlinear alternative theorem and the Guo-Krasnosel'skii fixed-point theorem. In Section 4, by using a mixed monotone method, we obtain some new uniqueness criteria for the singular boundary value problem (1.1)-(1.2).

2 Preliminaries and lemmas

For the convenience of the reader, we present here the necessary definitions from fractional calculus theory. These definitions can be found in the recent literature such as [1, 4] and [15].

Definition 2.1 [1, 4] The fractional-order integral of the function $h \in L^1([a,b],\mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,ds,$$

where Γ is the gamma function. When $a = 0$, we write $I^\alpha h(t) = [h * \varphi_\alpha](t)$, where $\varphi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$, and $\varphi_\alpha(t) = 0$ for $t \le 0$, and $\varphi_\alpha \to \delta(t)$ as $\alpha \to 0$, where δ is the delta function.

Definition 2.2 [1, 4] For a function h given on the interval $[a,b]$, the αth Riemann-Liouville fractional-order derivative of h is defined by

$$\big(D_{a+}^\alpha h\big)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} h(s)\,ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

From the definition of the Riemann-Liouville derivative, we can obtain the statement.

Lemma 2.1 [15] *Let $\alpha > 0$. If we assume $u \in C(0,1) \cap L(0,1)$, then differential equation*

$$D_{0^+}^\alpha u(t) = 0,$$

has

$$u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N}, \quad C_i \in \mathbb{R}, i = 1,2,\ldots,N,$$

as unique solutions, where N is the smallest integer greater than or equal to α.

Lemma 2.2 [15] *Assume that $h \in C(0,1) \cap L(0,1)$ with a derivative of order $q > 0$ that belongs to $C(0,1) \cap L(0,1)$. Then*

$$I_{0^+}^q D_{0^+}^q h(t) = h(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N}$$

for some $C_i \in \mathbb{R}$, $i = 1,2,\ldots,N$, where N is the smallest integer greater than or equal to q.

In the following, we present the Green's function of the FDE boundary value problem.

Lemma 2.3 *Let $h(t) \in C[0,1]$ and $1 < q \le 2$, then the unique solution of*

$$D_{0^+}^q u(t) + h(t) = 0, \quad 0 < t < 1, \tag{2.1}$$

$$u(0) = u'(1) = 0 \tag{2.2}$$

is given by

$$u(t) = \int_0^1 G(t,s)h(s)\,ds, \tag{2.3}$$

where $G(t,s)$ *is the Green's function given by*

$$G(t,s) = \begin{cases} \frac{(1-s)^{q-2}t^{q-1}-(t-s)^{q-1}}{\Gamma(q)}, & \text{if } 0 \le s \le t \le 1, \\ \frac{(1-s)^{q-2}t^{q-1}}{\Gamma(q)}, & \text{if } 0 \le t \le s \le 1. \end{cases} \tag{2.4}$$

Proof By Lemma 2.2, we can reduce the equation of problem (2.1) to an equivalent integral equation:

$$u(t) = -I_{0^+}^q h(t) + c_1t^{q-1} + c_2t^{q-2} = -\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)\,ds + c_1t^{q-1} + c_2t^{q-2} \tag{2.5}$$

for some constants $c_1, c_2 \in \mathbb{R}$.

So

$$u'(t) = (q-1)c_1t^{q-2} - \frac{q-1}{\Gamma(q)}\int_0^t (t-s)^{q-2}h(s)\,ds.$$

Applying the boundary condition (2.2), we have

$$c_1 = \frac{1}{\Gamma(q)}\int_0^1 (1-s)^{q-2}h(s)\,ds, \qquad c_2 = 0.$$

Therefore, the unique solution of problem (2.1)-(2.2) is

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)\,ds + c_1t^{q-1} + c_2t^{q-2} \\ &= -\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)\,ds + \frac{1}{\Gamma(q)}\int_0^1 (1-s)^{q-2}t^{q-1}h(s)\,ds \\ &= \int_0^t \left[\frac{(1-s)^{q-2}t^{q-1}-(t-s)^{q-1}}{\Gamma(q)}\right]h(s)\,ds + \int_t^1 \frac{(1-s)^{q-2}t^{q-1}}{\Gamma(q)}h(s)\,ds \\ &= \int_0^1 G(t,s)h(s)\,ds, \end{aligned}$$

which completes the proof. □

The following properties of the Green's function form the basis of our main work in this paper.

Lemma 2.4 *Let* $k(t) = \frac{t^{q-1}}{\Gamma(q)}$, $g(s) = \frac{s(1-s)^{q-2}}{\Gamma(q)}$. *The function* $G(t,s)$ *defined by* (2.4) *satisfies the following conditions*:

(i) $\Gamma(q)k(t)g(s) \le G(t,s) \le k(t)(1-s)^{q-2}$ *for* $t,s \in (0,1)$;

(ii) $t^{q-1}g(s) \le G(t,s) \le t^{q-2}g(s)$ *for* $t,s \in (0,1)$;

(iii) $G(t,s) >$ *for* $t,s \in (0,1)$.

