Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Positive solutions to boundary value problems of fractional difference equation with nonlocal conditions

Shugui Kang[1*], Yan Li[2] and Huiqin Chen[1]

*Correspondence: dtkangshugui@126.com
[1]School of Mathematics and Computer Sciences, Shanxi Datong University, Datong, Shanxi 037009, P.R. China
Full list of author information is available at the end of the article

Abstract
In this paper, we will use the Krasnosel'skii fixed point theorem to investigate a discrete fractional boundary value problem of the form $-\Delta^{\nu}y(t)=\lambda h(t+\nu-1)f(y(t+\nu-1))$, $y(\nu-2)=\Psi(y)$, $y(\nu+b)=\Phi(y)$, where $1<\nu\le 2$, $t\in[0,b]_{\mathbb{N}_0}$, $f:[0,\infty)\to[0,\infty)$ is a continuous function, $h:[\nu-1,\nu+b-1]_{\mathbb{N}_{\nu-1}}\to[0,\infty)$, $\Psi,\Phi:\mathbb{R}^{b+3}\to\mathbb{R}$ are given functionals, where Ψ, Φ are linear functionals, and λ is a positive parameter.

MSC: 26A33; 39A05; 39A12

Keywords: nonlocal conditions; positive solution; cone; fixed point theorem

1 Introduction

The theory of fractional differential equations and their applications has received intensive attention. However, the theory of fraction difference equations is still limited. But in the last few years, a number of papers on fractional difference equations have appeared [1–13]. Among them, Atici and Eloe [1] introduced and developed properties of discrete fractional calculus. In [2], Atici and Eloe studied a two-point boundary valve problem for a finite fractional difference equation. They obtained sufficient conditions for the existence of solutions for the following boundary value problem:

$$-\Delta^{\nu}y(t)=f\big(t+\nu-1,y(t+\nu-1)\big),\qquad y(\nu-2)=0=y(\nu+b+1),$$

where $t\in[0,b+1]_{\mathbb{N}_0}$, $1<\nu\le 2$, and $f:[\nu-1,\nu+b]_{\mathbb{N}_{\nu-1}}\times\mathbb{R}\to\mathbb{R}$ is a continuous function. Goodrich [3] deduced uniqueness theorems by means of the Lipschitz condition and deduced the existence of one or more positive solutions by using the cone theoretic techniques for this same boundary value problem. He showed that many of the classical existence and uniqueness theorems for second-order discrete boundary value problems extend to the fraction-order case. In [4], Goodrich obtained the existence of positive solutions to another boundary value problem. Goodrich [5] also considered a pair of discrete fractional boundary value problem of the form

$$\begin{aligned}&-\Delta^{\nu_1}y_1(t)=\lambda_1a_1(t+\nu_1-1)f_1\big(y_1(t+\nu_1-1),y_2(t+\nu_2-1)\big),\\&-\Delta^{\nu_2}y_2(t)=\lambda_2a_2(t+\nu_2-1)f_2\big(y_1(t+\nu_1-1),y_2(t+\nu_2-1)\big)\end{aligned}$$

$$y_1(\nu_1 - 2) = \Psi_1(y_1), \qquad y_2(\nu_2 - 2) = \Psi_2(y_2),$$
$$y_1(\nu_1 + b) = \Phi_1(y_1), \qquad y_2(\nu_2 + b) = \Phi_2(y_2),$$

where $t \in [0, b]_{\mathbb{N}_0}$, $\lambda_i > 0$, $a_i : \mathbb{R} \to [0, \infty)$, $\nu_i \in (1, 2]$ for each $i = 1, 2$. $\Psi_i, \Phi_i : \mathbb{R}^{b+3} \to \mathbb{R}$ are given functionals, and $f_i : [0, \infty) \times [0, \infty) \to [0, \infty)$ is continuous for each admissible i.

Extensive literature exists on boundary value problems of fractional difference equations [9–13]. Ferreiraa [9] provided sufficient conditions for the existence and uniqueness of solution to some discrete fractional boundary value problems of order less than 1. Goodrich [11–13] studied a ν order ($1 < \nu \le 2$) discrete fractional three-point boundary value problem and semipositone discrete fractional boundary value problems.

In this paper, we consider the following boundary value problems of fractional difference equation with nonlocal conditions:

$$-\Delta^{\nu} y(t) = \lambda h(t + \nu - 1) f\bigl(y(t + \nu - 1)\bigr), \tag{1}$$
$$y(\nu - 2) = \Psi(y), \tag{2}$$
$$y(\nu + b) = \Phi(y), \tag{3}$$

where $t \in [0, b]_{\mathbb{N}_0}$, $1 < \nu \le 2$, $f : [0, \infty) \to [0, \infty)$ is a continuous function. $h : [\nu - 1, \nu + b - 1]_{\mathbb{N}_{\nu-1}} \to [0, \infty)$, $\Psi, \Phi : \mathbb{R}^{b+3} \to \mathbb{R}$ are given linear functionals and λ is a positive parameter. The boundary conditions (2)-(3) are generally called nonlocal conditions. Our analysis relies on the Krasnosel'kill fixed-point theorem to get the main results of problem (1)-(3).

