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Value distribution of q-difference differential polynomials of entire functions

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Abstract

For a complex value $q \neq 0, 1$, and a transcendental entire function $f(z)$ with order, $0 < \sigma(f) < \infty$, we study the value distribution of q-difference differential polynomials $[f^n(z)(f(qz) - f(z))]^{(k)}$ and $[f(z)f(qz)]^{(k)}$.

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1 Introduction and main results

A meromorphic function $f(z)$ means meromorphic in the complex plane $\mathbb{C}$. If no poles occur, then $f(z)$ reduces to an entire function. For every real number $x \geq 0$, we define $\log^+ x := \max\{0, \log x\}$. Assume that $n(r, f)$ counts the number of poles of f in $|z| < r$, each pole according to its multiplicity, and that $\overline{n}(r, f)$ counts the number of the distinct poles of f in $|z| < r$, ignoring the multiplicity. The characteristic function of $f(z)$ is defined by

$$T(r,f) := m(r,f) + N(r,f),$$

where

$$N(r,f) := \int_0^r \frac{n(t,f) - n(0,f)}{t} \, dt + n(0,f) \log r$$

and

$$m(r,f) := \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f\left(re^{i\theta}\right) \right| d\theta.$$

The notation $\overline{N}(r, f)$ is similarly defined with $\overline{n}(r, f)$ instead of $n(r, f)$. More notations and definitions of the Nevanlinna value distribution theory of meromorphic functions can be found in [1, 2].

A meromorphic function $\alpha(z)$ is called a small function with respect to $f(z)$, if $T(r, \alpha) = S(r, f)$, where $S(r, f)$ denotes any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$ outside a possible exceptional set E of finite logarithmic measure. The order and the exponent of convergence of zeros of meromorphic function $f(z)$ is, respectively, defined as

$$\sigma(f) = \limsup_{r \to \infty} \frac{\log T(r,f)}{\log r}, \qquad \lambda(f) = \limsup_{r \to \infty} \frac{\log N(r, \frac{1}{f})}{\log r}.$$

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The difference operators for a meromorphic function $f(z)$ are defined as

$$\Delta_c f(z) = f(z + c) - f(z) \quad (c \neq 0),$$

$$\nabla_q f(z) = f(qz) - f(z) \quad (q \neq 0, 1).$$

A Borel exceptional value of $f(z)$ is any value $a \in \mathbb{C} \cup \{\infty\}$ satisfying $\lambda(f - a) < \sigma(f)$, where $\lambda(f - \infty)$ means $\lambda(\frac{1}{f})$.

Recently, the difference variant of the Nevanlinna theory has been established independently in [3–6]. Using these theories, value distributions of difference polynomials have been studied by many papers. For example, Laine and Yang [7] proved that if $f(z)$ is a transcendental entire function of finite order, c is a nonzero complex constant and $n \geq 2$, then $f^n(z)f(z + c)$ takes every nonzero value infinitely often.

Chen [8] considered the value distribution of $f(z)f(z + c)$ and obtained the following theorem.

Theorem A [8, Corollary 1.3] *Let $f(z)$ be a transcendental entire function of finite order, and let c be a nonzero complex constant. If $f(z)$ has the Borel exceptional value 0, then $H(z) = f(z)f(z + c)$ takes every nonzero value $a \in \mathbb{C}$ infinitely often.*

Chen [9] considered zeros of difference product $H_n(z) = f^n(z)\Delta_c f(z)$ and gave some conditions that guarantee $H_n(z)$ has finitely many zeros or infinitely many zeros.

Theorem B [9, Theorem 1] *Let $f(z)$ be a transcendental entire function of finite order and $c \in \mathbb{C} \setminus \{0\}$ be a constant such that $f(z + c) \not\equiv f(z)$. Set $H_n(z) = f^n(z)\Delta_c f(z)$ where $\Delta_c f(z) = f(z + c) - f(z)$, $n \geq 2$ is an integer. Then the following statements hold.*

(1) *If $f(z)$ satisfies $\sigma(f) \neq 1$, or has infinitely many zeros, then $H_n(z)$ has infinitely many zeros.*

(2) *If $f(z)$ has only finitely many zeros and $\sigma(f) = 1$, then $H_n(z)$ has only finitely many zeros.*

The zero distribution of differential polynomials is a classical topic in the theory of meromorphic functions. Hayman [10, Theorem 10] firstly considered the value distribution of $f^n f' - 1$, where f is a transcendental function.

