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Positive solutions of nonlinear boundary value problems for delayed fractional q-difference systems

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Abstract

In this paper, we investigate the existence and uniqueness of positive solutions to nonlinear boundary value problems for delayed fractional q-difference systems by applying the properties of the Green function and some well-known fixed-point theorems. As applications, some examples are presented to illustrate the main results.
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1 Introduction

In the past decades, fractional differential equations have been proved to be valuable tools in the investigation of many phenomena in various fields of science and engineering such as physics, mechanics, chemistry, biology, engineering, etc. Therefore, the subject of fractional differential equations has gained considerable attention by many researchers. Some recent results on fractional boundary value problems can be found in [1–4] and references therein. For example, Ahmad and Nieto [5] dealt with some existence results for a boundary value problem involving a nonlinear fractional order integrodifferential equation with integral boundary conditions based on a contraction mapping principle and Krasnoselskiii's fixed-point theorem. Ahmad *et al.* [6] investigated the existence and uniqueness of solutions for a class of Caputo-type fractional boundary value problems involving four-point nonlocal Riemann-Liouville integral boundary conditions of different order by means of standard tools of fixed-point theory and Leray-Schauder nonlinear alternative. Ouyang *et al.* [7] considered the following nonlinear system of fractional order differential equations with delays:

$$\left(D^{\alpha_i} u_i\right)(t) + f_i\left(t, u_1\left(\tau_{i1}(t)\right), \ldots, u_N\left(\tau_{iN}(t)\right)\right) = 0, \quad 0 < t < 1,$$

$$u_i^{(j)}(0) = 0, \quad j = 0, 1, \ldots, n_i - 2, i = 1, 2, \ldots, N,$$

$$u_i^{(n_i-1)}(1) = \eta_i, \quad i = 1, 2, \ldots, N,$$

where D^{α_i} is the standard Riemann-Liouville fractional derivative. By using some fixed-point theorems and some properties of the Green function, the existence of positive solutions was obtained.

The q-difference calculus or quantum calculus is an old subject that was initially developed by Jackson [8, 9]; basic definitions and properties of q-difference calculus can be found in the book mentioned in [10].

The fractional q-difference calculus had its origin in the works by Al-Salam [11] and Agarwal [12]. Recently, maybe due to the explosion in research within the fractional differential calculus setting, new developments in this theory of fractional q-difference calculus were made; for example, q-analogues of the integral and differential fractional operators properties such as the q-Laplace transform, q-Taylor's formula, Mittage-Leffler function [13–16], just to mention some.

More recently, boundary value problems of nonlinear fractional q-difference equations have gained popularity and importance. Many researchers pay attention to the existence and multiplicity of solutions or positive solutions for nonlinear boundary value problems of fractional q-difference equations by means of upper and lower solutions method and some fixed-point theorems such as the Krasnoselskii fixed-point theorem, the Leggett-Williams fixed-point theorem, and the Schauder fixed-point theorem; for examples, see [17–21] and the references therein. El-Shahed and Al-Askar [22] studied the existence of multiple positive solutions to the nonlinear q-fractional boundary value problems by using Guo-Krasnoselskii's fixed-point theorem in a cone. Graef and Kong [23] investigated the uniqueness, existence, and nonexistence of positive solutions for the boundary value problem with fractional q-derivatives in terms of different ranges of λ. Ma and Yang [24] obtained the existence of solutions for multi-point boundary value problems of nonlinear fractional q-difference equations by means of the Banach contraction principle and Krasnoselskii's fixed-point theorem. Zhao *et al.* [25] showed some existence results of positive solutions to nonlocal q-integral boundary value problems of a nonlinear fractional q-derivative equation using the generalized Banach contraction principle, the monotone iterative method, and Krasnoselskii's fixed-point theorem. Ferreira [26] and [27] dealt with the existence of positive solutions to nonlinear q-difference boundary value problems,

$$\big(D_q^\alpha u\big)(t) = -f\big(t, u(t)\big), \quad 0 \le t \le 1, 1 < \alpha \le 2,$$
$$u(0) = u(1) = 0,$$

and

$$\big(D_q^\alpha u\big)(t) = -f\big(t, u(t)\big), \quad 0 \le t \le 1, 2 < \alpha \le 3,$$
$$u(0) = (D_q u)(0) = 0, \qquad (D_q u)(1) = \beta \ge 0,$$

respectively. By applying a fixed-point theorem in cones, sufficient conditions for the existence of nontrivial solutions were enunciated.