The paper will be organized as follows. In Section 2, we will present basic definitions and demonstrate some lemmas in order to prove our main results. In Section 3, we establish some results for the existence of solutions to problem (1)-(3), and we provide an example to illustrate our main results.

2 Preliminaries

Let us first recall some basic lemmas which plays an important role in our discussions.

Definition 2.1 [1] We define

$$t^{(\nu)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)},$$

for any t and ν for which the right-hand side is defined. We also appeal to the convention that if $t + 1 - \nu$ is a pole of the Gamma function and $t + 1$ is not a pole, then $t^{(\nu)} = 0$.

Definition 2.2 [1] The νth fractional sum of a function f defined on the set $\mathbb{N}_a := \{a, a + 1, \ldots\}$, for $\nu > 0$, is defined to be

$$\Delta^{-\nu} f(t) = \Delta^{-\nu} f(t; a) := \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t - s - 1)^{(\nu-1)} f(s),$$

where $t \in \{a + \nu, a + \nu + 1, \ldots\} =: \mathbb{N}_{a+\nu}$. We also define the νth fractional difference, where $\nu > 0$ and $0 \le N - 1 < \nu \le N$, to be

$$\Delta^{\nu} f(t) = \Delta^{N} \Delta^{\nu - N} f(t),$$

where $t \in \mathbb{N}_{a+\nu}$.

Lemma 2.1 [1] *Let t and ν be any numbers for which $t^{(\nu)}$ and $t^{(\nu-1)}$ are defined. Then*

$$\Delta t^{(\nu)} = \nu t^{(\nu-1)}.$$

Lemma 2.2 [1] *Let $0 \le N-1 < \nu \le N$. Then*

$$\Delta^{-\nu}\Delta^{\nu} y(t) = y(t) + c_1 t^{(\nu-1)} + c_2 t^{(\nu-2)} + \cdots + c_N t^{(\nu-N)},$$

for some $c_i \in \mathbb{R}$, with $1 \le i \le N$.

Lemma 2.3 [4] *Let $1 < \nu \le 2$, and $h : [\nu-1, \nu+b-1]_{\mathbb{N}_{\nu-1}} \to \mathbb{R}$ be given. The unique solution of the FBVP*

$$-\Delta^{\nu} y(t) = g(t+\nu-1), \qquad y(\nu-2) = 0 = y(\nu+b)$$

is given by

$$y(t) = \sum_{s=0}^{b} G(t,s) g(s+\nu-1),$$

where $G : [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}} \times [0,b]_{\mathbb{N}_0} \to \mathbb{R}$ is defined by

$$G(t,s) = \frac{1}{\Gamma(\nu)} \begin{cases} \frac{t^{(\nu-1)}(\nu+b-s-1)^{(\nu-1)}}{(\nu+b)^{(\nu-1)}} - (t-s-1)^{(\nu-1)}, & 0 \le s < t-\nu+1 \le b, \\ \frac{t^{(\nu-1)}(\nu+b-s-1)^{(\nu-1)}}{(\nu+b)^{(\nu-1)}}, & 0 \le t-\nu+1 \le s \le b. \end{cases} \tag{4}$$

Lemma 2.4 [4] *The Green function $G(t,s)$ given in Lemma 2.3 satisfies:*

(i) $G(t,s) \ge 0$, *for each* $(t,s) \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}} \times [0,b]_{\mathbb{N}_0}$;

(ii) $\max_{t \in [\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}} G(t,s) = G(s+\nu-1, s)$ *for each* $s \in [0,b]_{\mathbb{N}_0}$; *and*

(iii) *there exists a number $\gamma \in (0,1)$ such that*

$$\min_{\frac{\nu+b}{4} \le t \le \frac{3(\nu+b)}{4}} G(t,s) \ge \gamma \max_{t \in [\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}} G(t,s) = \gamma G(s+\nu-1, s),$$

for $s \in [0,b]_{\mathbb{N}_0}$.

Theorem 2.1 *Let $f : [0,\infty) \to [0,\infty)$, and $\Psi, \Phi \in C([\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}, \mathbb{R})$ be given. A function $y(t)$ is a solution of the discrete FBVP* (1)-(3) *if and only if $y(t)$ is a fixed point of the operator*

$$Ty(t) := \alpha(t)\Psi(y) + \beta(t)\Phi(y) + \lambda \sum_{s=0}^{b} G(t,s) h(s+\nu-1) f\bigl(y(s+\nu-1)\bigr), \tag{5}$$

where

$$\alpha(t) := \frac{1}{\Gamma(\nu-1)} \left[t^{(\nu-2)} - \frac{1}{b+2} t^{(\nu-1)} \right], \tag{6}$$

$$\beta(t) := \frac{t^{(\nu-1)}}{(\nu+b)^{(\nu-1)}}, \tag{7}$$

and $G(t,s)$ is as given in Lemma 2.3.