Recently, Liu, Liu and Cao [11] investigated the zeros of $[f(z)^n f(z + c)]^{(k)} - \alpha(z)$ and $[f(z)^n \Delta_c f(z)]^{(k)} - \alpha(z)$, where $\alpha(z)$ is a nonzero small function with respect to $f(z)$.

Theorem C [11, Theorems 1.1 and 1.3] *Let $f(z)$ be a transcendental entire function of finite order and $\alpha(z)$ be a nonzero small function with respect to $f(z)$. If $n \geq k + 2$, then $[f(z)^n f(z + c)]^{(k)} - \alpha(z)$ has infinitely many zeros. If $f(z)$ is not a periodic function with period c and $n \geq k + 3$, then $[f(z)^n \Delta_c f(z)]^{(k)} - \alpha(z)$ has infinitely many zeros.*

The main purpose of this paper is to consider a transcendental entire function $f(z)$ with positive and finite order and obtain some results on the value distributions of the q-difference differential polynomials $[f^n(z)(f(qz) - f(z))]^{(k)}$ and $[f(z)f(qz)]^{(k)}$. The first theorem will consider what conditions guarantee that $[f^n(z)\nabla_q f(z)]^{(k)}$ has infinitely many zeros.

Theorem 1.1 *Let $f(z)$ be a transcendental entire function of finite and positive order $\sigma(f)$, $q \in \mathbb{C} \setminus \{0, 1\}$ be a constant such that $q^{\sigma(f)} \neq 1$ and $f(z) \not\equiv f(qz)$. Set $H(z) = [f^n(z)\nabla_q f(z)]^{(k)}$, $n \geq 1$ is an integer. If $f(z)$ has finitely many zeros, then $H(z) - \alpha(z)$ has infinitely many zeros, where $\alpha(z)$ is a nonzero small entire function with respect to $f(z)$.*

In the following, we will study the value distribution of $[f(z)f(qz)]^{(k)}$.

Theorem 1.2 *Let $f(z)$ be a transcendental entire function of finite and positive order $\sigma(f)$, a be a finite Borel exceptional value of $f(z)$, $q \in \mathbb{C} \setminus \{0, 1\}$ be a constant such that $q^{\sigma(f)} \neq \pm 1$. Set $H(z) = [f(z)f(qz)]^{(k)}$, then the following statements hold.*

(1) *If $a = 0$, then 0 is also the Borel exceptional value of $H(z)$. So that $H(z)$ has no nonzero finite Borel exceptional value.*

(2) *If $a \neq 0$, then $H(z)$ has no finite Borel exceptional value.*

(3) *$H(z)$ takes every nonzero value $c \in \mathbb{C}$ infinitely often and satisfies $\lambda(H - c) = \sigma(f)$.*

Using the similar method of the proof of Theorem 1.2(1), we get the following result immediately.

Corollary 1.1 *Let $f(z)$ be a transcendental entire function of finite and positive order $\sigma(f)$, $q \in \mathbb{C} \setminus \{0, 1\}$ be a constant such that $q^{\sigma(f)} \neq -n$. If 0 is a Borel exceptional value of $f(z)$, then 0 is also the Borel exceptional value of $[f^n(z)f(qz)]^{(k)}$.*

2 Some lemmas

The following are the well-known Weierstrass factorization and Hadamard factorization theorems.

Lemma 2.1 [12] *If an entire function $f(z)$ has a finite exponent of convergence $\lambda(f)$ for its zero-sequence, then $f(z)$ has a representation in the form*

$$f(z) = Q(z)e^{g(z)},$$

satisfying $\lambda(Q) = \sigma(Q) = \lambda(f)$. Further, if $f(z)$ is of finite order, then $g(z)$ in the above form is a polynomial of degree less or equal to the order of $f(z)$.

