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Existence results for Riemann-Liouville fractional neutral evolution equations

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Abstract

In this paper, by using the fractional power of operators and the theory of measure of noncompactness, we discuss a class of fractional neutral evolution equations with Riemann-Liouville fractional derivative. We establish sufficient conditions for the existence of mild solutions for fractional neutral evolution equations in the cases C_0 semigroup is compact or noncompact. We give an example to illustrate the applications of the abstract results.

Keywords: fractional neutral evolution equations; Riemann-Liouville fractional derivative; mild solutions; analytic semigroup; measure of noncompactness

1 Introduction

A strong motivation for studying fractional differential equations comes from the fact that fractional order derivatives and integrals have extensive applications in viscoelasticity, analytical chemistry, electromagnetic, neuron modeling, and biological sciences, and the theory of fractional calculus has attracted great interest from the mathematical science research community. For more details of the theory and applications in this field, see the monographs of Samako *et al.* [1], Kilbas *et al.* [2], Miller and Ross [3], Podlubny [4] and the references therein.

Practical problems require definitions of fractional derivatives allowing the utilization of physically interpretable initial conditions. Initial conditions for Caputo fractional derivatives are expressed in terms of initial integer order derivatives. Heymans and Podlubny [5] demonstrated that it is possible to attribute physical meaning to initial conditions expressed in terms of Riemann-Liouville fractional derivatives or integrals on the field of the viscoelasticity, and such initial conditions are more appropriate than physically interpretable initial conditions.

Recently, fractional evolution equations with the Caputo fractional derivative with difference conditions were studied by many authors (see, *e.g.*, [6–15]), but much less is known about the fractional evolution equations with Riemann-Liouville fractional derivative; see [16–18].

In [16], Zhou, Zhang and Shen researched the existence of solutions for a fractional evolution equation with the Riemann-Liouville fractional derivative of the form

$$\begin{cases} {}^L D_{0^+}^q x(t) = Ax(t) + (Fx)(t), & t \in J' := (0, a], 0 < q < 1, \\ (I_{0^+}^{1-q} x)(0) + g(x) = x_0, \end{cases}$$

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where $^{L}D_{0+}^{\alpha}$ is the Riemann-Liouville fractional derivative of order q, I_{0+}^{1-q} is the Riemann-Liouville integral of order $1-q$, A is the infinitesimal generator of a C_0-semigroup $\{T(t), t \geq 0\}$ on a Banach space X. $F : C(J',X) \to L(J',X)$ and $g : C(J',X) \to L(J',X)$ are given functions satisfying some assumptions.

However, fractional neutral evolution equations with a Riemann-Liouville fractional derivative have not been considered in the literature. Motivated by the above work, the aim of this paper is to study the existence of solutions for fractional neutral evolution equations with a Riemann-Liouville fractional derivative of the form

$$\begin{cases} ^{L}D_{0+}^{q}[x(t) - h(t,x(t))] = -Ax(t) + f(t,x(t)), & t \in J' = (0,b], 0 < q < 1, \\ I_{0+}^{1-q}[x(t) - h(t,x(t))]|_{t=0} = x_0 \in X, \end{cases} \tag{1}$$

where $^{L}D_{0+}^{q}$ is the Riemann-Liouville fractional derivative of order q with the lower limit zero, I_{0+}^{1-q} is the Riemann-Liouville integral of order $1-q$. X is a Banach space, $-A : D(A) \subseteq X \to X$ is the infinitesimal generator of an analytic semigroup $\{T(t), t \geq 0\}$ on a Banach space X. h and f are given functions satisfying some assumptions.

Neutral differential equations arise in many areas of applied mathematics and for this reason these equations have received much attention in the last few decades. The literature related to fractional neutral differential equations is very extensive; see for instance [12–14, 19]. The results we obtained in this paper are a generalization and continuation of the recent results on this issue.

The rest of this paper is organized as follows. In Section 2, some notations and preparation results are given. In Section 3, by using the fractional power of operators and the technique of using a measure of noncompactness, we give existence results for problem (1) under both compactness and noncompactness conditions on the C_0 semigroup. We present an example to demonstrate our main results in Section 4. Finally, the manuscript ends with our conclusions.

2 Preliminaries

In this section, we introduce the notations, definitions, and preliminary facts that will be used in the remainder of this paper.

In this paper, we assume that X is a Banach space. Let $C(J,X)$ denote the Banach space of all X-value continuous functions from $J = [0,b]$ into X with the norm $\|x\|_{C(J,X)} = \sup_{t \in J} \|x(t)\|_X$. For measurable functions $m : J \to \mathbb{R}$, we define the norm $\|m\|_{L^p(J,\mathbb{R})} := (\int_J |m(t)|^p \, dt)^{\frac{1}{p}}, 1 \leq p < \infty$. $L^p(J,\mathbb{R})$ $(1 \leq p < \infty)$ is the Banach space of all Lebesgue measurable functions from J into $\mathbb{R}$ with $\|m\|_{L^p(J,\mathbb{R})} < \infty$. Let $L^p(J,X)$ denote the Banach space of functions $m : J \to X$ which are Bochner integrable normed by $\|m\|_{L^p(J,X)}$. Let $J' = (0,b]$, to define the mild solution of (1), we also consider the Banach space $C_{1-q}(J,X) = \{x \in C(J',X) : t^{1-q}x(t) \in C(J,X)\}$ with the norm $\|x\|_{C_{1-q}} = \sup_{t \in J}\{t^{1-q}\|x(t)\|_X\}$. It is easy to see $(C_{1-q}(J,X), \|\cdot\|_{C_{1-q}})$ is a Banach space.

First, let us recall the following definitions and properties of the fractional calculus. For more details, one can refer to [2–4].

Definition 2.1 The fractional integral of order q with the lower limit zero for a function f is defined as

$$I_{0+}^{q}f(t) = \frac{1}{\Gamma(q)}\int_0^t \frac{f(s)}{(t-s)^{1-q}}\,ds, \quad t > 0, q > 0, \tag{2}$$

provided the right side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 The Riemann-Liouville derivative of order q with the lower limit zero for a function $f : [0, \infty) \to \mathbb{R}$ can be written as

$$^L D_{0+}^q f(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{f(s)}{(t-s)^{q-n+1}} \, ds, \quad t > 0, n-1 < q < n. \tag{3}$$

Definition 2.3 The Caputo derivative of order q for a function $f : [0, \infty) \to \mathbb{R}$ can be written as

$$^c D_{0+}^q f(t) = {}^L D^q \left[f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right], \quad t > 0, n-1 < q < n. \tag{4}$$

Remark 2.4

(i) If $f(t) \in C^n[0, \infty)$, then

$$^c D_{0+}^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{q+1-n}} \, ds = I_{0+}^{n-q} f^{(n)}(t), \quad t > 0, n-1 < q < n.$$

(ii) The Caputo derivative of a constant is equal to zero.