Proof From Lemma 2.2, we find that a general solution to problem (1)-(3) is

$$y(t) = -\Delta^{-\nu}\lambda h(t+\nu-1)f\big(y(t+\nu-1)\big) + c_1 t^{(\nu-1)} + c_2 t^{(\nu-2)}.$$

From the boundary condition (2), we get

$$\begin{aligned} y(\nu-2) &= -\Delta^{-\nu}\lambda h(t+\nu-1)f\big(y(t+\nu-1)\big)|_{t=\nu-2} + c_1(\nu-2)^{(\nu-1)} + c_2(\nu-2)^{(\nu-2)} \\ &= -\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{(\nu-1)}\lambda h(s+\nu-1)f\big(y(s+\nu-1)\big)|_{t=\nu-2} + c_2\Gamma(\nu-1) \\ &= c_2\Gamma(\nu-1) \\ &= \Psi(y), \end{aligned}$$

so

$$c_2 = \frac{\Psi(y)}{\Gamma(\nu-1)}.$$

On the other hand, applying the boundary condition (3) to $y(t)$ implies that

$$\begin{aligned} y(\nu+b) &= -\Delta^{-\nu}\lambda h(t+\nu-1)f\big(y(t+\nu-1)\big)|_{t=\nu+b} + c_1(\nu+b)^{(\nu-1)} + c_2(\nu+b)^{(\nu-2)} \\ &= -\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{(\nu-1)}\lambda h(s+\nu-1)f\big(y(s+\nu-1)\big)|_{t=\nu+b} \\ &\quad + c_1(\nu+b)^{(\nu-1)} + \frac{\Psi(y)}{\Gamma(\nu-1)}(\nu+b)^{(\nu-2)} \\ &= \Phi(y). \end{aligned}$$

Namely

$$\begin{aligned} c_1(\nu+b)^{(\nu-1)} = {}& \Phi(y) - \frac{\Psi(y)}{\Gamma(\nu-1)}(\nu+b)^{(\nu-2)} \\ & + \frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{(\nu-1)}\lambda h(s+\nu-1)f\big(y(s+\nu-1)\big)|_{t=\nu+b}, \end{aligned}$$

so

$$\begin{aligned} c_1 &= \frac{\Phi(y)}{(\nu+b)^{(\nu-1)}} - \frac{\Psi(y)(\nu+b)^{(\nu-2)}}{\Gamma(\nu-1)(\nu+b)^{(\nu-1)}} \\ &\quad + \frac{1}{\Gamma(\nu)(\nu+b)^{(\nu-1)}}\sum_{s=0}^{b}(\nu+b-s-1)^{(\nu-1)}\lambda h(s+\nu-1)f\big(y(s+\nu-1)\big) \\ &= \frac{\Phi(y)}{(\nu+b)^{(\nu-1)}} - \frac{\Psi(y)}{(b+2)\Gamma(\nu-1)} \\ &\quad + \frac{1}{\Gamma(\nu)(\nu+b)^{(\nu-1)}}\sum_{s=0}^{b}(\nu+b-s-1)^{(\nu-1)}\lambda h(s+\nu-1)f\big(y(s+\nu-1)\big). \end{aligned}$$

Finally, we get

$$\begin{aligned}
y(t) &= -\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-s-1)^{(\nu-1)}\lambda h(s+\nu-1)f\bigl(y(s+\nu-1)\bigr) \\
&\quad+\left(\frac{\Phi(y)}{(\nu+b)^{(\nu-1)}}-\frac{\Psi(y)}{(b+2)\Gamma(\nu-1)}\right)t^{(\nu-1)} \\
&\quad+\frac{t^{(\nu-1)}}{\Gamma(\nu)(\nu+b)^{(\nu-1)}}\sum_{s=0}^{b}(\nu+b-s-1)^{(\nu-1)}\lambda h(s+\nu-1)f\bigl(y(s+\nu-1)\bigr) \\
&\quad+\frac{\Psi(y)}{\Gamma(\nu-1)}t^{(\nu-2)} \\
&= \Psi(y)\left(-\frac{t^{(\nu-1)}}{(b+2)\Gamma(\nu-1)}+\frac{t^{(\nu-2)}}{\Gamma(\nu-1)}\right)+\Phi(y)\cdot\frac{t^{(\nu-1)}}{(\nu+b)^{(\nu-1)}} \\
&\quad+\sum_{s=0}^{t-\nu}\left(\frac{t^{(\nu-1)}(\nu+b-s-1)^{(\nu-1)}}{\Gamma(\nu)(\nu+b)^{(\nu-1)}}-\frac{(t-s-1)^{(\nu-1)}}{\Gamma(\nu)}\right)\lambda h(s+\nu-1)f\bigl(y(s+\nu-1)\bigr) \\
&\quad+\sum_{s=t-\nu+1}^{b}\frac{t^{(\nu-1)}(\nu+b-s-1)^{(\nu-1)}}{\Gamma(\nu)(\nu+b)^{(\nu-1)}}\lambda h(s+\nu-1)f\bigl(y(s+\nu-1)\bigr) \\
&= \Psi(y)\alpha(t)+\Phi(y)\beta(t)+\lambda\sum_{s=0}^{b}G(t,s)h(s+\nu-1)f\bigl(y(s+\nu-1)\bigr).
\end{aligned}$$

Consequently, we see that $y(t)$ is a solution of (1)-(3) if and only if $y(t)$ is a fixed point of (5), as desired. □

Lemma 2.5 *The function $\alpha(t)$ is strictly decreasing in t, for $t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}$. In addition, $\min_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}}\alpha(t)=0$, and $\max_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}}\alpha(t)=1$. On the other hand, the function $\beta(t)$ is strictly increasing in t, for $t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}$. In addition, $\min_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}}\beta(t)=0$, and $\max_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}}\beta(t)=1$.*