Proof (i) In the following, we consider $\Gamma(q)G(t,s)$.

When $s \leq t$, we have

$$\begin{aligned} \Gamma(q)G(t,s) &= (1-s)^{q-2}t^{q-1} - (t-s)^{q-1} \\ &= t(t-ts)^{q-2} - (t-s)^{q-1} \\ &\geq t(t-ts)^{q-2} - (t-ts)^{q-1} \\ &= ts(t-ts)^{q-2} \\ &= t^{q-1}s(1-s)^{q-2}. \end{aligned} \tag{2.6}$$

On the other hand, we have

$$\begin{aligned} \Gamma(q)G(t,s) &= (1-s)^{q-2}t^{q-1} - (t-s)^{q-1} \\ &= t(t-ts)^{q-2} - (t-s)^{q-2}(t-s) \\ &\leq t(t-ts)^{q-2} - (t-ts)^{q-2}(t-s) \\ &= s(t-ts)^{q-2} \\ &= \frac{s}{t}t^{q-1}(1-s)^{q-2} \\ &\leq t^{q-1}(1-s)^{q-2}. \end{aligned} \tag{2.7}$$

When $s \geq t$, we get

$$\Gamma(q)G(t,s) = (1-s)^{q-2}t^{q-1} \geq s(1-s)^{q-2}t^{q-1}. \tag{2.8}$$

On the other hand, we have

$$\Gamma(q)G(t,s) = (1-s)^{q-2}t^{q-1}. \tag{2.9}$$

From (2.6)-(2.9), we have (i).

(ii) When $s \leq t$, we get

$$\Gamma(q)G(t,s) \geq t^{q-1}s(1-s)^{q-2}.$$

Thus,

$$\Gamma(q)G(t,s)t^{2-q} = \left[(1-s)^{q-2}t^{q-1} - (t-s)^{q-1}\right]t^{2-q} \geq t^{q-1}s(1-s)^{q-2}t^{2-q} = ts(1-s)^{q-2}.$$

On the other hand, we have

$$\Gamma(q)G(t,s) \leq t^{q-2}s(1-s)^{q-2}.$$

So

$$\Gamma(q)G(t,s)t^{2-q} \leq \left[t^{q-2}s(1-s)^{q-2}\right]t^{2-q} = s(1-s)^{q-2}.$$

When $s \geq t$, we get

$$\Gamma(q)G(t,s) \geq t^{q-1}s(1-s)^{q-2}.$$

Thus,

$$\Gamma(q)G(t,s)t^{2-q} \geq ts(1-s)^{q-2}.$$

On the other hand, we have

$$\Gamma(q)G(t,s)t^{q-2} = (1-s)^{q-2}t^{q-1}t^{2-q} = t(1-s)^{q-2} \leq s(1-s)^{q-2}.$$

Therefore we have (ii). Clearly $G(t,s) > 0$ holds trivially. The proof is finished. □

Lemma 2.5 *The function $G^*(t,s) := t^{2-q}G(t,s)$ has the following properties*:

$$tg(s) \leq G^*(t,s) \leq g(s),$$

where

$$G^*(t,s) = \begin{cases} \frac{[(1-s)^{q-2}t^{q-1}-(t-s)^{q-1}]t^{2-q}}{\Gamma(q)}, & \text{if } 0 \leq s \leq t \leq 1, \\ \frac{t(1-s)^{q-2}}{\Gamma(q)}, & \text{if } 0 \leq t \leq s \leq 1. \end{cases}$$

Let $y(t) = t^{2-q}u(t)$, by $u(t) = \int_0^1 G(t,s)h(s)\,ds$, we get

$$y(t) = \int_0^1 t^{2-q}G(t,s)h(s)\,ds = \int_0^1 G^*(t,s)h(s)\,ds.$$

The following three theorems are fundamental in the proofs of our main results.

Lemma 2.6 [35] *Let X be a Banach space, and let $P \subset X$ be a cone in X. Assume Ω_1, Ω_2 are open subsets of X with $\theta \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $\mathcal{A}: P \to P$ be a completely continuous operator such that either*

(i) $\|\mathcal{A}u\| \leq \|u\|$, $u \in P \cap \partial\Omega_1$, $\|\mathcal{A}u\| \geq \|u\|$, $u \in P \cap \partial\Omega_2$, *or*

(ii) $\|\mathcal{A}u\| \geq \|u\|$, $u \in P \cap \partial\Omega_1$, $\|\mathcal{A}u\| \leq \|u\|$, $u \in P \cap \partial\Omega_2$.