(iii) If f is an abstract function with values in X, then integrals which appear in Definitions 2.1 and 2.2 are taken in Bochner's sense.

Throughout this paper, let $-A$ be the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t \geq 0}$ of uniformly bounded linear operators on X; this means that there exists $M \geq 1$, such that $M := \sup_{t \in [0, \infty)} \|T(t)\| < \infty$. Let $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A. Then for $0 < \eta \leq 1$, it is possible to define the fractional power A^η as a closed linear operator on its domain $D(A^\eta)$. Moreover, the subspace $D(A^\eta)$ is dense in X and the expression

$$\|x\|_\eta = \|A^\eta x\|, \quad x \in D(A^\eta),$$

defines a norm on $D(A^\eta)$. We denote by X_η the Banach space $D(A^\eta)$ normed with $\|x\|_\eta$.

For analytic semigroup $\{T(t)\}_{t \geq 0}$, the following property will be used:

(i) for any $\eta \in (0, 1]$, there exists a positive constant C_η such that

$$\|A^\eta T(t)\| \leq \frac{C_\eta}{t^\eta}, \quad 0 < t \leq b.$$

For more details as regards the above preliminaries, we refer the reader to [20, 21].

Next, we recall some definitions and properties of measure of noncompactness. For more details, we refer the reader to [22, 23] and the references therein.

Definition 2.5 Let X be a Banach space, $\mathcal{P}(X)$ denote the collection of all nonempty subsets of X, and $(\mathcal{A}, \geq)$ a partially ordered set. A map $\beta : \mathcal{P}(X) \to \mathcal{A}$ is called a measure of noncompactness on X, MNC for short, if

$$\beta(\overline{co}\Omega) = \beta(\Omega)$$

for every $\Omega \in \mathcal{P}(X)$, where $\overline{co}\Omega$ is the closure of convex hull of Ω.

Definition 2.6 A measure of noncompactness β is called

(a) monotone if $\Omega_0, \Omega_1 \in \mathcal{P}(X)$, $\Omega_0 \subseteq \Omega_1$ implies $\beta(\Omega_0) \leq \beta(\Omega_1)$;

(b) nonsingular if $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ for every $a \in X$, $\Omega \in \mathcal{P}(X)$;

(c) invariant with respect to the union with compact sets if $\beta(K \cup \Omega) = \beta(\Omega)$ for every relatively compact set $K \subseteq X$ and $\Omega \in \mathcal{P}(X)$;

(d) semi-additive if $\beta(\Omega_0 \cup \Omega_1) = \max(\beta(\Omega_0), \beta(\Omega_1))$ for every $\Omega_0, \Omega_1 \in \mathcal{P}(X)$;

(e) regular if the condition $\beta(\Omega) = 0$ is equivalent to the relative compactness of Ω.

One of the most important examples of the measure of noncompactness is the Kuratowski measure of noncompactness α defined on each bounded subset B of X by

$$\alpha(B) = \inf\{\varepsilon > 0 : B \text{ is covered by a finite number of sets with diameter } \leq \varepsilon\}.$$

It is well known that the Kuratowski measure of noncompactness α enjoys the above properties (a)-(e) and other properties (see [24–26]). We have

(f) $\alpha(B_1 + B_2) \leq \alpha(B_1) + \alpha(B_2)$, where $B_1 + B_2 = \{x + y : x \in B_1, y \in B_2\}$;

(g) $\alpha(\lambda B) \leq |\lambda|\alpha(B)$ for any $\lambda \in R$.

For any $W \subset C(J, X)$, we define

$$\int_0^t W(s)\,ds = \left\{ \int_0^t u(s)\,ds : u \in W \right\}, \quad \text{for } t \in J = [0, b],$$

where $W(s) = \{u(s) \in X : u \in W\}$.

Proposition 2.7 *If $W \subset C(J, X)$ is bounded and equicontinuous, then $\overline{co}\,W \subset C(J, X)$ is also bounded and equicontinuous.*

Proposition 2.8 [27] *If $W \subset C(J, X)$ is bounded and equicontinuous, then $t \mapsto \alpha(W(t))$ is continuous on J and*

$$\alpha(W) = \max_{t \in J} \alpha\big(W(t)\big), \qquad \alpha\left(\int_0^t W(s)\,ds \right) \leq \int_0^t \alpha\big(W(s)\big)\,ds, \quad \text{for } t \in [0, b].$$

Proposition 2.9 [28] *X is a Banach space, if $B \subset X$ is bounded, then for each $\varepsilon > 0$, there is a sequence $\{u_n\}_{n=1}^{\infty} \subset B$, such that*

$$\alpha(B) \leq 2\alpha\big(\{u_n\}_{n=1}^{\infty}\big) + \varepsilon.$$

Proposition 2.10 [29] *Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of Bochner integrable functions from J into X with $\|u_n(t)\| \leq m(t)$ for almost all $t \in J$ and every $n \geq 1$, where $m \in L(J, R^+)$, then the function $\varphi(t) = \alpha(\{u_n(t)\}_{n=1}^{\infty})$ belongs to $L(J, R^+)$ and satisfies*

$$\alpha\left(\int_0^t u_n(s)\,ds : n \geq 1 \right) \leq 2 \int_0^t \varphi(s)\,ds.$$

Lemma 2.11 *Let B be a closed and convex subset of a Banach space X. Suppose $F : B \to B$ is a continuous operator and $F(B)$ is bounded. For each bounded subset $B_0 \subset B$, set*

$$F^1(B_0) = F(B_0), \qquad F^n(B_0) = F\big(\overline{co}\big(F^{n-1}(B_0)\big)\big), \quad n = 2, 3, \dots.$$

If there exist a constant $0 \le k < 1$ and a positive integer n_0 such that, for any bounded subset $B_0 \subset B$,

$$\alpha\big(F^{n_0}(B_0)\big) \le k\alpha(B_0),$$

then there exists a nonempty, compact, and convex subset $\widehat{D} \subset B$ such that $F : \widehat{D} \to \widehat{D}$ and $\alpha(F(\widehat{D})) = 0$.

According to Lemma 2.4 in [30], we can obtain the results immediately.

The map $F : B \subseteq X \to X$ is said to be an α-contraction if there exists a positive constant $k < 1$ such that $\alpha(F(B_0)) \le k\alpha(B_0)$ for any bounded closed subset $B_0 \subseteq B$.