In [28], Liang and Zhang discussed the following nonlinear q-fractional three-point boundary value problem:

$$\big(D_q^\alpha u\big)(t) + f\big(t, u(t)\big) = 0, \quad t \in [0, 1], \alpha \in (2, 3],$$
$$u(0) = (D_q u)(0) = 0, \qquad (D_q u)(1) = \beta(D_q u)(\eta).$$

By using a fixed-point theorem in partially ordered sets, the authors obtained sufficient conditions for the existence and uniqueness of positive and nondecreasing solutions to the above boundary value problem.

In [29], Ahmad *et al.* studied the following nonlocal boundary value problems of nonlinear fractional q-difference equations,

$$\left({}^c D_q^\alpha u\right)(t) = f\left(t, u(t)\right), \quad t \in [0,1], \alpha \in (1,2],$$

$$a_1 u(0) - b_1 (D_q u)(0) = c_1 u(\eta_1), \qquad a_2 u(1) + b_2 (D_q u)(1) = c_2 u(\eta_2),$$

where ${}^c D_q^\alpha$ denotes the Caputo fractional q-derivative of order α, and $a_i, b_i, c_i, \eta_i \in \mathbb{R}$ ($i = 1,2$). The existence of solutions for the problem was shown by applying some well-known tools of fixed-point theory, such as Banach contraction principle, Krasnoselskii's fixed-point theorem, and Leray-Schauder nonlinear alternative.

In [30], Alsaedi *et al.* were concerned with the following nonlinear fractional q-difference equations with nonlocal integral boundary conditions:

$$\left({}^c D_q^\beta u\right)(t) + \lambda u(t) = f\left(t, u(t)\right), \quad t \in [0,1], \beta \in (1,2],$$

$$u(0) = 0, \qquad u(1) = (I_q u)(\eta) = \int_0^\eta u(s)\, d_q s, \quad 0 < \eta < 1.$$

The existence results were obtained by applying some well-known fixed-point theorems.

Motivated by the above works, in this paper, we consider the following system of nonlinear fractional q-difference equations with delays:

$$\left(D_q^{\alpha_i} u_i\right)(t) + f_i\left(t, u_1\left(\tau_{i1}(t)\right), \ldots, u_N\left(\tau_{iN}(t)\right)\right) = 0, \quad 0 < t < 1,$$

$$\left(D_q^j u_i\right)(0) = 0, \quad j = 0, 1, \ldots, n_i - 2, i = 1, 2, \ldots, N, \tag{1.1}$$

$$\left(D_q^{n_i - 1} u_i\right)(1) = \eta_i, \quad i = 1, 2, \ldots, N,$$

where $D_q^{\alpha_i}$ is the fractional q-derivative of the Riemann-Liouville type, $\alpha_i \in (n_i - 1, n_i]$ for some $n_i > 2$, $\eta_i \geq 0$ for $i = 1, 2, \ldots, N$, $0 \leq \tau_{ij}(t) \leq t$ for $i, j = 1, 2, \ldots, N$, and f_i is a nonlinear function from $[0,1] \times \mathbb{R}_+^N$ to $\mathbb{R}_+ = [0, \infty)$. The purpose of this paper is to establish sufficient conditions on the existence of positive solutions for fractional q-difference system (1.1) by using some properties of the Green function and some fixed-point theorems such as the Banach contraction principle, Krasnoselskii's fixed-point theorem, and the Leray-Schauder nonlinear alternative. By a positive solution for the fractional q-difference system (1.1) we mean a mapping with positive components on $[0,1]$ such that (1.1) is satisfied. Obviously, (1.1) includes the usual system of fractional q-difference equations when $\tau_{ij}(t) \equiv t$ for all i and j. Therefore, the obtained results generalize and include some existing ones.

2 Preliminaries

For convenience of the reader, we present some necessary definitions and lemmas of fractional q-calculus theory to facilitate analysis of problem (1.1). These details can be found in the recent literature; see [10] and references therein.

Let $q \in (0,1)$ and define

$$[a]_q = \frac{q^a - 1}{q - 1}, \quad a \in \mathbb{R}.$$

The q-analogue of the power $(a - b)^n$ with $n \in \mathbb{N}_0$ is

$$(a - b)^{(0)} = 1, \qquad (a - b)^{(n)} = \prod_{k=0}^{n-1}(a - bq^k), \quad n \in \mathbb{N}, a, b \in \mathbb{R}.$$

More generally, if $\alpha \in \mathbb{R}$, then

$$(a - b)^{(\alpha)} = a^\alpha \prod_{n=0}^{\infty} \frac{a - bq^n}{a - bq^{\alpha+n}}.$$

Note that, if $b = 0$, then $a^{(\alpha)} = a^\alpha$. The q-gamma function is defined by

$$\Gamma_q(x) = \frac{(1 - q)^{(x-1)}}{(1 - q)^{x-1}}, \quad x \in \mathbb{R} \setminus \{0, -1, -2, \ldots\},$$

and satisfies $\Gamma_q(x + 1) = [x]_q \Gamma_q(x)$.