Then $\mathcal{A}$ has a fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Let P be a normal cone of a Banach space E, and $e \in P$ with $\|e\| \leq 1$, $e \neq \theta$. Define

$$Q_e = \{x \in P | x \neq \theta, \text{there exist constants } m, M > 0 \text{ such that } me \leq x \leq Me\}. \tag{2.10}$$

Definition 2.3 [36] Assume $A : Q_e \times Q_e \to Q_e$. A is said to be mixed monotone if $A(x,y)$ is nondecreasing in x and nonincreasing in y, *i.e.*, if $x_1 \leq x_2$ $(x_1, x_2 \in Q_e)$ implies $A(x_1,y) \leq A(x_2,y)$ for any $y \in Q_e$, and $y_1 \leq y_2$ $(y_1, y_2 \in Q_e)$ implies $A(x,y_1) \geq A(x,y_2)$ for any $x \in Q_e$. $x^* \in Q_e$ is said to be a fixed point of A if $A(x^*,x^*) = x^*$.

Lemma 2.7 [36] *Suppose that $A : Q_e \times Q_e \to Q_e$ is a mixed monotone operator and $\exists$ a constant β $(0 \le \beta < 1)$ such that*

$$A\left(tx, \frac{1}{t}y\right) \ge t^{\beta} A(x,y), \quad \forall x, y \in Q_e, 0 < t < 1.$$

Then A has a unique fixed point $x^* \in Q_e$.

Lemma 2.8 [37] *Assume Ω is a relative subset of a convex set K in a normed space X. Let $A : \overline{\Omega} \to K$ be a compact map with $0 \in \Omega$. Then either*
(A1) *A has a fixed point in $\overline{\Omega}$, or*
(A2) *there is a $x \in \partial\Omega$ and a $\lambda < 1$ such that $x = \lambda A(x)$.*

3 Positive solutions of a singular problem

In this section, we establish some new existence results for the singular fractional differential equation (1.1)-(1.2). We always assume that $f : [0,1] \times (0,\infty) \to [0,\infty)$ is continuous in this section. Given $a \in L^1(0,1)$, we write a $a \succ 0$ if $a \ge 0$ for $t \in [0,1]$ and it is positive in a set of positive measure.

Theorem 3.1 *Suppose that the following hypotheses hold*:
(H1) *for each constant $L > 0$, there exists a continuous function $\phi_L \succ 0$ such that $f(t,x) \ge \phi_L(t)$; for all $t \in [0,1]$ and $x \in (0,L]$, one has $0 < \int_0^1 g(s)\phi_L(s)\,ds < \infty$;*
(H2) *there exist continuous, nonnegative functions $b(x)$ and $d(x)$ such that*

$$0 \le f(t,x) = b(x) + d(x) \quad \textit{for all } (t,x) \in [0,1] \times (0,\infty),$$

and $b(x) > 0$ is nonincreasing and $\frac{d(x)}{b(x)}$ is nondecreasing in $x \in (0,\infty)$;
(H3) *there exists a constant $K_0 > 0$ such that $b(lm) \le K_0 b(l)b(m)$ for all $l, m \ge 0$;*
(H4) $\int_0^1 b(s^{q-1})\,ds < \infty$;
(H5) *there exists a constant $r > 0$ such that*

$$\big(b(r) + d(r)\big)K_0 \int_0^1 g(s)t^{q-2}b(s)\,ds < r.$$

Then problem (1.1)-(1.2) has at least one positive solution x with $0 < \|x\| < r$.

Proof Since (H5) holds, we can choose $n_0 \in \{1,2,\ldots\}$ such that

$$\big(b(r) + d(r)\big)K_0 \int_0^1 g(s)t^{q-2}b(s)\,ds + \frac{1}{n_0} < r.$$

Let $N_0 = \{n_0, n_0 + 1, \ldots\}$. Fix $n \in N_0$ and consider the family of integral equations

$$u(t) := \lambda \int_0^1 G(t,s)f_n\big(s,u(s)\big)\,ds + \frac{1}{n}, \tag{3.1}$$

where $\lambda \in [0,1]$ and

$$f_n(t,u) = \begin{cases} f(t,u), & \text{if } u \ge \frac{1}{n}, \\ f(t,\frac{1}{n}), & \text{if } u \le \frac{1}{n}. \end{cases}$$

We claim that any solution u of (3.1) for any $\lambda \in [0,1]$ must satisfy $\|u\| \neq r$. Otherwise, assume that u is a solution of (3.1) for some $\lambda \in [0,1]$ such that $\|u\| = r$. Then $u(t) \geq \frac{1}{n}$ for $t \in [0,1]$. Note that

$$\|u\| \leq \frac{1}{n} + \lambda \int_0^1 g(s)t^{q-2} f_n\bigl(s, u(s)\bigr)\,ds. \tag{3.2}$$

Hence, for all $t \in [0,1]$, we have

$$\begin{aligned} u(t) &\geq \frac{1}{n} + \lambda \int_0^1 g(s)t^{q-1} f_n\bigl(s, u(s)\bigr)\,ds \\ &\geq \frac{1}{n} + t^{q-1}\left(\|u\| - \frac{1}{n}\right) \\ &\geq t^{q-1}\|u\| = t^{q-1}r. \end{aligned}$$