Lemma 2.12 (Darbo-Sadovskii's fixed point theorem) [24] *If B is a bounded closed convex subset of a Banach space X, the continuous map $F : B \to B$ is an α-contraction, then the map F has at least one fixed point in B.*

On the basis of [12, 16], we give the following definition of a mild solution of system (1).

Definition 2.13 By a mild solution of system (1), we mean the function $x \in C_{1-q}(J, X)$ which satisfies

$$x(t) = t^{q-1}T_q(t)x_0 + h\big(t, x(t)\big) + \int_0^t (t-s)^{q-1}AT_q(t-s)h\big(s, x(s)\big)\,ds$$

$$+ \int_0^t (t-s)^{q-1}T_q(t-s)f\big(s, x(s)\big)\,ds, \quad t \in (0, b],$$

where

$$T_q(t) = q\int_0^\infty \theta M_q(\theta)T\big(t^q\theta\big)\,d\theta,$$

and M_q is a probability density function which is defined by

$$M_q(\theta) = \sum_{n=1}^\infty \frac{(-\theta)^{n-1}}{(n-1)!\Gamma(1-qn)}, \quad 0 < q < 1, \theta \in \mathbb{C}.$$

Lemma 2.14 [12] *The operator $T_q(t)$ has the following properties.*

(i) *For any fixed $t \ge 0$, $T_q(t)$ is linear and bounded operators, i.e., for any $x \in X$,*

$$\big\|T_q(t)x\big\| \le \frac{M}{\Gamma(q)}\|x\|;$$

(ii) *$T_q(t)$ $(t \ge 0)$ is strongly continuous, which means that, for $\forall x \in X$ and $0 < t' < t'' \le b$, we have*

$$\big\|T_q\big(t''\big)x - T_q\big(t'\big)x\big\| \to 0 \quad \text{as } t'' \to t';$$

(iii) *for every $t > 0$, $T_q(t)$ is a compact operator if $T(t)$ is also compact.*

Lemma 2.15 [12] *For any $\beta \in (0,1)$, $x \in D(A^\beta)$ and $\eta \in (0,1]$, we have*

$$AT_q(t)x = A^{1-\beta}T_q(t)A^\beta x, \quad 0 \le t \le b,$$

and

$$\left\| A^\eta T_q(t) \right\| \le \frac{qC_\eta \Gamma(2-\eta)}{t^{q\eta}\Gamma(1+q(1-\eta))}, \quad 0 < t \le b.$$

3 Existence of mild solutions

Before stating and proving the main results, we introduce the following assumptions.

(H_1) $T(t)$ is a compact operator for every $t > 0$.

(H_2) For almost all $t \in J$, the function $f(t,\cdot) : X \to X$ is continuous and for each $z \in X$, the function $f(\cdot,z) : J \to X$ is strongly measurable.

(H_3) There exist a function $\phi(t) \in L^{\frac{1}{p}}(J,\mathbb{R}^+)$, $p \in (0,q)$, and a constant $c > 0$ such that

$$\left\| f(t,x(t)) \right\| \le \phi(t) + ct^{1-q}\left\| x(t) \right\|, \quad \forall x \in C_{1-q}(J,X), t \in J,$$

and the operator $G : \Phi \to L^{\frac{1}{p}}(J,X)$ is continuous, where

$$(Gx)(t) = f(t,x(t)), \quad \forall x \in \Phi, \Phi \text{ is any bounded set of } C_{1-q}(J,X).$$

(H_4) $h : J' \times X \to X$ is a continuous function and there exist a constant $\beta \in (0,1)$ and $L > 0$ such that $h \in D(A^\beta)$ and for any $z,y \in X$, $t \in J'$, the function $A^\beta h(\cdot,z)$ is strongly measurable and $A^\beta h$ satisfies

$$\left\| A^\beta h(t,x(t)) - A^\beta h(t,y(t)) \right\| \le Lt^{1-q}\left\| x(t) - y(t) \right\|, \quad \text{for all } x,y \in C_{1-q}(J,X), t \in J',$$

and the inequality

$$\left\| A^\beta h(t,x(t)) \right\| \le L\left(1 + \left\| x \right\|_{C_{1-q}}\right), \quad \forall t \in J', x \in C_{1-q}(J,X).$$

For brevity, let

$$M_0 = \left\| A^{-\beta} \right\|,$$

$$M_1 = b^{1-q}M_0 L + \frac{C_{1-\beta}\Gamma(1+\beta)Lb^{1-q+q\beta}}{\beta\Gamma(1+q\beta)} + \frac{Mcb}{\Gamma(1+q)},$$

$$M_2 = \frac{M}{\Gamma(q)}\left\| x_0 \right\| + b^{1-q}M_0 L + \frac{C_{1-\beta}\Gamma(1+\beta)Lb^{1-q+q\beta}}{\beta\Gamma(1+q\beta)} + \frac{M}{\Gamma(q)}\left(\frac{1-p}{q-p}b\right)^{1-p}\left\| \phi \right\|_{L^{\frac{1}{p}}(J,X)}.$$

Define an operator F on $C_{1-q}(J,X)$ by

$$(Fx)(t) = t^{q-1}T_q(t)x_0 + h(t,x(t)) + \int_0^t (t-s)^{q-1}AT_q(t-s)h(s,x(s))\,ds$$

$$+ \int_0^t (t-s)^{q-1}T_q(t-s)f(s,x(s))\,ds, \quad t \in (0,b].$$

For any $x \in C_{1-q}(J, X)$, let

$$(Fx)(t) = (F_1 x)(t) + (F_2 x)(t),$$

where

$$(F_1 x)(t) = t^{q-1} T_q(t) x_0 + \int_0^t (t-s)^{q-1} T_q(t-s) f(s, x(s)) \, ds, \quad t \in (0, b],$$

$$(F_2 x)(t) = h(t, x(t)) + \int_0^t (t-s)^{q-1} A T_q(t-s) h(s, x(s)) \, ds, \quad t \in (0, b].$$

Assume that $M_1 < 1$, and let

$$B_r = \left\{ x \in C_{1-q}(J, X) : \|x\|_{C_{1-q}} \le r \right\}, \quad \text{where } r \ge \frac{M_2}{1 - M_1}.$$

Lemma 3.1 *If the assumptions* (H$_1$)-(H$_3$) *are satisfied and* $M_1 < 1$, *then* $F_1(B_r)$ *is relatively compact set in* $C_{1-q}(J, X)$.