Lemma 2.2 [13] *Suppose that $f_1(z), f_2(z), \ldots, f_n(z)$ ($n \geq 2$) are meromorphic functions and $g_1(z), g_2(z), \ldots, g_n(z)$ are entire functions satisfying the following conditions:*

(1) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$;

(2) *$g_j(z) - g_k(z)$ are not constants for $1 \leq j < k \leq n$;*

(3) *For $1 \leq j \leq n$, $1 \leq h < k \leq n$, $T(r, f_j) = o(T(r, e^{g_h - g_k}))$ ($r \to \infty$, $r \notin E$).*

Then $f_j(z) \equiv 0$ ($j = 1, 2, \ldots, n$).

3 The proofs

3.1 Proof of Theorem 1.1

Since $f(z)$ is a transcendental entire function of finite order and has finitely many zeros, then by Lemma 2.1, $f(z)$ can be written as

$$f(z) = g(z)e^{h(z)},$$

where $g(z)$ $(\not\equiv 0)$, $h(z)$ are polynomials. Set

$$h(z) = a_s z^s + \cdots + a_0,$$

where $a_s, a_{s-1}, \ldots, a_0$ are constants, and $a_s \neq 0$. Since $\sigma(f) \neq 0$, then $\sigma(f) = \deg(h(z)) = s \geq 1$. So

$$
\begin{aligned}
H(z) &= \left[f^n(z) \nabla_q f(z) \right]^{(k)} \\
&= \left[g^n(z) g(qz) e^{nh(z)+h(qz)} - g^{n+1}(z) e^{(n+1)h(z)} \right]^{(k)} \\
&= g_1(z) e^{nh(z)+h(qz)} - g_2(z) e^{(n+1)h(z)},
\end{aligned}
$$

where $g_1(z)$, $g_2(z)$ are nonzero polynomials and

$$(n+1)h(z) - \left(nh(z) + h(qz)\right) = \left(1 - q^s\right) a_s z^s + \left(1 - q^{s-1}\right) a_{s-1} z^{s-1} + \cdots + (1-q) a_1 z.$$

Since $q^{\sigma(f)} = q^s \neq 1$, then $(n+1)h(z) - (nh(z) + h(qz))$ is not a constant. So $H(z)$ is a transcendental entire function, suppose that $H(z) - \alpha(z)$ has finitely many zeros, then by Lemma 2.1, $H(z) - \alpha(z)$ can be written as

$$H(z) - \alpha(z) = g_3(z) e^{h_1(z)}.$$

Here $g_3(z)$ $(\not\equiv 0)$, $h_1(z)$ are polynomials. Combing the above equalities, we obtain

$$g_1(z) e^{nh(z)+h(qz)} - g_2(z) e^{(n+1)h(z)} - g_3(z) e^{h_1(z)} - \alpha(z) = 0. \tag{3.1}$$

Note that $(n+1)h(z) - (nh(z) + h(qz))$ is not a constant and $g_1(z)$, $g_2(z)$, $g_3(z)$ are nonzero polynomials. If $(nh(z) + h(qz)) - h_1(z)$ and $(n+1)h(z) - h_1(z)$ are not constants, then by (3.1) and Lemma 2.2, we obtain

$$g_1(z) \equiv 0, \qquad g_2(z) \equiv 0, \qquad g_3(z) \equiv 0, \qquad \alpha(z) \equiv 0.$$

This is a contradiction.

If $(nh(z) + h(qz)) - h_1(z) = c$, where c is a constant, then (3.1) can be rewritten as

$$\left(g_1(z) - e^{-c} g_3(z)\right) e^{nh(z)+h(qz)} - g_2(z) e^{(n+1)h(z)} - \alpha(z) = 0. \tag{3.2}$$

By (3.2) and Lemma 2.2, we obtain

$$g_1(z) - e^{-c} g_3(z) \equiv 0, \qquad g_2(z) \equiv 0, \qquad \alpha(z) \equiv 0.$$

This is a contradiction.

If $(n+1)h(z) - h_1(z) = c$, where c is a constant, then using the same method as above, we also obtain a contradiction.

Hence $H(z) - \alpha(z)$ has infinitely many zeros.