Thus we have from condition (H2), for all $t \in [0,1]$,

$$\begin{aligned} u(t) &= \lambda \int_0^1 G(t,s) f_n\bigl(s, u(s)\bigr)\,ds + \frac{1}{n} \\ &= \lambda \int_0^1 G(t,s) f\bigl(s, u(s)\bigr)\,ds + \frac{1}{n} \\ &\leq \int_0^1 g(s)t^{q-2} f\bigl(s, u(s)\bigr)\,ds + \frac{1}{n} \\ &\leq \int_0^1 g(s)t^{q-2} b\bigl(u(s)\bigr)\left(1 + \frac{d(u(s))}{b(u(s))}\right)ds + \frac{1}{n} \\ &\leq \left(1 + \frac{d(r)}{b(r)}\right) \int_0^1 g(s)t^{q-2} K_0 b(r) b(s)\,ds + \frac{1}{n} \\ &\leq \bigl(b(r) + d(r)\bigr) K_0 \int_0^1 g(s)t^{q-2} b(s)\,ds + \frac{1}{n_0}. \end{aligned} \tag{3.3}$$

Therefore,

$$r = \bigl\|u(t)\bigr\| \leq \bigl(b(r) + d(r)\bigr) K_0 \int_0^1 g(s)t^{q-2} b(s)\,ds + \frac{1}{n_0}.$$

This is a contradiction and the claim is proved.

Now the Leray-Schauder nonlinear alternative guarantees that the integral equation

$$u(t) = \int_0^1 G(t,s) f_n\bigl(s, u(s)\bigr)\,ds + \frac{1}{n} \tag{3.4}$$

has a solution, denoted by u_n, in $\overline{B}_r = \{x \in C(J) : \|u\| \leq r\}$.

Next we claim that $u_n(t)$ has a uniform sharper lower bound, *i.e.*, there exists a function $\rho \in C([0,1])$ that is unrelated to n such that $\rho(t) > 0$ for a.e. $t \in [0,1]$ and for any $n \in N_0$,

$$u_n(t) \geq \rho(t), \quad t \in [0,1]. \tag{3.5}$$

By (H1), there exists a continuous function $\phi_r \succ 0$ such that $f(t,x) \geq \phi_r(t)$ for all $t \in [0,1]$ and $\|x\| \leq r$. In view of $u_n(t) \leq r$, so we have

$$\begin{aligned} u_n(t) &= \int_0^1 G(t,s) f_n\bigl(s, u_n(s)\bigr)\, ds + \frac{1}{n} \\ &= \int_0^1 G(t,s) f\bigl(s, u_n(s)\bigr)\, ds + \frac{1}{n} \\ &\geq \int_0^1 G(t,s)\phi_r(s)\, ds \\ &\geq t^{q-1}\int_0^1 g(s)\phi_r(s)\, ds. \end{aligned}$$

We choose $\rho(t) = t^{q-1}\int_0^1 g(s)\phi_r(s)\, ds$. Then (3.5) holds.

In order to pass from the solutions u_n of the truncation equation (3.4) to that of the original equation (1.1)-(1.2), we need the following fact:

$$\{u_n\}_{n\in N_0} \text{ is an equicontinuous family on } [0,1]. \tag{3.6}$$

In fact, for any $t_1, t_2 \in [0,1]$, we have

$$\begin{aligned} \bigl|u_n(t_1) - u_n(t_2)\bigr| &= \left|\int_0^1 \bigl[G(t_1,s) - G(t_2,s)\bigr] f\bigl(s, u_n(s)\bigr)\, ds\right| \\ &\leq \int_0^1 \bigl|G(t_1,s) - G(t_2,s)\bigr| f\bigl(s, u_n(s)\bigr)\, ds \\ &\leq \left(1 + \frac{d(r)}{b(r)}\right) \int_0^1 \bigl|G(t_1,s) - G(t_2,s)\bigr| b\left(s\int_0^1 g(\tau)\phi_r(\tau)\, d\tau\right) ds \\ &\leq \left(1 + \frac{d(r)}{b(r)}\right) \int_0^1 \bigl|G(t_1,s) - G(t_2,s)\bigr| K_0 b(s) b\left(\int_0^1 g(\tau)\phi_r(\tau)\, d\tau\right) ds \\ &\leq \left(1 + \frac{d(r)}{b(r)}\right) K_0 b\left(\int_0^1 g(\tau)\phi_r(\tau)\, d\tau\right) \int_0^1 \bigl|G(t_1,s) - G(t_2,s)\bigr| b\bigl(s^{q-1}\bigr)\, ds. \end{aligned}$$

By continuity of $G(t,\cdot)$ and the mean value theorem for integrals, there exists a $\xi \in (0,1)$ such that

$$\bigl|u_n(t_1) - u_n(t_2)\bigr| \leq \left(1 + \frac{d(r)}{b(r)}\right) K_0 b\left(\int_0^1 g(\tau)\phi_r(\tau)\, d\tau\right) \int_0^1 \bigl|G(t_1,\xi) - G(t_2,\xi)\bigr| b(s)\, ds.$$