Proof Note that for every $t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}$,

$$\begin{aligned}
\Delta_t\alpha(t) &= \Delta_t\left(\frac{t^{(\nu-2)}}{\Gamma(\nu-1)}-\frac{t^{(\nu-1)}}{(b+2)\Gamma(\nu-1)}\right) \\
&= \frac{1}{\Gamma(\nu-1)}\left((\nu-2)t^{(\nu-3)}-(\nu-1)\frac{t^{(\nu-2)}}{b+2}\right) \\
&< 0.
\end{aligned}$$

So, the first claim about $\alpha(t)$ holds. On the other hand,

$$\begin{aligned}
\alpha(\nu-2) &= \frac{(\nu-2)^{(\nu-2)}}{\Gamma(\nu-1)}-\frac{(\nu-2)^{(\nu-1)}}{(b+2)\Gamma(\nu-1)}=1, \\
\alpha(\nu+b) &= \frac{(\nu+b)^{(\nu-2)}}{\Gamma(\nu-1)}-\frac{(\nu+b)^{(\nu-1)}}{(b+2)\Gamma(\nu-1)} \\
&= \frac{\Gamma(\nu+b+1)}{\Gamma(\nu+b+1-\nu+2)\Gamma(\nu-1)}-\frac{\Gamma(\nu+b+1)}{\Gamma(\nu+b+1-\nu+1)(b+2)\Gamma(\nu-1)}
\end{aligned}$$

$$= \frac{\Gamma(\nu+b+1)}{\Gamma(b+3)\Gamma(\nu-1)} - \frac{\Gamma(\nu+b+1)}{\Gamma(b+2)(b+2)\Gamma(\nu-1)}$$
$$= 0.$$

It follows that

$$\max_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}} \alpha(t) = 1, \qquad \min_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}} \alpha(t) = 0.$$

In a similar way, it may be shown that $\beta(t)$ satisfies the properties given in the statement of this lemma. We omit the details. □

Corollary 2.1 *Let* $I = [\frac{b+\nu}{4}, \frac{3(b+\nu)}{4}]$. *There are constants* $M_\alpha, M_\beta \in (0,1)$ *such that* $\min_{t\in I} \alpha(t) = M_\alpha\|\alpha\|$, $\min_{t\in I} \beta(t) = M_\beta\|\beta\|$, *where* $\|\cdot\|$ *is the usual maximum norm.*

Lemma 2.6 [14] *Let E be a Banach space, and let* $\mathcal{K} \subset E$ *be a cone in E. Assume the bounded sets* Ω_1, Ω_2 *are open subsets of E with* $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, *and let*

$$S : \mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathcal{K}$$

be a completely continuous operator such that either

(1) $\|Su\| \leq \|u\|$, $u \in \mathcal{K} \cap \partial\Omega_1$, $\|Su\| \geq \|u\|$, $u \in \mathcal{K} \cap \partial\Omega_2$; *or*

(2) $\|Su\| \geq \|u\|$, $u \in K \cap \partial\Omega_1$, $\|Su\| \leq \|u\|$, $u \in K \cap \partial\Omega_2$.

Then S has a fixed point in $\mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3 Main results

In the sequel, we let

$$\sigma := 2 \max_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}} \sum_{s=0}^{b} G(t,s)h(s+\nu-1), \tag{8}$$

$$\tau := \min_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}} \sum_{s=[\frac{b+\nu}{4}-\nu+1]}^{[\frac{3(b+\nu)}{4}-\nu+1]} G(t,s)h(s+\nu-1), \tag{9}$$

$$\widetilde{\gamma} := \min\{\gamma, M_\alpha, M_\beta\}, \tag{10}$$

where γ is the number given in Lemma 2.4(iii). Observe that $\widetilde{\gamma} \in (0,1)$. We now present the conditions on Ψ, Φ, and f that we presume in the sequel:

(S1) $\lim_{y\to 0} \frac{f(y)}{y} = \infty$, $\lim_{y\to\infty} \frac{f(y)}{y} = \infty$.

(S2) $\lim_{y\to 0} \frac{f(y)}{y} = 0$, $\lim_{y\to\infty} \frac{f(y)}{y} = 0$.

(S3) $\lim_{y\to 0} \frac{f(y)}{y} = l$, $0 < l < \infty$.

(S4) $\lim_{y\to\infty} \frac{f(y)}{y} = L$, $0 < L < \infty$.

(G1) The functionals Ψ, Φ are linear. In particular, we assume that

$$\Psi(y) = \sum_{i=\nu-2}^{\nu+b} c_{i-\nu+2}y(i), \qquad \Phi(y) = \sum_{k=\nu-2}^{\nu+b} d_{k-\nu+2}y(k),$$

for $c_{i-\nu+2}, d_{k-\nu+2} \in \mathbb{R}$.

(G2) We have both $\sum_{i=\nu-2}^{\nu+b} c_{i-\nu+2}G(i,s) \geq 0$ and $\sum_{k=\nu-2}^{\nu+b} d_{k-\nu+2}G(k,s) \geq 0$, for each $s \in [0,b]_{\mathbb{N}_0}$, and

$$\sum_{i=\nu-2}^{\nu+b} c_{i-\nu+2} + \sum_{k=\nu-2}^{\nu+b} d_{k-\nu+2} \leq \frac{1}{2}.$$

(G3) Each of $\Psi(\alpha)$, $\Psi(\beta)$, $\Phi(\alpha)$, $\Phi(\beta)$ is nonnegative.