Proof Using (H$_1$)-(H$_3$) and Lemma 2.14, one can easily prove that $F_1 x \in C_{1-q}(J, X)$ for any $x \in C_{1-q}(J, X)$. Then F_1 is well defined on $C_{1-q}(J, X)$. Let

$$\Omega = \left\{ y \in C(J, X) : y(t) = t^{1-q}(F_1 x)(t), x \in B_r \right\}.$$

For the sake of convenience, we divide the proof into several steps.

Step 1: For any $y \in \Omega$, $\|y\|_{C(J,X)} \le r$.

For any $x \in B_r$ and $t \in J$, taking into account the imposed assumptions, applying Lemma 2.14 and the Hölder inequality, we obtain

$$t^{1-q} \|(F_1 x)(t)\| \le \|T_q(t) x_0\| + t^{1-q} \left\| \int_0^t (t-s)^{q-1} T_q(t-s) f(s, x(s)) \, ds \right\|$$

$$\le \frac{M}{\Gamma(q)} \|x_0\| + \frac{M t^{1-q}}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left[\phi(s) + c s^{1-q} \|x(s)\| \right] ds$$

$$\le \frac{M}{\Gamma(q)} \|x_0\| + \frac{M}{\Gamma(q)} \left(\frac{1-p}{q-p} b \right)^{1-p} \|\phi\|_{L^{\frac{1}{p}}(J,X)} + \frac{M c b r}{\Gamma(q+1)}$$

$$\le r.$$

Hence, for any $y \in \Omega$, $\|y\|_{C(J,X)} \le r$.

Step 2: $\Omega \subseteq C(J, X)$ is equicontinuous.

For any $y \in \Omega$, let $0 = t_1 < t_2 \le b$. Taking (H$_3$) into account, we get

$$\|y(t_2) - y(0)\|$$

$$\le \|T_q(t_2) x_0 - T_q(0) x_0\| + \left\| t_2^{1-q} \int_0^{t_2} (t_2 - s)^{q-1} T_q(t_2 - s) f(s, x(s)) \, ds \right\|$$

$$\le \|T_q(t_2) x_0 - T_q(0) x_0\| + \frac{M}{\Gamma(q)} \left(\frac{1-p}{q-p} t_2 \right)^{1-p} \|\phi\|_{L^{\frac{1}{p}}(J,X)} + \frac{M c r t_2}{\Gamma(q+1)}$$

$$\to 0, \quad \text{as } t_2 \to 0^+.$$

For $0 < t_1 < t_2 \le b$, selecting $\varepsilon > 0$ sufficiently small. Taking (H$_3$) and (H$_4$) into account, we have

$$\left\| y(t_2) - y(t_1) \right\|$$

$$\le \left\| T_q(t_2)x_0 - T_q(t_1)x_0 \right\| + \left\| t_2^{1-q} \int_{t_1}^{t_2} (t_2 - s)^{q-1} T_q(t_2 - s) f\left(s, x(s)\right) ds \right\|$$

$$+ \left\| \int_0^{t_1} \left[t_2^{1-q}(t_2 - s)^{q-1} - t_1^{1-q}(t_1 - s)^{q-1} \right] T_q(t_2 - s) f\left(s, x(s)\right) ds \right\|$$

$$+ \left\| t_1^{1-q} \int_0^{t_1} (t_1 - s)^{q-1} \left[T_q(t_2 - s) - T_q(t_1 - s) \right] f\left(s, x(s)\right) ds \right\|$$

$$\le \left\| T_q(t_2)x_0 - T_q(t_1)x_0 \right\| + \frac{Mb^{1-q}}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} \left[\phi(s) + cr \right] ds$$

$$+ \frac{M}{\Gamma(q)} \int_0^{t_1} \left[t_1^{1-q}(t_1 - s)^{q-1} - t_2^{1-q}(t_2 - s)^{q-1} \right] \left(\phi(s) + cr \right) ds$$

$$+ \left\| \int_0^{t_1 - \varepsilon} t_1^{1-q}(t_1 - s)^{q-1} \left[T_q(t_2 - s) - T_q(t_1 - s) \right] f\left(s, x(s)\right) ds \right\|$$

$$+ \left\| \int_{t_1 - \varepsilon}^{t_1} t_1^{1-q}(t_1 - s)^{q-1} \left[T_q(t_2 - s) - T_q(t_1 - s) \right] f\left(s, x(s)\right) ds \right\|$$

$$\le I_1 + I_2 + I_3 + I_4 + I_5,$$

where

$$I_1 = \left\| T_q(t_2)x_0 - T_q(t_1)x_0 \right\|,$$

$$I_2 = \frac{Mb^{1-q}}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} \left[\phi(s) + cr \right] ds,$$

$$I_3 = \frac{M}{\Gamma(q)} \int_0^{t_1} \left[t_1^{1-q}(t_1 - s)^{q-1} - t_2^{1-q}(t_2 - s)^{q-1} \right] \left(\phi(s) + cr \right) ds,$$

$$I_4 = \sup_{s \in [0, t_1 - \varepsilon]} \left\| T_q(t_2 - s) - T_q(t_1 - s) \right\| \left[\left(\frac{1-p}{q-p} b \right)^{1-p} \| \phi \|_{L^{\frac{1}{p}}(J, X)} + \frac{bcr}{q} \right],$$

$$I_5 = \frac{2Mb^{1-q}}{\Gamma(q)} \int_{t_1 - \varepsilon}^{t_1} (t_1 - s)^{q-1} \left[\phi(s) + cr \right] ds.$$

According to Lemma 2.14(ii), it is easy to see that $I_1 \to 0$ as $t_2 \to t_1$. Applying the absolute continuity of the Lebesgue integral, we see that I_2 and I_5 tend to zero independently of $x \in B_r$ as $t_2 \to t_1$, $\varepsilon \to 0$. Noting that

$$0 < \left[t_1^{1-q}(t_1 - s)^{q-1} - t_2^{1-q}(t_2 - s)^{q-1} \right] \left(\phi(s) + cr \right) < t_1^{1-q}(t_1 - s)^{q-1} \left(\phi(s) + cr \right),$$

and that $\int_0^{t_1} t_1^{1-q}(t_1 - s)^{q-1} (\phi(s) + cr) \, ds$ exists, by the Lebesgue dominated convergence theorem, we see that I_3 tends to zero independently of $x \in B_r$ as $t_2 \to t_1$. Since (H$_1$) and Lemma 2.14 imply the continuity of $T_q(t)$ ($t > 0$) in t in the uniform operator topology, it is easy to see that I_4 tends to zero independently of $x \in B_r$ as $t_2 \to t_1$. Thus, $\| y(t_2) - y(t_1) \|$ tends to zero independently of $x \in B_r$ as $t_2 \to t_1$, which means that Ω is equicontinuous.