By the continuity of $G(\cdot,s)$ and (H4), then (3.6) holds. By the Arzela-Ascoli theorem, there exist a subsequence N_1 of N_0 and $u \in C([0,1])$ such that $\{u_n\}_{n\in N_1}$ is uniformly convergent to u and u satisfies $\rho(t) \leq u(t) \leq r$ for any $t \in [0,1]$. In view of $u_n(t) = \int_0^1 G(t,s) f_n(s, u_n(s))\, ds$, by the Lebesgue dominated convergence theorem, we have $u(t) = \int_0^1 G(t,s) f(s, u(s))\, ds$. Therefore, (1.1)-(1.2) have one positive solution u with $0 < \|u\| < r$. This completes the proof. □

Theorem 3.2 *Suppose that* (H2), (H3), (H4), *and* (H5) *are satisfied. Furthermore assume that*:

(H6) *There exists a positive number* $R > r$ *such that*

$$\frac{1}{\Gamma(q)}\left(1-\frac{1}{q}\right)^{q-1} b(R)\int_0^1 s(1-s)^{q-1}\left\{1+\frac{d(sR)}{b(sR)}\right\} ds \geq R;$$

then problem (1.1)-(1.2) *has a solution* $\widetilde{u}$ *with* $r < \|\widetilde{u}\| \leq R$.

Proof To show the existence of $\widetilde{u}$, we will use Lemma 2.6. Define

$$K = \{u \in C[0,1] : u(t) \geq t\|u\|, \forall t \in [0,1]\}. \tag{3.7}$$

Clearly K is a cone of $C[0,1]$. Let

$$\Omega_1 = \{u \in C[0,1] : \|u\| < r\}, \qquad \Omega_2 = \{u \in C[0,1] : \|u\| < R\}.$$

Next, let $\mathcal{A} : K \cap (\overline{\Omega}_2 \backslash \Omega_1) \to C[0,1]$ be defined by

$$(\mathcal{A}y)(t) := \int_0^1 G^*(t,s) f\left(s, s^{q-2}y(s)\right) ds. \tag{3.8}$$

First we show that $\mathcal{A}$ maps $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$. If $y \in K \cap (\overline{\Omega}_2 \backslash \Omega_1)$, then for $t \in [0,1]$ we have

$$(\mathcal{A}y)(t) \leq \frac{1}{\Gamma(q)}\int_0^1 s(1-s)^{q-2} f\left(s, s^{q-2}y(s)\right) ds$$

and

$$(\mathcal{A}y)(t) \geq \frac{t}{\Gamma(q)}\int_0^1 s(1-s)^{q-2} f\left(s, s^{q-2}y(s)\right) ds,$$

this implies that $(\mathcal{A}y)(t) \geq t\|\mathcal{A}y\|$, *i.e.* $\mathcal{A}y \in K$.

Next, we show that $\mathcal{A}$ is equicontinuous. The proof will be given in several steps.

Step 1: We will show that $\mathcal{A}$ is continuous.

In fact, let $x_n, x \in K \cap (\overline{\Omega}_2 \backslash \Omega_1)$, $n = 1, 2, 3, \ldots$ with $\lim_{n\to\infty} \|x_n - x\| = 0$. It is obvious that $r < \|x_n\| \leq R$, $r < \|x\| \leq R$, $x_n(t) \geq tr$, $x(t) \geq tr$. We have

$$x_n(t) \in [tr, R], \quad n \in \{1, 2, \ldots\}, t \in [0,1],$$
$$x(t) \in [tr, R], \quad t \in [0,1].$$

Notice also that

$$\gamma_n(s) = \left\|f\left(s, x_n(s)\right) - f\left(s, x(s)\right)\right\| \to 0, \quad \text{as } n \to \infty, \text{for } \forall s \in [0,1]$$

and

$$\gamma_n(s) \leq 2b(tr)\left\{1+\frac{d(R)}{b(R)}\right\} \leq 2K_0 b(t)b(r)\left\{1+\frac{d(R)}{b(R)}\right\}.$$

Now these together with the Lebesgue dominated convergence theorem guarantee that

$$\|(\mathcal{A}x_n)(t)-(\mathcal{A}x)(t)\| \le (q-1)\int_0^1 q(s)\gamma_n(s)\,ds \to 0, \quad \text{as } n\to\infty.$$

Hence $\mathcal{A}: K\cap(\overline{\Omega}_2\backslash\Omega_1)\to K$ is continuous.

Step 2: We will prove that the operator $\mathcal{A}: K\cap(\overline{\Omega}_2\backslash\Omega_1)\to K$ is compact.