First, we let $\mathcal{B}$ represent all maps from $[\nu-2, \nu_+ b]_{\mathbb{N}_{\nu-2}}$ into $\mathbb{R}$, and equipped with the maximum norm $\|\cdot\|$. Clearly, $\mathcal{B}$ is a Banach space. We define the cone $\mathcal{K} \subset \mathcal{B}$ by

$$\mathcal{K} := \Big\{ y \in \mathcal{B} | y(t) \geq 0, \min_{t \in [\frac{b+\nu}{4}, \frac{3(\nu+b)}{4}]} y(t) \geq \widetilde{\gamma}\|y\|, \Psi(y) \geq 0, \Phi(y) \geq 0 \Big\}. \tag{11}$$

Lemma 3.1 *Assume that* (G1)-(G3) *hold, and let* T *be the operator defined in* (5). *Then* $T : \mathcal{K} \to \mathcal{K}$.

Proof For every $y \in \mathcal{K}$, by (G1), we show first that

$$\begin{aligned}
\Psi(Ty) &= \sum_{i=\nu-2}^{\nu+b} c_{i-\nu+2}(Ty)(i) = \sum_{i=\nu-2}^{\nu+b} c_{i-\nu+2} \sum_{s=0}^{b} G(i,s)\lambda h(s+\nu-1) f\big(y(s+\nu-1)\big) \\
&\quad + \sum_{i=\nu-2}^{\nu+b} \sum_{j=\nu-2}^{\nu+b} c_{i-\nu+2} c_{j-\nu+2} y(j)\alpha(i) + \sum_{i=\nu-2}^{\nu+b} \sum_{k=\nu-2}^{\nu+b} c_{i-\nu+2} d_{k-\nu+2} y(k)\beta(i) \\
&= \Psi\left(\sum_{s=0}^{b} G(t,s)\lambda h(s+\nu-1) f\big(y(s+\nu-1)\big) \right) + \Psi(\alpha)\Psi(y) + \Psi(\beta)\Phi(y).
\end{aligned}$$

By assumptions (G2) and (G3) together with the nonnegativity of $f(y)$ and the fact that $y \in \mathcal{K}$, we can get $\Psi(Ty) \geq 0$:

$$\begin{aligned}
\Phi(Ty) &= \sum_{k=\nu-2}^{\nu+b} d_{k-\nu+2}(Ty)(k) = \sum_{k=\nu-2}^{\nu+b} d_{k-\nu+2} \sum_{s=0}^{b} G(k,s)\lambda h(s+\nu-1) f\big(y(s+\nu-1)\big) \\
&\quad + \sum_{k=\nu-2}^{\nu+b} \sum_{l=\nu-2}^{\nu+b} d_{k-\nu+2} c_{l-\nu+2} y(l)\alpha(k) + \sum_{k=\nu-2}^{\nu+b} \sum_{i=\nu-2}^{\nu+b} d_{k-\nu+2} d_{i-\nu+2} y(i)\beta(k) \\
&= \Phi\left(\sum_{s=0}^{b} G(t,s)\lambda h(s+\nu-1) f\big(y(s+\nu-1)\big) \right) + \Phi(\alpha)\Psi(y) + \Phi(\beta)\Phi(y).
\end{aligned}$$

It also shows that $\Phi(Ty) \geq 0$.

On the other hand, it follows from both Lemma 2.4 and Corollary 2.1 that

$$\begin{aligned}
\min_{t \in [\frac{b+\nu}{4}, \frac{3(\nu+b)}{4}]} (Ty)(t) &\geq \min_{t \in [\frac{b+\nu}{4}, \frac{3(\nu+b)}{4}]} \lambda \sum_{s=0}^{b} G(t,s) h(s+\nu-1) f\big(y(s+\nu-1)\big) \\
&\quad + \min_{t \in [\frac{b+\nu}{4}, \frac{3(\nu+b)}{4}]} \alpha(t)\Psi(y) + \min_{t \in [\frac{b+\nu}{4}, \frac{3(\nu+b)}{4}]} \beta(t)\Phi(y) \\
&\geq \widetilde{\gamma} \max_{t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}} \lambda \sum_{s=0}^{b} G(t,s) h(s+\nu-1) f\big(y(s+\nu-1)\big)
\end{aligned}$$

$$\begin{aligned}
&+ M_\alpha \|\alpha\| \Psi(y) + M_\beta \|\beta\| \Phi(y) \\
&\geq \widetilde{\gamma} \max_{t \in [\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}} \lambda \sum_{s=0}^{b} G(t,s) h(s+\nu-1) f\bigl(y(s+\nu-1)\bigr) \\
&\quad + \widetilde{\gamma} \|\alpha\| \Psi(y) + \widetilde{\gamma} \|\beta\| \Phi(y) \\
&\geq \widetilde{\gamma} \|Ty\|.
\end{aligned}$$

Hence

$$\min_{t \in [\frac{b+\nu}{4}, \frac{3(\nu+b)}{4}]} (Ty)(t) \geq \widetilde{\gamma} \|Ty\|.$$