Step 3: For any $t \in [0, b]$, $\Omega(t) = \{y(t), y \in \Omega\}$ is relatively compact in X.

This is trivial for $t = 0$, since $\Omega(0) = \{\frac{x_0}{\Gamma(q)}\}$. So it is only necessary to consider $0 < t \leq b$. Let $0 < t \leq b$ be fixed. For $\forall \varepsilon \in (0, t)$, $\forall \delta > 0$, define

$$
\begin{aligned}
y^{\varepsilon,\delta}(t) &= q \int_\delta^\infty \theta M_q(\theta) T(t^q \theta) x_0 \, d\theta \\
&\quad + q t^{1-q} \int_0^{t-\varepsilon} \int_\delta^\infty (t-s)^{q-1} \theta M_q(\theta) T((t-s)^q \theta) f(s, x(s)) \, d\theta \, ds \\
&\leq T(\varepsilon^q \delta) \left\{ q \int_\delta^\infty \theta M_q(\theta) T(t^q \theta - \varepsilon^q \delta) x_0 \, d\theta \right. \\
&\quad \left. + q t^{1-q} \int_0^{t-\varepsilon} \int_\delta^\infty (t-s)^{q-1} \theta M_q(\theta) T((t-s)^q \theta - \varepsilon^q \delta) f(s, x(s)) \, d\theta \, ds \right\}.
\end{aligned}
$$

Then from the compactness of $T(\varepsilon^q \delta)$ $(\varepsilon^q \delta > 0)$, we find that the set $\Omega^{\varepsilon,\delta}(t) = \{y^{\varepsilon,\delta}(t), y \in \Omega\}$ is relatively compact in X for $\forall \varepsilon \in (0, t)$ and $\forall \delta > 0$.

Moreover, we have

$$
\begin{aligned}
&\left\| y(t) - y^{\varepsilon,\delta}(t) \right\| \\
&\leq \left\| q \int_0^\delta \theta M_q(\theta) T(t^q \theta) x_0 \, d\theta \right\| \\
&\quad + \left\| q t^{1-q} \int_0^t \int_0^\delta (t-s)^{q-1} \theta M_q(\theta) T((t-s)^q \theta) f(s, x(s)) \, d\theta \, ds \right\| \\
&\quad + \left\| q t^{1-q} \int_{t-\varepsilon}^t \int_\delta^\infty (t-s)^{q-1} \theta M_q(\theta) T((t-s)^q \theta) f(s, x(s)) \, d\theta \, ds \right\| \\
&\leq q M \|x_0\| \int_0^\delta \theta M_q(\theta) \, d\theta + q M b^{1-q} \int_0^t (t-s)^{q-1} [\phi(s) + cr] \, ds \int_0^\delta \theta M_q(\theta) \, d\theta \\
&\quad + q M b^{1-q} \int_{t-\varepsilon}^t (t-s)^{q-1} [\phi(s) + cr] \, ds \int_0^\infty \theta M_q(\theta) \, d\theta \\
&\leq q M \|x_0\| \int_0^\delta \theta M_q(\theta) \, d\theta + q M b^{1-q} \int_0^t (t-s)^{q-1} [\phi(s) + cr] \, ds \int_0^\delta \theta M_q(\theta) \, d\theta \\
&\quad + \frac{M b^{1-q}}{\Gamma(q)} \int_{t-\varepsilon}^t (t-s)^{q-1} [\phi(s) + cr] \, ds \to 0, \quad \text{as } \varepsilon \to 0, \delta \to 0.
\end{aligned}
$$

Therefore, there are relatively compact sets arbitrarily close to the set $\Omega(t)$, $t > 0$. Hence the set $\Omega(t)$, $t > 0$ is also relatively compact in X.

As a consequence of Step 1-Step 3, with the Arzola-Ascoli theorem, we can conclude that $\Omega \subseteq C(J, X)$ is a relatively compact set. Keeping in mind the relationship of Ω and $F_1(B_r)$, one can easily prove that $F_1(B_r)$ is a relatively compact set in $C_{1-q}(J, X)$. The proof is complete. □

3.1 The case that $T(t)$ is compact

Theorem 3.2 *Assume that hypotheses* (H_1)-(H_4) *hold and* $M_1 < 1$, *then system* (1) *has at least one mild solution.*

Proof Using (H$_1$)-(H$_4$), Lemma 2.14 and Lemma 2.15, one can easily prove that $Fx \in C_{1-q}(J,X)$ for any $x \in C_{1-q}(J,X)$. Then F is well defined on $C_{1-q}(J,X)$. We will show that F satisfies all conditions of Lemma 2.12. The proof will be given in several steps.

For any $x \in B_r$ and $t \in J$, taking into account the imposed assumptions, applying Lemma 2.14 and the Hölder inequality, we obtain

$$
\begin{aligned}
t^{1-q}\big\|(Fx)(t)\big\| &\leq \big\|T_q(t)x_0\big\| + \big\|t^{1-q}h\big(t,x(t)\big)\big\| \\
&\quad + \left\|t^{1-q}\int_0^t (t-s)^{q-1}AT_q(t-s)h\big(s,x(s)\big)\,ds\right\| \\
&\quad + \left\|t^{1-q}\int_0^t (t-s)^{q-1}T_q(t-s)f\big(s,x(s)\big)\,ds\right\| \\
&\leq \frac{M}{\Gamma(q)}\|x_0\| + \big\|t^{1-q}A^{-\beta}A^{\beta}h\big(t,x(t)\big)\big\| \\
&\quad + \left\|t^{1-q}\int_0^t (t-s)^{q-1}A^{1-\beta}T_q(t-s)A^{\beta}h\big(s,x(s)\big)\,ds\right\| \\
&\quad + \frac{Mt^{1-q}}{\Gamma(q)}\int_0^t (t-s)^{q-1}\big[\phi(s) + cs^{1-q}\|x(s)\|\big]\,ds \\
&\leq \frac{M}{\Gamma(q)}\|x_0\| + b^{1-q}M_0L(1+r) + \frac{C_{1-\beta}\Gamma(1+\beta)Lb^{1-q+q\beta}(1+r)}{\beta\Gamma(1+q\beta)} \\
&\quad + \frac{M}{\Gamma(q)}\left(\frac{1-p}{q-p}b\right)^{1-p}\|\phi\|_{L^{\frac{1}{p}}(J,X)} + \frac{Mcbr}{\Gamma(q+1)} \\
&\leq r.
\end{aligned}
$$

Thus, F maps B_r into B_r.