The q-derivative of a function f is here defined by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1 - q)x}, \qquad (D_q f)(0) = \lim_{x \to 0}(D_q f)(x),$$

and q-derivatives of higher order by

$$(D_q^0 f)(x) = f(x) \quad \text{and} \quad (D_q^n f)(x) = D_q(D_q^{n-1}f)(x), \quad n \in \mathbb{N}.$$

The q-integral of a function f defined in the interval $[0, b]$ is given by

$$(I_q f)(x) = \int_0^x f(t)\, d_q t = x(1 - q) \sum_{n=0}^{\infty} f(xq^n)q^n, \quad x \in [0, b].$$

If $a \in [0, b]$ and f is defined in the interval $[0, b]$, its integral from a to b is defined by

$$\int_a^b f(t)\, d_q t = \int_0^b f(t)\, d_q t - \int_0^a f(t)\, d_q t.$$

Similarly to what is done for derivatives, an operator I_q^n can be defined, namely,

$$(I_q^0 f)(x) = f(x) \quad \text{and} \quad (I_q^n f)(x) = I_q(I_q^{n-1}f)(x), \quad n \in \mathbb{N}.$$

The fundamental theorem of calculus applies to these operators I_q and D_q, *i.e.*,

$$(D_q I_q f)(x) = f(x),$$

and if f is continuous at $x = 0$, then

$$(I_q D_q f)(x) = f(x) - f(0).$$

Basic properties of the two operators can be found in the book [10]. We now point out three formulas that will be used later ($_iD_q$ denotes the derivative with respect to variable i)

$$\left[a(t-s)\right]^{(\alpha)} = a^\alpha (t-s)^{(\alpha)}, \qquad {}_tD_q(t-s)^{(\alpha)} = [\alpha]_q(t-s)^{(\alpha-1)},$$

$$\left({}_xD_q \int_0^x f(x,t)\,d_q t\right)(x) = \int_0^x {}_xD_q f(x,t)\,d_q t + f(qx,x).$$

We note that if $\alpha > 0$ and $a \le b \le t$, then $(t-a)^{(\alpha)} \ge (t-b)^{(\alpha)}$ [26].

Definition 2.1 ([12]) Let $\alpha \ge 0$ and f be function defined on $[0,1]$. The fractional q-integral of the Riemann-Liouville type is $I_q^0 f(x) = f(x)$ and

$$\left(I_q^\alpha f\right)(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x-qt)^{(\alpha-1)} f(t)\,d_q t, \quad \alpha > 0, x \in [0,1].$$

Definition 2.2 ([14]) The fractional q-derivative of the Riemann-Liouville type of order $\alpha \ge 0$ is defined by $D_q^0 f(x) = f(x)$ and

$$\left(D_q^\alpha f\right)(x) = \left(D_q^m I_q^{m-\alpha} f\right)(x), \quad \alpha > 0,$$

where m is the smallest integer greater than or equal to α.

Definition 2.3 ([14]) The fractional q-derivative of the Caputo type of order $\alpha \ge 0$ is defined by

$$\left({}^c D_q^\alpha f\right)(x) = \left(I_q^{m-\alpha} D_q^m f\right)(x), \quad \alpha > 0,$$

where m is the smallest integer greater than or equal to α.

Lemma 2.4 ([12, 15]) *Let $\alpha, \beta \ge 0$ and f be a function defined on $[0,1]$. Then the following formulas hold*:
   (1) $(I_q^\beta I_q^\alpha f)(x) = I_q^{\alpha+\beta} f(x)$,
   (2) $(D_q^\alpha I_q^\alpha f)(x) = f(x)$.