Finally, for every $y \in \mathcal{K}$,

$$(Ty)(t) = \alpha(t)\Psi(y) + \beta(t)\Phi(y) + \lambda \sum_{s=0}^{b} G(t,s) h(s+\nu-1) f\bigl(y(s+\nu-1)\bigr) \geq 0.$$

So, we conclude that $T : \mathcal{K} \to \mathcal{K}$, and the proof is complete. □

Lemma 3.2 *Suppose that conditions* (G1)-(G3) *hold, and there exist two different positive numbers a, b such that*

$$\max_{0 \leq y \leq a} f(y) \leq \frac{a}{\lambda\sigma}; \tag{12}$$

$$\min_{\widetilde{\gamma} b \leq y \leq b} f(y) \geq \frac{b}{\lambda\tau}. \tag{13}$$

Then, problem (1)-(3) *has at least one positive solution $\overline{y} \in \mathcal{K}$ such that* $\min\{a,b\} \leq \|\overline{y}\| \leq \max\{a,b\}$.

Proof Let $\Omega_a := \{y \in \mathcal{K} : \|y\| < a\}$. Then, for any $y \in \mathcal{K} \cap \partial\Omega_a$, we have

$$\begin{aligned}
(Ty)(t) &= \alpha(t)\Psi(y) + \beta(t)\Phi(y) + \lambda \sum_{s=0}^{b} G(t,s) h(s+\nu-1) f\bigl(y(s+\nu-1)\bigr) \\
&\leq \Psi(y) + \Phi(y) + \lambda \sum_{s=0}^{b} G(t,s) h(s+\nu-1) \frac{a}{\lambda\sigma} \\
&\leq \sum_{i=\nu-2}^{\nu+b} c_{i-\nu+2} y(i) + \sum_{k=\nu-2}^{\nu+b} d_{k-\nu+2} y(k) + \lambda \frac{\sigma}{2} \cdot \frac{a}{\lambda\sigma} \\
&\leq \left(\sum_{i=\nu-2}^{\nu+b} c_{i-\nu+2} + \sum_{k=\nu-2}^{\nu+b} d_{k-\nu+2} \right) \|y\| + \frac{a}{2} \\
&\leq \frac{a}{2} + \frac{a}{2} \\
&= \|y\|,
\end{aligned}$$

that is, $\|Ty\| \leq \|y\|$ for $y \in \partial\Omega_a \cap \mathcal{K}$.

On the other hand, we let $\Omega_b := \{y \in \mathcal{K} : \|y\| < b\}$. For any $y \in \mathcal{K} \cap \partial\Omega_b$, we have

$$\begin{aligned}(Ty)(t) &= \alpha(t)\Psi(y) + \beta(t)\Phi(y) + \lambda \sum_{s=0}^{b} G(t,s)h(s+\nu-1)f\bigl(y(s+\nu-1)\bigr) \\ &\geq \lambda \sum_{s=[\frac{b+\nu}{4}-\nu+1]}^{[\frac{3(b+\nu)}{4}-\nu+1]} G(t,s)h(s+\nu-1)f\bigl(y(s+\nu-1)\bigr) \\ &\geq \lambda \sum_{s=[\frac{b+\nu}{4}-\nu+1]}^{[\frac{3(b+\nu)}{4}-\nu+1]} G(t,s)h(s+\nu-1)\frac{b}{\lambda\tau} \\ &\geq \lambda\tau \cdot \frac{b}{\lambda\tau} \\ &= \|y\|,\end{aligned}$$

that is, $\|Ty\| \geq \|y\|$ for $y \in \mathcal{K} \cap \partial\Omega_b$. By means of Lemma 2.6, there exists $\overline{y} \in \mathcal{K}$ such that $T\overline{y} = \overline{y}$. □

Theorem 3.1 *Suppose that conditions* (S1), *and* (G1)-(G3) *hold. Then, for every* $\lambda \in (0, \lambda^*)$, *problem* (1)-(3) *has at least two positive solutions, where*

$$\lambda^* = \frac{1}{\sigma} \sup_{r>0} \frac{r}{\max_{0\leq y\leq r} f(y)}. \tag{14}$$

Proof Define function $q(r) = \frac{r}{\sigma \max_{0\leq y\leq r} f(y)}$. In view of the continuity of the function $f(y)$, we have $q(r) \in C((0,\infty),(0,\infty))$. From $\lim_{y\to 0} \frac{f(y)}{y} = \infty$, we see that $\lim_{r\to 0} \frac{r}{\sigma f(r)} = 0$, that is,

$$\frac{r}{\sigma f(r)} \geq \frac{r}{\sigma \max_{0\leq y\leq r} f(y)} = q(r) > 0,$$

so

$$\lim_{r\to 0} q(r) = 0.$$

By $\lim_{y\to\infty} \frac{f(y)}{y} = \infty$, we see further that $\lim_{r\to\infty} q(r) = 0$. Thus, there exists $r_0 > 0$ such that $q(r_0) = \max_{r>0} q(r) = \lambda^*$. For any $\lambda \in (0, \lambda^*)$, by means of the intermediate value theorem, there exist two points l_1, l_2 $(0 < l_1 < r_0 < l_2 < \infty)$ such that $q(l_1) = q(l_2) = \lambda$. Thus, we have

$$f(y) \leq \frac{l_1}{\lambda\sigma}, \quad y \in [0, l_1]; \qquad f(y) \leq \frac{c_2}{\lambda\sigma}, \quad y \in [0, l_2].$$