Indeed, for $x\in K\cap(\overline{\Omega}_2\backslash\Omega_1)$,

$$\begin{aligned}\|\mathcal{A}x\| &\le (q-1)\left\{1+\frac{d(R)}{b(R)}\right\}\int_0^1 q(s)b(sr)\,ds\\ &\le (q-1)\left\{1+\frac{d(R)}{b(R)}\right\}K_0 b(r)\int_0^1 q(s)b(s)\,ds,\end{aligned}$$

and for $t,t'\in[0,1]$, we have

$$\begin{aligned}\|\mathcal{A}x(t)-\mathcal{A}x(t')\| &\le \left\{1+\frac{d(R)}{b(R)}\right\}\int_0^1 \left|G(t,s)-G(t',s)\right|b(sr)\,ds\\ &\le \left\{1+\frac{d(R)}{b(R)}\right\}K_0 b(r)\int_0^1 \left|G(t,s)-G(t',s)\right|b(s)\,ds.\end{aligned}$$

By continuity of $G(t,\cdot)$ and the mean value theorem for integrals, there exists a $\eta\in(0,1)$ such that

$$\|\mathcal{A}x(t)-\mathcal{A}x(t')\| \le \left\{1+\frac{d(R)}{b(R)}\right\}K_0 b(r)\left|G(t,\eta)-G(t',\eta)\right|\int_0^1 b(s)\,ds.$$

By continuity of $G(\cdot,s)$, using condition (H4), and the Arzela-Ascoli theorem guarantees that $\mathcal{A}: K\cap(\overline{\Omega}_2\backslash\Omega_1)\to K$ is compact.

Now we prove that

$$\|\mathcal{A}x\|\le\|x\|, \quad \forall x\in K\cap\partial\Omega_1. \tag{3.9}$$

In fact, for any $x\in K\cap\partial\Omega_1$, we have for $t\in[0,1]$,

$$\begin{aligned}\mathcal{A}x(t) &= \int_0^1 G(t,s)f\bigl(s,x(s)\bigr)\,ds\\ &\le \int_0^1 G(t,s)b\bigl(x(s)\bigr)\left\{1+\frac{d(x(s))}{b(x(s))}\right\}ds\\ &\le \left\{1+\frac{d(r)}{b(r)}\right\}\int_0^1 g(s)t^{q-2}b(sr)\,ds\\ &\le \left\{1+\frac{d(r)}{b(r)}\right\}K_0 b(r)\int_0^1 g(s)t^{q-2}b(s)\,ds\\ &= \bigl\{b(r)+d(r)\bigr\}K_0\int_0^1 g(s)t^{q-2}b(s)\,ds\\ &< r\\ &= \|x\|.\end{aligned}$$

Therefore, $\|\mathcal{A}x\| \leq \|x\|$, *i.e.*, (3.9) holds. On the other hand, we prove that

$$\|\mathcal{A}x\| \geq \|x\|, \quad \forall x \in K \cap \partial\Omega_2. \tag{3.10}$$

In fact, for any $x \in K \cap \partial\Omega_2$, we have for $t \in [0,1]$,

$$\begin{aligned}
\mathcal{A}x\left(1-\frac{1}{q}\right) &= \int_0^1 G\left(1-\frac{1}{q}, s\right) f\bigl(s, x(s)\bigr)\, ds \\
&\geq \frac{1}{\Gamma(q)}\left(1-\frac{1}{q}\right)^{q-1} \int_0^1 s(1-s)^{q-1} b\bigl(x(s)\bigr)\left\{1 + \frac{d(x(s))}{b(x(s))}\right\} ds \\
&\geq \frac{1}{\Gamma(q)}\left(1-\frac{1}{q}\right)^{q-1} b(R) \int_0^1 s(1-s)^{q-1}\left\{1 + \frac{d(sR)}{b(sR)}\right\} ds \\
&\geq R \\
&= \|x\|.
\end{aligned}$$

This implies (3.10) holds.

It follows from Lemma 2.6, (3.9), and (3.10) that $\mathcal{A}$ has a fixed point $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$. Clearly, this fixed point is a positive solution of (1.1)-(1.2) satisfying $r < \|\widetilde{u}\| \leq R$. This completes the proof. □

Theorem 3.3 *Suppose that* (H1)-(H6) *are satisfied. Then problem* (1.1)-(1.2) *has two solutions u and $\widetilde{u}$ with* $0 < \|u\| < r < \|\widetilde{u}\| \leq R$.

4 Uniqueness of solution for a singular problem

Throughout this section we assume that

(H7) $f(t,x) = q_1(t)[g_1(x) + h_1(x)]$, $t \in (0,1)$, where

$$g_1 : [0, +\infty) \to [0, +\infty) \text{ is continuous and nondecreasing;}$$
$$h_1 : (0, +\infty) \to (0, +\infty) \text{ is continuous and nonincreasing.}$$

By property (i) of the Green's function in Lemma 2.4, we assume there exist $a, m, n \in C[0,1]$ with $a(t), m(t), n(t) > 0$ for $t \in (0,1)$ such that

$$a(t)m(s) \leq G(t,s)\Gamma(q) \leq a(t)n(s), \quad t, s \in [0,1], \tag{4.1}$$

where $a(t) = t^{q-1}$, $m(s) = s(1-s)^{q-2}$, $n(s) = (1-s)^{q-2}$. Clearly $\|a\| = \sup_{t\in J} a(t) < 1$.