Next, we will show that F is continuous in B_r.

Let $\{x_n\} \in B_r$ with $x_n \to x$ in B_r, we have

$$
\begin{aligned}
t^{1-q}\big\|(Fx_n)(t) &- (Fx)(t)\big\| \\
&\leq t^{1-q}\big\|h\big(t,x_n(t)\big) - h\big(t,x(t)\big)\big\| \\
&\quad + t^{1-q}\left\|\int_0^t (t-s)^{q-1}AT_q(t-s)\big[h\big(s,x_n(s)\big) - h\big(s,x(s)\big)\big]\,ds\right\| \\
&\quad + t^{1-q}\left\|\int_0^t (t-s)^{q-1}T_q(t-s)\big[f\big(s,x_n(s)\big) - f\big(s,x(s)\big)\big]\,ds\right\| \\
&\leq \big\|t^{1-q}A^{-\beta}\big[A^{\beta}h\big(t,x_n(t)\big) - A^{\beta}h\big(t,x(t)\big)\big]\big\| \\
&\quad + t^{1-q}\left\|\int_0^t (t-s)^{q-1}A^{1-\beta}T_q(t-s)\big[A^{\beta}h\big(s,x_n(s)\big) - A^{\beta}h\big(s,x(s)\big)\big]\,ds\right\| \\
&\quad + \frac{Mt^{1-q}}{\Gamma(q)}\int_0^t (t-s)^{q-1}\big\|f\big(s,x_n(s)\big) - f\big(s,x(s)\big)\big\|\,ds \\
&\leq b^{1-q}M_0L\|x_n - x\|_{C_{1-q}} + \frac{b^{1-q+q\beta}C_{1-\beta}\Gamma(1+\beta)L}{\beta\Gamma(1+q\beta)}\|x_n - x\|_{C_{1-q}} \\
&\quad + \frac{M}{\Gamma(q)}\left(\frac{1-p}{q-p}b\right)^{1-p}\|Gx_n - Gx\|_{L^{\frac{1}{p}}(J,X)},
\end{aligned}
$$

which implies

$$\|Fx_n - Fx\|_{C_{1-q}} \to 0 \quad \text{as } n \to \infty.$$

This means that F is continuous in B_r.

According to Lemma 3.1, $F_1(B_r)$ is relatively compact in $C_{1-q}(J, X)$, then $\alpha(F_1(B_r)) = 0$.

For any $x_1, x_2 \in B_r$, we have

$$
\begin{aligned}
t^{1-q}&\big\|(F_2 x_1)(t) - (F_2 x_2)(t)\big\| \\
&\le t^{1-q}\big\|A^{-\beta}\big[A^{\beta}h\big(t, x_1(t)\big) - A^{\beta}h\big(t, x_2(t)\big)\big]\big\| \\
&\quad + t^{1-q}\bigg\|\int_0^t (t-s)^{q-1} A^{1-\beta} T_q(t-s)\big[A^{\beta}h\big(s, x_1(s)\big) - A^{\beta}h\big(s, x_2(s)\big)\big]\,ds\bigg\| \\
&\le b^{1-q}M_0 L\|x_1 - x_2\|_{C_{1-q}} + \frac{b^{1-q+q\beta}C_{1-\beta}\Gamma(1+\beta)L}{\beta\Gamma(1+q\beta)}\|x_1 - x_2\|_{C_{1-q}},
\end{aligned}
$$

which implies that

$$\|F_2 x_1 - F_2 x_2\|_{C_{1-q}} \le \left[b^{1-q}M_0 L + \frac{b^{1-q(1-\beta)}C_{1-\beta}\Gamma(1+\beta)L}{\beta\Gamma(1+q\beta)}\right]\|x_1 - x_2\|_{C_{1-q}}.$$

Hence

$$\alpha\big(F_2(B_r)\big) \le \left[b^{1-q}M_0 L + \frac{b^{1-q(1-\beta)}C_{1-\beta}\Gamma(1+\beta)L}{\beta\Gamma(1+q\beta)}\right]\alpha(B_r).$$

Therefore

$$
\begin{aligned}
\alpha\big(F(B_r)\big) &\le \alpha\big(F_1(B_r)\big) + \alpha\big(F_2(B_r)\big) \\
&\le \left[b^{1-q}M_0 L + \frac{b^{1-q(1-\beta)}C_{1-\beta}\Gamma(1+\beta)L}{\beta\Gamma(1+q\beta)}\right]\alpha(B_r).
\end{aligned}
$$

Noting that $M_1 < 1$, we find that the operator F is an α-contraction in B_r. It follows from Lemma 2.12 that F has at least one fixed point in B_r. Then problem (1) has at least one mild solution in B_r. The proof is complete. $\qquad\square$

3.2 The case that $T(t)$ is not compact

For any $x \in B_r$, set $y(t) = t^{1-q}x(t)$, define $\mathcal{F}_1$ as follows:

$$
\begin{aligned}
(\mathcal{F}_1 y)(t) &= t^{1-q}(F_1 x)(t) \\
&= T_q(t)x_0 + t^{1-q}\int_0^t (t-s)^{q-1}T_q(t-s)f\big(s, x(s)\big)\,ds, \quad t \in (0, b],
\end{aligned}
$$

and

$$(\mathcal{F}_1 y)(t) = \frac{x_0}{\Gamma(q)}, \quad t = 0.$$

If $T(t)$ is noncompact, we will need the following assumptions.

$(H_1)'$ $T(t)$ $(t > 0)$ is continuous in the uniform operator topology for $t > 0$.