On the other hand, in view of (S1), we see that there exist d_1, d_2 $(0 < d_1 < l_1 < r_0 < l_2 < d_2 < \infty)$ such that

$$\frac{f(y)}{y} \geq \frac{1}{\lambda\widetilde{\gamma}\tau}, \quad y \in (0, d_1] \cup [d_2\widetilde{\gamma}, \infty).$$

That is,

$$f(y) \geq \frac{d_1}{\lambda\tau}, \quad y \in [d_1\widetilde{\gamma}, d_1]; \qquad f(y) \geq \frac{d_2}{\lambda\tau}, \quad y \in [d_2\widetilde{\gamma}, d_2].$$

An application of Lemma 3.2 leads to two distinct solutions of (1)-(3) which satisfy $d_1 \leq \|\overline{y_1}\| \leq l_1, l_2 \leq \|\overline{y_2}\| \leq d_2$. □

Theorem 3.2 *Suppose* (S2) *and* (G1)-(G3) *hold. Then for any* $\lambda > \lambda^{**}$, *equations* (1)-(3) *have at least two positive solutions. Here*

$$\lambda^{**} = \frac{1}{\tau} \inf_{r>0} \frac{r}{\min_{\widetilde{\gamma} r \leq y \leq r} f(y)}. \tag{15}$$

The proof is similar to Theorem 3.1 and hence is omitted.

Theorem 3.3 *Assume that* (S3), (S4) *and* (G1)-(G3) *hold. For each* λ *satisfying*

$$\frac{1}{\widetilde{\gamma}\tau L} < \lambda < \frac{1}{\sigma l}, \tag{16}$$

or

$$\frac{1}{\widetilde{\gamma}\tau l} < \lambda < \frac{1}{\sigma L}, \tag{17}$$

equations (1)-(3) *have a positive solution.*

Proof Suppose (16) holds. Let $\eta > 0$ be such that

$$\frac{1}{\widetilde{\gamma}\tau(L-\eta)} \leq \lambda \leq \frac{1}{\sigma(l+\eta)}.$$

Note that $l > 0$. There exists $H_1 > 0$ such that $f(y) \leq (l+\eta)y$ for $0 < y \leq H_1$. So, for $y \in \mathcal{K}$ and $\|y\| = H_1$, we have

$$\begin{aligned}
(Ty)(t) &= \alpha(t)\Psi(y) + \beta(t)\Phi(y) + \lambda \sum_{s=0}^{b} G(t,s)h(s+\nu-1)f\big(y(s+\nu-1)\big) \\
&\leq \Psi(y) + \Phi(y) + \lambda(l+\eta) \sum_{s=0}^{b} G(t,s)h(s+\nu-1)y(s+\nu-1) \\
&= \sum_{i=\nu-2}^{\nu+b} c_{i-\nu+2}y(i) + \sum_{k=\nu-2}^{\nu+b} d_{k-\nu+2}y(k) + \lambda(l+\eta)\sum_{s=0}^{b} G(t,s)h(s+\nu-1)y(s+\nu-1) \\
&\leq \left(\sum_{i=\nu-2}^{\nu+b} c_{i-\nu+2} + \sum_{k=\nu-2}^{\nu+b} d_{k-\nu+2} \right)\|y\| + \lambda(l+\eta)\cdot\frac{\sigma}{2}\|y\| \\
&\leq \frac{H_1}{2} + \frac{H_1}{2} \\
&= \|y\|.
\end{aligned}$$

That is, $\|Ty\| \leq \|y\|$ for $y \in \mathcal{K}$ and $\|y\| = H_1$.

Next, since $L > 0$, there exists a $\overline{H}_2 > 0$ such that $f(y) \geq (L-\eta)y$ for $y \geq \widetilde{\gamma}\overline{H}_2$. Let $H_2 = \max\{2H_1, \overline{H}_2\}$. Then, for $y \in \mathcal{K}$ with $\|y\| = H_2$,

$$\begin{aligned}(Ty)(t) &= \alpha(t)\Psi(y) + \beta(t)\Phi(y) + \lambda\sum_{s=0}^{b} G(t,s)h(s+\nu-1)f\bigl(y(s+\nu-1)\bigr)\\&\geq \lambda\sum_{s=0}^{b} G(t,s)h(s+\nu-1)f\bigl(y(s+\nu-1)\bigr)\\&\geq \lambda(L-\eta)\sum_{s=[\frac{b+\nu}{4}-\nu+1]}^{[\frac{3(b+\nu)}{4}-\nu+1]} G(t,s)h(s+\nu-1)y(s+\nu-1)\\&\geq \lambda(L-\eta)\widetilde{\gamma}\|y\|\sum_{s=[\frac{b+\nu}{4}-\nu+1]}^{[\frac{3(b+\nu)}{4}-\nu+1]} G(t,s)h(s+\nu-1)\\&\geq \lambda(L-\eta)\widetilde{\gamma}\|y\|\tau\\&\geq \|y\|,\end{aligned}$$

that is, $\|Ty\| \geq \|y\|$ for $y \in \mathcal{K}$ and $\|y\| = H_2$.