3.2 Proof of Theorem 1.2

Since $f(z)$ is a transcendental entire function of finite and positive order with a Borel exceptional value a, then by Lemma 2.1, $f(z)$ can be written as

$$f(z) = a + g(z)e^{\alpha z^s}.$$

Here s is a positive integer, α is a nonzero constant, $g(z)$ is a nonzero entire function satisfying $\sigma(g) < \sigma(f) = s$, thus

$$\begin{aligned}
f(z)f(qz) &= \left(a + g(z)e^{\alpha z^s}\right)\left(a + g(qz)e^{\alpha q^s z^s}\right) \\
&= a^2 + ag(qz)e^{\alpha q^s z^s} + ag(z)e^{\alpha z^s} + g(z)g(qz)e^{\alpha(1+q^s)z^s}.
\end{aligned}$$

Since $q^{\sigma(f)} = q^s \neq \pm 1$, $g(z)$ is a nonzero entire function satisfying $\sigma(g) < s$, $f(z)f(qz)$ is a transcendental entire function and $\sigma(f(z)f(qz)) = s$, and by the classical result of Nevanlinna theory, we get $\sigma(H) = \sigma(f(z)f(qz)) = s$. Then

$$\begin{aligned}
H(z) &= \left[f(z)f(qz)\right]^{(k)} \\
&= \left[a^2 + ag(qz)e^{\alpha q^s z^s} + ag(z)e^{\alpha z^s} + g(z)g(qz)e^{\alpha(1+q^s)z^s}\right]^{(k)} \\
&= g_1(z)e^{\alpha q^s z^s} + g_2(z)e^{\alpha z^s} + g_3(z)e^{\alpha(1+q^s)z^s}.
\end{aligned}$$

Here $g_1(z), g_2(z), g_3(z)$ are differential polynomials of $g(z), g(qz)$ and $\sigma(g_i(z)) < s$, $i = 1, 2, 3$.

Case 1. $a = 0$, by the above equality, we get

$$H(z) = g_3(z)e^{\alpha(1+q^s)z^s}.$$

Since $q^{\sigma(f)} = q^s \neq -1$ and $\sigma(g_3(z)) < s$, this implies that 0 is the Borel exceptional value of $H(z)$.

Case 2. $a \neq 0$, suppose $H(z)$ has a finite Borel exceptional value b, then by Lemma 2.1, $H(z)$ can be written as

$$H(z) = b + h(z)e^{\beta z^s}.$$

Here β is a nonzero constant, $h(z)$ is a nonzero entire function satisfying $\sigma(h) < \sigma(H) = s$. Combing the above equalities, we obtain

$$g_1(z)e^{\alpha q^s z^s} + g_2(z)e^{\alpha z^s} + g_3(z)e^{\alpha(1+q^s)z^s} - h(z)e^{\beta z^s} - b = 0. \tag{3.3}$$

Since $q^{\sigma(f)} = q^s \neq \pm 1$, we have $\alpha q^s \neq \alpha \neq \alpha(1 + q^s)$.

If $\alpha q^s \neq \beta$, $\alpha \neq \beta$, $\alpha(1 + q^s) \neq \beta$. By (3.3) and Lemma 2.2, we obtain

$$g_1(z) \equiv 0, \qquad g_2(z) \equiv 0, \qquad g_3(z) \equiv 0, \qquad h(z) \equiv 0, \qquad b \equiv 0.$$

This is a contradiction.

If $\alpha q^s = \beta$, then (3.3) can be rewritten as

$$\left(g_1(z) - h(z)\right)e^{\alpha q^s z^s} + g_2(z)e^{\alpha z^s} + g_3(z)e^{\alpha(1+q^s)z^s} - b = 0. \tag{3.4}$$

By (3.4) and Lemma 2.2, we obtain

$$g_1(z) - h(z) \equiv 0, \qquad g_2(z) \equiv 0, \qquad g_3(z) \equiv 0, \qquad b \equiv 0.$$

Which is a contradiction.

If $\alpha = \beta$ or $\alpha(1 + q^s) = \beta$, then using the same method as above, we also obtain a contradiction.

Case 3. From Case 1 and Case 2, we get that if $f(z)$ has a finite Borel exceptional value, then any nonzero finite value c must not be the Borel exceptional value of $H(z)$, so $H(z)$ takes every nonzero value $c \in \mathbb{C}$ infinitely often, since $\sigma(H) = s$, then $\lambda(H - c) = \sigma(H) = \sigma(f)$.

The proof of Theorem 1.2 is completed.

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