Theorem 2.5 ([26]) *Let $\alpha > 0$ and p be a positive integer. Then the following equality holds*:

$$\left(I_q^\alpha D_q^\alpha f\right)(x) = \left(D_q^\alpha I_q^\alpha f\right)(x) - \sum_{k=0}^{p-1} \frac{x^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)} \left(D_q^k f\right)(0).$$

Theorem 2.6 (Banach contraction mapping theorem [31]) *Let M be a complete metric space and let $T : M \to M$ be a contraction mapping. Then T has a unique fixed point.*

Theorem 2.7 ([32, 33]) *Let C be a closed and convex subset of a Banach space X. Assume that U is a relatively open subset of C with $0 \in U$ and $T : \overline{U} \to C$ is completely continuous. Then at least one of the following two properties holds:*

(i) *T has a fixed point in $\overline{U}$,*

(ii) *there exist $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda Tu$.*

Theorem 2.8 (Krasnoselskii fixed-point theorem [31, 34]) *Let P be a cone in a Banach space X. Assume that Ω_1 and Ω_2 are open subsets of X with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subseteq \Omega_2$. Suppose that $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1)$ is a completely continuous operator such that either*

(i) *$\|Tu\| \leq \|u\|$ for $u \in P \cap \partial \Omega_1$ and $\|Tu\| \geq \|u\|$ for $u \in P \cap \partial \Omega_2$, or*

(ii) *$\|Tu\| \geq \|u\|$ for $u \in P \cap \partial \Omega_1$ and $\|Tu\| \leq \|u\|$ for $u \in P \cap \partial \Omega_2$.*

Then T has a fixed point in $\overline{\Omega}_2 \setminus \Omega_1$.

3 Existence of positive solutions

Throughout this paper, we let $E = C([0,1], \mathbb{R}^N)$. Then $(E, \| \cdot \|_E)$ is a Banach space, where

$$\|u\|_E = \max_{1 \leq i \leq N} \max_{0 \leq t \leq 1} |u_i(t)| \quad \text{for } u = (u_1, \dots, u_N)^T \in E.$$

In this section, we always assume that $f = (f_1, \dots, f_N)^T \in C([0,1] \times \mathbb{R}_+^N, \mathbb{R}_+^N)$.

Lemma 3.1 *Fractional q-difference systems* (1.1) *is equivalent to the following system of q-integral equations*:

$$u_i(t) = \int_0^1 G_i(t, qs) f_i\big(s, u_1(\tau_{i1}(s)), \dots, u_N(\tau_{iN}(s))\big) \, d_q s + \frac{\eta_i t^{\alpha_i - 1}}{[\alpha_i - 1]_q \cdots [\alpha_i - n_i + 1]_q} \quad (3.1)$$

for $i = 1, 2, \dots, N$, where

$$G_i(t, s) = \frac{1}{\Gamma_q(\alpha_i)} \begin{cases} t^{\alpha_i - 1}(1-s)^{(\alpha_i - n_i)} - (t-s)^{(\alpha_i - 1)}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha_i - 1}(1-s)^{(\alpha_i - n_i)}, & 0 \leq t \leq s \leq 1. \end{cases} \quad (3.2)$$

Proof It is easy to see that if $(u_1, u_2, \dots, u_N)^T$ satisfies (3.1), then it also satisfies (1.1). So, assume that $(u_1, u_2, \dots, u_N)^T$ is a solution to (1.1). In view of Lemma 2.4 and Theorem 2.5, integrating both sides of the first equation of (1.1) of order α_i with respect to t, we can see that

$$u_i(t) = -\frac{1}{\Gamma_q(\alpha_i)} \int_0^t (t - qs)^{(\alpha_i - 1)} f_i\big(s, u_1(\tau_{i1}(s)), \dots, u_N(\tau_{iN}(s))\big) \, d_q s$$

$$+ c_{1i} t^{\alpha_i - 1} + c_{2i} t^{\alpha_i - 2} + \cdots + c_{n_i i} t^{\alpha_i - n_i}$$

for $0 \leq t \leq 1$, $i = 1, 2, \dots, N$. It follows that

$$(D_q u_i)(t) = -\frac{[\alpha_i - 1]_q}{\Gamma_q(\alpha_i)} \int_0^t (t - qs)^{(\alpha_i - 2)} f_i\big(s, u_1(\tau_{i1}(s)), \dots, u_N(\tau_{iN}(s))\big) \, d_q s$$

$$+ [\alpha_i - 1]_q c_{1i} t^{\alpha_i - 2} + [\alpha_i - 2]_q c_{2i} t^{\alpha_i - 3} + \cdots + [\alpha_i - n_i + 1]_q c_{n_i - 1, i} t^{\alpha_i - n_i}$$

for $0 \le t \le 1$, $i = 1, 2, \ldots, N$. Combining with the boundary conditions in (1.1), this yields

$$c_{n_i-1,i} = 0, \quad i = 1, 2, \ldots, N.$$

Similarly, one can obtain $c_{n_i-2,i} = c_{n_i-3,i} = \cdots = c_{2,i} = 0$, $i = 1, 2, \ldots, N$. Also we have