(H_5) There exists a constant $\widehat{L} > 0$ such that for any bounded $D \subset C_{1-q}(J, X)$,

$$\alpha\big(f\big(t, D(t)\big)\big) \le \widehat{L}\alpha\big(t^{1-q}D(t)\big), \quad \text{for a.e. } t \in J.$$

Theorem 3.3 *If assumptions* $(H_1)'$, (H_2)-(H_5) *are satisfied and* $M_1 < 1$, *then problem* (1) *has at least one mild solution.*

Proof Define

$$\widetilde{B}_r = \big\{h \in C(J, X) : \|h\|_{C(J, X)} \le r\big\}.$$

For any $x \in B_r$, $y(t) = t^{1-q}x(t)$, it is easy to see that $y \in \widetilde{B}_r$. Let

$$\Omega = \big\{\mathcal{F}_1 y : (\mathcal{F}_1 y)(t) = t^{1-q}(F_1 x)(t), x \in B_r\big\}.$$

According to $(H_1)'$, Step 1 and Step 2 of Lemma 3.1, we find that $\mathcal{F}_1(\widetilde{B}_r) \subseteq \widetilde{B}_r$ and $\mathcal{F}_1(\widetilde{B}_r)$ is equicontinuous. Let $B = \overline{co}\mathcal{F}_1(\widetilde{B}_r)$ then $B \subseteq \widetilde{B}_r$ is a bounded, closed, and convex set. According to Proposition 2.7, B is also equicontinuous. Since

$$B = \overline{co}\mathcal{F}_1(\widetilde{B}_r) \subseteq \widetilde{B}_r,$$

we have $\mathcal{F}_1(B) \subseteq B$. Using a similar method as in the proof of Theorem 3.2, we can easily see that $\mathcal{F}_1 : B \to B$ is continuous. For any bounded subset $B_0 \subset B$, set

$$\mathcal{F}_1^1(B_0) = \mathcal{F}_1(B_0), \qquad \mathcal{F}_1^n(B_0) = \mathcal{F}_1\big(\overline{co}\big(\mathcal{F}^{n-1}(B_0)\big)\big), \quad n = 2, 3, \ldots.$$

We know from Propositions 2.8-2.10, for any $\varepsilon > 0$, that there is a sequence $\{u_n\}_{n=1}^\infty \subset B_0$, such that

$$\begin{aligned}
\alpha\big((\mathcal{F}_1^1 B_0)(t)\big) &\le 2\alpha\big(\{(\mathcal{F}_1 u_n)(t)\}_{n=1}^\infty\big) + \varepsilon \\
&= 2\alpha\left(T_q(t)x_0 + t^{1-q}\int_0^t (t-s)^{q-1}T_q(t-s)f\big(s, \{x_n(s)\}_{n=1}^\infty\big)\,ds\right) + \varepsilon \\
&\le \frac{4Mb^{1-q}}{\Gamma(q)}\int_0^t (t-s)^{q-1}\alpha\big(f\big(s, \{x_n(s)\}_{n=1}^\infty\big)\big)\,ds + \varepsilon \\
&\le \frac{4Mb^{1-q}\widehat{L}}{\Gamma(q)}\int_0^t (t-s)^{q-1}\alpha\big(s^{1-q}\{x_n(s)\}_{n=1}^\infty\big)\,ds + \varepsilon \\
&= \frac{4Mb^{1-q}\widehat{L}}{\Gamma(q)}\int_0^t (t-s)^{q-1}\alpha\big(\{u_n(s)\}_{n=1}^\infty\big)\,ds + \varepsilon \\
&\le \frac{4M\widehat{L}b^{1-q}\alpha(B_0)t^q}{\Gamma(q+1)} + \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\alpha\big((\mathcal{F}_1^1 B_0)(t)\big) \le \frac{4M\widehat{L}b^{1-q}\alpha(B_0)t^q}{\Gamma(q+1)}.$$

For any $\varepsilon > 0$, there is a sequence $\{y_n\}_{n=1}^{\infty} \subset \overline{co}(F^1(B_0))$, such that

$$\alpha\big((\mathcal{F}_1^2 B_0)(t)\big) \leq 2\alpha\big(\{(\mathcal{F}_1 y_n)(t)\}_{n=1}^{\infty}\big) + \varepsilon$$

$$= 2\alpha\left(T_q(t)x_0 + t^{1-q}\int_0^t (t-s)^{q-1} T_q(t-s)f\big(s, \{x_n(s)\}_{n=1}^{\infty}\big)\, ds\right) + \varepsilon$$

$$\leq \frac{4Mb^{1-q}}{\Gamma(q)}\int_0^t (t-s)^{q-1}\alpha\big(f\big(s, \{x_n(s)\}_{n=1}^{\infty}\big)\big)\, ds + \varepsilon$$

$$\leq \frac{4Mb^{1-q}\widehat{L}}{\Gamma(q)}\int_0^t (t-s)^{q-1}\alpha\big(s^{1-q}\{x_n(s)\}_{n=1}^{\infty}\big)\, ds + \varepsilon$$

$$= \frac{4Mb^{1-q}\widehat{L}}{\Gamma(q)}\int_0^t (t-s)^{q-1}\alpha\big(\{y_n(s)\}_{n=1}^{\infty}\big)\, ds + \varepsilon$$

$$\leq \frac{4Mb^{1-q}\widehat{L}}{\Gamma(q)}\int_0^t (t-s)^{q-1}\alpha\big((\mathcal{F}_1^1 B_0)(s)\big)\, ds + \varepsilon$$

$$\leq \frac{4M\widehat{L}b^{1-q}}{\Gamma(q)}\int_0^t (t-s)^{q-1} \times \frac{4MLb^{1-q}s^q}{\Gamma(q+1)}\alpha(B_0)\, ds + \varepsilon$$

$$\leq \frac{(4M\widehat{L}b^{1-q}t^q)^2}{\Gamma(2q+1)}\alpha(B_0) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\alpha\big((\mathcal{F}_1^2 B_0)(t)\big) \leq \frac{(4M\widehat{L}b^{1-q}t^q)^2}{\Gamma(2q+1)}\alpha(B_0).$$

It can be shown by mathematical induction that

$$\alpha\big((\mathcal{F}_1^n B_0)(t)\big) \leq \frac{(4M\widehat{L}b^{1-q}t^q)^n}{\Gamma(nq+1)}\alpha(B_0)$$

$$\leq \frac{(4M\widehat{L}b)^n}{\Gamma(nq+1)}\alpha(B_0).$$

By using the well-known Stirling formula, we get

$$\Gamma(1+nq) = \sqrt{2\pi nq}\left(\frac{nq}{e}\right)^{nq} e^{\frac{\theta}{12nq}}, \quad 0 < \theta < 1.$$

Then

$$\lim_{n\to\infty}\frac{(4M\widehat{L}b)^n}{\Gamma(nq+1)} = \lim_{n\to\infty}\frac{(4M\widehat{L}b)^n}{\sqrt{2\pi nq}(\frac{nq}{e})^{nq} e^{\frac{\theta}{12nq}}} = 0.$$

Then there exists a $\gamma \in (0,1)$ such that

$$\alpha\big((\mathcal{F}_1^n B_0)(t)\big) \leq \gamma\alpha(B_0).$$

It can be shown that $\mathcal{F}_1^n B_0 \subseteq B$. Since B is bounded and equicontinuous, $\mathcal{F}_1^n B_0$ is also bounded and equicontinuous. It follows from Proposition 2.8 that

$$\alpha\big(\mathcal{F}_1^n B_0\big) = \max_{t\in J}\alpha\big((\mathcal{F}_1^n B_0)(t)\big) \leq \gamma\alpha(B_0).$$