Suppose that x is a solution of (1.1)-(1.2), then

$$x(t) := \int_0^1 G(t,s) f\bigl(s, x(s)\bigr)\, ds, \quad t \in [0,1].$$

By (4.1), we have

$$a(t)\int_0^1 \frac{1}{\Gamma(q)} m(s) f\bigl(s, x(s)\bigr)\, ds \leq x(t) \leq a(t)\int_0^1 \frac{1}{\Gamma(q)} n(s) f\bigl(s, x(s)\bigr)\, ds.$$

So if $x(t)$ is a solution of problem (1.1)-(1.2), then $x \in Q_e$ which was defined in (2.10), where $e(t) = t^{q-1} = a(t)$.

Let $P = \{x \in C[0,1] : x(t) \geq 0, \forall t \in [0,1]\}$. Clearly P is a normal cone of the Banach space $C[0,1]$.

Theorem 4.1 *Suppose that* (H7) *is satisfied, and there exists* $\beta \in (0,1)$ *such that*

$$g_1(tx) \geq t^{\beta} g_1(x) \tag{4.2}$$

and

$$h_1\bigl(t^{-1}x\bigr) \geq t^{\beta} h_1(x) \tag{4.3}$$

for any $t \in (0,1)$ *and* $x > 0$, *and* $q_1 \in C((0,1),(0,\infty))$ *satisfies*

$$\int_0^1 \frac{1}{\Gamma(q)} n(s) a^{-\beta}(s) q_1(s)\, ds < +\infty; \tag{4.4}$$

then problem (1.1)-(1.2) *has a unique positive solution* x^*.

Proof Since (4.3) holds, let $t^{-1}x = y$; one has

$$h_1(y) \geq t^{\beta} h_1(ty).$$

Then

$$h_1(ty) \leq t^{-\beta} h_1(y), \quad \forall t \in (0,1), y > 0. \tag{4.5}$$

Let $y = 1$. The above inequality is

$$h_1(t) \leq t^{-\beta} h_1(1), \quad \forall t \in (0,1). \tag{4.6}$$

From (4.3), (4.5), and (4.6), one has

$$\begin{aligned} &h_1\bigl(t^{-1}x\bigr) \geq t^{\beta} h_1(x), \qquad h_1\left(\frac{1}{t}\right) \geq t^{\beta} h_1(1), \qquad h_1(tx) \leq t^{-\beta} h_1(x), \\ &h_1(t) \leq t^{-\beta} h_1(1), \quad t \in (0,1), x > 0. \end{aligned} \tag{4.7}$$

Similarly, from (4.2), one has

$$g_1(tx) \geq t^{\beta} g_1(x), \qquad g_1(t) \geq t^{\beta} g_1(1), \quad t \in (0,1), x > 0. \tag{4.8}$$

Let $t = \frac{1}{x}$, $x > 1$, so one has

$$g_1(x) \leq x^{\beta} g_1(1), \quad t \geq 1. \tag{4.9}$$

Let $e(t) = a(t)$, and we define

$$Q_e = \left\{x \in P \,\middle|\, \frac{1}{M} a(t) \leq x(t) \leq M a(t), t \in [0,1]\right\}, \tag{4.10}$$

where $M > 1$ is chosen such that

$$M > \max\left\{\left\{\int_0^1 \lambda \frac{1}{\Gamma(q)} n(s)q_1(s)a^{-\beta}(s)\left[g_1(1) + h_1(1)\right] ds\right\}^{\frac{1}{1-\beta}},\right.$$
$$\left.\left\{\int_0^1 \lambda \frac{1}{\Gamma(q)} m(s)q_1(s)a^{\beta}(s)\left[g_1(1) + h_1(1)\right] ds\right\}^{-\frac{1}{1-\beta}}\right\}. \tag{4.11}$$

For any $x, y \in Q_e$ we define

$$A_\lambda(x,y)(t) = \lambda \int_0^1 G(t,s)q_1(s)\left[g_1\left(x(s)\right) + h_1\left(y(s)\right)\right] ds, \quad \forall t \in [0,1]. \tag{4.12}$$

First we show that $A_\lambda : Q_e \times Q_e \to Q_e$. Let $x, y \in Q_e$ and from (4.8) we have

$$g_1\left(x(t)\right) \le g_1\left(Ma(t)\right) \le g_1(M) \le M^\beta g_1(1), \quad t \in (0,1),$$

and from (4.7) we have

$$h_1\left(y(t)\right) \le h_1\left(\frac{1}{M}a(t)\right) \le a^{-\beta}(t)h_1\left(\frac{1}{M}\right) \le M^\beta a^{-\beta}(t)h_1(1), \quad t \in (0,1).$$