In view of Lemma 2.6, we see that equations (1)-(3) have a positive solution. The other case is similarly proved. □

Example 3.1 Consider the following boundary value problems:

$$\triangle^{\frac{13}{10}} y(t) = -\lambda\left(y^{\frac{1}{2}}\left(t+\frac{3}{10}\right) + y^2\left(t+\frac{3}{10}\right)\right), \tag{18}$$

$$y\left(-\frac{7}{10}\right) = \frac{1}{12}y\left(\frac{13}{10}\right) - \frac{1}{25}y\left(\frac{53}{10}\right), \tag{19}$$

$$y\left(\frac{213}{10}\right) = \frac{1}{30}y\left(\frac{83}{10}\right) - \frac{1}{100}y\left(\frac{73}{10}\right), \tag{20}$$

where $b = 20$, $\nu = \frac{13}{10}$, and we take

$$f(y) = y^{\frac{1}{2}} + y^2,$$
$$\psi(y) = \frac{1}{12}y\left(\frac{13}{10}\right) - \frac{1}{25}y\left(\frac{53}{10}\right), \qquad \phi(y) = \frac{1}{30}y\left(\frac{83}{10}\right) - \frac{1}{100}y\left(\frac{73}{10}\right),$$

$f: [0,+\infty) \times [0,+\infty) \to [0,+\infty)$, and y is defined on the time scale $\{-\frac{7}{10}, \frac{3}{10}, \ldots, \frac{213}{10}\}$, f and ψ, ϕ satisfy conditions of Theorem 3.1.

A computation shows that $\lambda^* \approx 5.33 \times 10^{-3}$. Then, for every $\lambda \in (0, \lambda^*)$, problem (18)-(20) has at least two positive solutions.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
SK conceived of the study, and participated in its design and coordination. YL drafted the manuscript. HC participated in the design of the study and the sequence correction. All authors read and approved the final manuscript.

Author details
[1]School of Mathematics and Computer Sciences, Shanxi Datong University, Datong, Shanxi 037009, P.R. China.
[2]Department of Mathematics, Shanxi Normal University, Linfen, Shanxi 041000, P.R. China.

Acknowledgements
The authors are very grateful to the reviewers for their valuable suggestions and useful comments, which led to an improvement of this paper. Project supported by the National Natural Science Foundation of China (Grant No. 11271235) and Shanxi Province (2008011002-1) and Shanxi Datong University Institute (2009-Y-15, 2010-B-01, 2013K5) and the Development Foundation of Higher Education Department of Shanxi Province (20101109, 20111020).

Received: 29 July 2013 Accepted: 11 December 2013 Published: 07 Jan 2014

References

1. Atici, FM, Eloe, PW: Discrete fractional calculus with the nabla operator. Electron. J. Qual. Theory Differ. Equ. (Spec. Ed. I) **3**, 1-12 (2009)
2. Atici, FM, Eloe, PW: Two-point boundary value problems for finite fractional difference equations. J. Differ. Equ. Appl. **17**(4), 445-456 (2011)
3. Goodrich, CS: Some new existence results for fractional difference equations. Int. J. Dyn. Syst. Differ. Equ. **3**, 145-162 (2011)
4. Goodrich, CS: Solutions to a discrete right-focal fractional boundary value problem. Int. J. Differ. Equ. **5**(2), 195-216 (2010)
5. Goodrich, CS: Existence of a positive solution to a system of discrete fractional boundary value problems. Comput. Math. Appl. **217**, 4740-4753 (2011)
6. Goodrich, CS: Existence and uniqueness of solutions to a fractional difference equation with nonlocal conditions. Comput. Math. Appl. **61**, 191-202 (2011)
7. Bastos, NRO: Discrete-time fractional variational problems. Signal Process. **91**, 513-524 (2011)
8. Benchohra, M, Hamani, S, Ntouyas, SK: Boundary value problems for differential equations with fractional order and nonlocal conditions. Nonlinear Anal. **71**, 2391-2396 (2009)
9. Ferreira, RAC: Existence and uniqueness of solution to some discrete fractional boundary value problems of order less than one. J. Differ. Equ. Appl. **19**, 712-718 (2013)
10. Ferreira, RAC, Torres, DFM: Fractional *h*-difference equations arising from the calculus of variations. Appl. Anal. Discrete Math. **5**, 110-121 (2011)
11. Goodrich, CS: On a discrete fractional three-point boundary value problem. J. Differ. Equ. Appl. **18**, 397-415 (2012)
12. Goodrich, CS: On a first-order semipositone discrete fractional boundary value problem. Arch. Math. **99**, 509-518 (2012)
13. Goodrich, CS: On semipositone discrete fractional boundary value problems with nonlocal boundary conditions. J. Differ. Equ. Appl. **19**, 1758-1780 (2013)
14. Agarwal, R, Meehan, M, O'Regan, D: Fixed Point Theory and Applications. Cambridge University Press, Cambridge (2001)

10.1186/1687-1847-2014-7
Cite this article as: Kang et al.: Positive solutions to boundary value problems of fractional difference equation with nonlocal conditions. *Advances in Difference Equations* 2014, 2014:7