Then, by Lemma 2.11, there exists a $\overline{D} \subset B \subset \widetilde{B}_r$ such that

$$\alpha\big(\mathcal{F}_1(\overline{D})\big) = 0.$$

Let

$$Q = \big\{ x \in C_{1-q}(J, X) : x(t) = t^{q-1} y(t), y \in \overline{D} \big\}.$$

For any $x \in Q$, we have $\|x\|_{C_{1-q}} = \|y\|_{C(J,X)} \leq r$, which means that $Q \subseteq B_r$. Since $\mathcal{F}_1(\overline{D})$ is relatively compact, we can easily prove that $F_1(Q)$ is relatively compact, that is

$$\alpha\big(F_1(Q)\big) = 0.$$

On the other hand, for any $x_1, x_2 \in Q$, we have

$$
\begin{aligned}
t^{1-q} &\big\|(F_2 x_1)(t) - (F_2 x_2)(t)\big\| \\
&\leq t^{1-q} \big\| A^{-\beta} \big[A^\beta h\big(t, x_1(t)\big) - A^\beta h\big(t, x_2(t)\big) \big] \big\| \\
&\quad + t^{1-q} \bigg\| \int_0^t (t-s)^{q-1} A^{1-\beta} T_q(t-s) \big[A^\beta h\big(s, x_1(s)\big) - A^\beta h\big(s, x_2(s)\big) \big] \, ds \bigg\| \\
&\leq b^{1-q} M_0 L \|x_1 - x_2\|_{C_{1-q}} + \frac{b^{1-q(1-\beta)} C_{1-\beta} \Gamma(1+\beta) L}{\beta \Gamma(1+q\beta)} \|x_1 - x_2\|_{C_{1-q}},
\end{aligned}
$$

which implies that

$$\|F_2 x_1 - F_2 x_2\|_{C_{1-q}} \leq \left[b^{1-q} M_0 L + \frac{b^{1-q(1-\beta)} C_{1-\beta} \Gamma(1+\beta) L}{\beta \Gamma(1+q\beta)} \right] \|x_1 - x_2\|_{C_{1-q}}.$$

Hence

$$\alpha\big(F_2(Q)\big) \leq \left[b^{1-q} M_0 L + \frac{b^{1-q(1-\beta)} C_{1-\beta} \Gamma(1+\beta) L}{\beta \Gamma(1+q\beta)} \right] \alpha(Q).$$

Therefore

$$\alpha\big(F(Q)\big) \leq \alpha\big(F_1(Q)\big) + \alpha\big(F_2(Q)\big) \leq \left[b^{1-q} M_0 L + \frac{b^{1-q(1-\beta)} C_{1-\beta} \Gamma(1+\beta) L}{\beta \Gamma(1+q\beta)} \right] \alpha(Q).$$

Noting that $M_1 < 1$, thus the operator F is an α-contraction in Q. It follows from Lemma 2.12 that F has at least one fixed point in Q. Then problem (1) has at least one mild solution in Q. The proof is complete. □

4 An example

Let $X = L^2([0, \pi], \mathbb{R})$. As an application of our results, consider the following fractional partial differential equations:

$$
\begin{cases}
{}^L D_{0+}^{\frac{1}{2}} \big[z(t, x) - \sin\big(\int_0^\pi b(t, y, x)[z(t, y) + \frac{\partial}{\partial y} z(t, y)] \, dy \big) \big] \\
\quad = \frac{\partial^2}{\partial x^2} z(t, x) + f(t, z(t, x)), \quad t \in (0, b], x \in [0, \pi], \\
z(t, 0) = z(t, \pi) = 0, \quad t \in (0, b], \\
I_{0+}^{\frac{1}{2}} z(t, x)|_{t=0} = z_0(x), \quad x \in [0, \pi],
\end{cases}
\tag{5}
$$

where $^{L}D_{0+}^{\frac{1}{2}}$ is the Riemann-Liouville fractional derivative of order $\frac{1}{2}$, $b > 0$, $I_{0+}^{\frac{1}{2}}$ is a Riemann-Liouville integral of order $\frac{1}{2}$. Define the operator A by $Az = -z''$, with the domain

$$D(A) = \left\{ z(\cdot) \in X : z, z' \text{ are absolutely continuous}, z'' \in X, z(t, 0) = z(t, \pi) = 0 \right\}.$$

Then $-A$ generates a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ which is compact, analytic. Moreover, A can be written as

$$Az = \sum_{n=1}^{\infty} n^2 < z, \quad e_n > e_n, z \in D(A),$$

where $e_n(x) = \sqrt{\frac{2}{\pi}} \sin nx$, $0 \leq x \leq \pi$, $n = 1, 2, \ldots$ is an orthonormal basis of X. We have

$$T(t)z = \sum_{n=1}^{\infty} e^{-n^2 t} < z, \quad e_n > e_n, z \in X, \quad \text{and} \quad \left\| T(t) \right\| \leq e^{-t} \leq 1 = M, \quad t \geq 0.$$

For each $z \in X$,

$$A^{-\frac{1}{2}}z = \sum_{n=1}^{\infty} \frac{1}{n} < z, \quad e_n > e_n;$$

in particular, $\|A^{-\frac{1}{2}}\| = 1$.

The operator $A^{\frac{1}{2}}$ is given by

$$A^{\frac{1}{2}}z = \sum_{n=1}^{\infty} n < z, \quad e_n > e_n$$

on the space

$$D\left(A^{\frac{1}{2}}\right) = \left\{ z(\cdot) \in X, \sum_{n=1}^{\infty} n < z, e_n > e_n \in X \right\}.$$

Choose $q = \beta = \frac{1}{2}$, define

$$z(t, x) = z(t)(x), \qquad f\left(t, z(t, x)\right) = f\left(t, z(t)\right)(x),$$

$$h(t, z)(x) = \sin\left(\int_0^{\pi} b(t, y, x)\left[z(t, y) + \frac{\partial}{\partial y} z(t, y) \right] dy \right),$$

then system (5) can be written in the abstract form given by (1). Assume that (H_2)-(H_4) are satisfied and $M_1 < 1$; then by Theorem 3.2, system (5) has at least one mild solution.

5 Conclusions

In this manuscript, by using the fractional power of operators and the theory of a measure of noncompactness, the existence of fractional neutral evolution equations with a Riemann-Liouville fractional derivative were investigated. We give an example to illustrate the applications of the abstract results.

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