So we have

$$\begin{aligned}
A_\lambda(x,y)(t) &\le \lambda \int_0^1 \frac{1}{\Gamma(q)} a(t)n(s)q_1(s)\left[g_1\left(x(s)\right) + h_1\left(y(s)\right)\right] ds \\
&\le \lambda a(t)M^\beta \int_0^1 \frac{1}{\Gamma(q)} n(s)q_1(s)\left[g_1(1) + a^{-\beta}(s)h_1(1)\right] ds \\
&\le \lambda a(t)M^\beta \int_0^1 \frac{1}{\Gamma(q)} n(s)q_1(s)\left[g_1(1) + a^{-\beta}h_1(1)\right] ds \\
&\le \lambda a(t)M^\beta \int_0^1 \frac{1}{\Gamma(q)} n(s)q_1(s)a^{-\beta}(s)\left[g_1(1) + h_1(1)\right] ds \\
&\le Ma(t), \quad \forall t \in [0,1].
\end{aligned}$$

On the other hand, for any $x, y \in Q_e$, from (4.7) and (4.8), we have

$$g_1\left(x(t)\right) \ge g_1\left(\frac{1}{M}a(t)\right) \ge a^\beta(t)g_1\left(\frac{1}{M}\right) \ge a^\beta(t)\frac{1}{M^\beta}g_1(1), \quad t \in (0,1)$$

and

$$h_1\left(y(t)\right) \ge h_1\left(Ma(t)\right) \ge h_1(M) = h_1\left(\frac{1}{\frac{1}{M}}\right) \ge \frac{1}{M^\beta}h_1(1), \quad t \in (0,1),$$

so we have

$$\begin{aligned}
A_\lambda(x,y)(t) &\ge \lambda \int_0^1 \frac{1}{\Gamma(q)} a(t)m(s)q_1(s)\left[g_1\left(x(s)\right) + h_1\left(y(s)\right)\right] ds \\
&\ge \lambda a(t)M^{-\beta} \int_0^1 \frac{1}{\Gamma(q)} m(s)q_1(s)\left[a^\beta(s)g_1(1) + h_1(1)\right] ds
\end{aligned}$$

$$\begin{aligned}&\geq \lambda a(t)M^{-\beta}\int_0^1 \frac{1}{\Gamma(q)}m(s)q_1(s)a^{\beta}(s)\bigl[g_1(1)+h_1(1)\bigr]\,ds\\&\geq \frac{1}{M}a(t),\quad \forall t\in[0,1].\end{aligned}$$

Thus A_λ is well defined and $A_\lambda(Q_e\times Q_e)\subseteq Q_e$.

Next, for any $l\in(0,1)$ and $x,y\in Q_e$ we have

$$\begin{aligned}A_\lambda\bigl(lx,l^{-1}y\bigr)(t) &= \lambda\int_0^1 G(t,s)q_1(s)\bigl[g_1\bigl(lx(s)\bigr)+h_1\bigl(l^{-1}y(s)\bigr)\bigr]\,ds\\&\geq \lambda\int_0^1 G(t,s)q_1(s)\bigl[l^{\beta}g_1\bigl(x(s)\bigr)+l^{\beta}h_1\bigl(y(s)\bigr)\bigr]\,ds\\&= l^{\beta}A(x,y)(t),\quad \forall t\in[0,1].\end{aligned}$$

Thus the conditions of Lemma 2.7 hold. Therefore there exists a unique $x^*\in Q_e$ such that $A_\lambda(x^*,x^*)=x^*$. This completes the proof. □

Example 1 Consider the boundary value problem

$$D_{0^+}^{q}u(t)=u^{-a}(t)+\nu u^{b}(t),\quad 0<t<1, \tag{4.13}$$

$$u(0)=u'(1)=0, \tag{4.14}$$

where $0<a,b<1$, $\nu\geq 0$.

We let

$$\beta=\max\{a,b\}<1,\qquad q_1(t)=1,\qquad g_1(x)=\nu x^{b},\qquad h_1(x)=x^{-a}.$$

Thus, we have

$$g_1(tx)=t^{b}g_1(x)\geq t^{\beta}g_1(x),\qquad h_1\bigl(t^{-1}x\bigr)=t^{a}h_1(x)\geq t^{\beta}h_1(x).$$

For any $t\in(0,1)$ and $x>0$, and

$$\int_0^1 \frac{1}{\Gamma(q)}n(s)a^{-\beta}(s)q_1(s)\,ds<+\infty.$$

Since $\beta<1$, and $1<q\leq 2$, thus all conditions in Theorem 4.1 are satisfied. Applying Theorem 4.1, we can find that (4.13)-(4.14) has a unique positive solution $x^*(t)$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the manuscript. All authors read and approved the final manuscript.

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