$$\begin{aligned}
\left(D_q^{n_i-1} u_i\right)(t) = {}&-\frac{[\alpha_i - 1]_q \cdots [\alpha_i - n_i + 1]_q}{\Gamma_q(\alpha_i)} \\
&\times \int_0^t (t - qs)^{(\alpha_i - n_i)} f_i\big(s, u_1(\tau_{i1}(s)), \ldots, u_N(\tau_{iN}(s))\big)\, d_q s \\
&+ [\alpha_i - 1]_q \cdots [\alpha_i - n_i + 1]_q c_{1i} t^{\alpha_i - 2}, \quad i = 1, 2, \ldots, N.
\end{aligned} \tag{3.3}$$

Then it follows from (3.3) and the boundary condition $(D_q^{n_i-1} u)(1) = \eta_i$ that

$$\begin{aligned}
c_{1i} = {}&\frac{\eta_i}{[\alpha_i - 1]_q \cdots [\alpha_i - n_i + 1]_q} \\
&+ \frac{1}{\Gamma_q(\alpha_i)} \int_0^1 (1 - qs)^{(\alpha_i - n_i)} f_i\big(s, u_1(\tau_{i1}(s)), \ldots, u_N(\tau_{iN}(s))\big)\, d_q s.
\end{aligned}$$

Therefore, for $i = 1, 2, \ldots, N$,

$$\begin{aligned}
u_i(t) = {}&-\frac{1}{\Gamma_q(\alpha_i)} \int_0^t (t - qs)^{(\alpha_i - 1)} f_i\big(s, u_1(\tau_{i1}(s)), \ldots, u_N(\tau_{iN}(s))\big)\, d_q s \\
&+ \frac{\eta_i t^{\alpha_i - 1}}{[\alpha_i - 1]_q \cdots [\alpha_i - n_i + 1]_q} \\
&+ \frac{t^{\alpha_i - 1}}{\Gamma_q(\alpha_i)} \int_0^1 (1 - qs)^{(\alpha_i - n_i)} f_i\big(s, u_1(\tau_{i1}(s)), \ldots, u_N(\tau_{iN}(s))\big)\, d_q s \\
= {}&\frac{1}{\Gamma_q(\alpha_i)} \int_0^t \big(t^{\alpha_i - 1}(1 - qs)^{(\alpha_i - n_i)} - (t - qs)^{(\alpha_i - 1)}\big) f_i\big(s, u_1(\tau_{i1}(s)), \ldots, u_N(\tau_{iN}(s))\big)\, d_q s \\
&+ \frac{1}{\Gamma_q(\alpha_i)} \int_0^1 t^{\alpha_i - 1}(1 - qs)^{(\alpha_i - n_i)} f_i\big(s, u_1(\tau_{i1}(s)), \ldots, u_N(\tau_{iN}(s))\big)\, d_q s \\
&+ \frac{\eta_i t^{\alpha_i - 1}}{[\alpha_i - 1]_q \cdots [\alpha_i - n_i + 1]_q} \\
= {}&\int_0^1 G_i(t, qs) f_i\big(s, u_1(\tau_{i1}(s)), \ldots, u_N(\tau_{iN}(s))\big)\, d_q s + \frac{\eta_i t^{\alpha_i - 1}}{[\alpha_i - 1]_q \cdots [\alpha_i - n_i + 1]_q},
\end{aligned}$$

where G_i is defined as in (3.2). The proof is completed. □

Some properties of the Green functions $G_i(t, s)$ needed in the sequel are now stated and proved.

Lemma 3.2 *Function $G_i(t, s)$ defined above satisfies the following conditions*:
(a) $G_i(t, qs) \ge 0$ *and* $G_i(t, qs) \le G_i(1, qs)$ *for all* $0 \le t, s \le 1$;
(b) $G_i(t, qs) \ge g_i(t) G_i(1, qs)$ *for all* $0 \le t, s \le 1$ *with* $g(t) = t^{\alpha_i - 1}$.

Proof We start by defining the two functions

$$\varphi_i(t,s) = t^{\alpha_i-1}(1-s)^{(\alpha_i-n_i)} - (t-s)^{(\alpha_i-1)}, \quad 0 \le s \le t \le 1,$$

and

$$\psi_i(t,s) = t^{\alpha_i-1}(1-s)^{(\alpha_i-n_i)}, \quad 0 \le t \le s \le 1.$$

It is clear that $\psi_i(t,qs) \ge 0$ and $\psi_i(0,qs) = 0$. On the other hand, for $t \ne 0$

$$\begin{aligned}
\varphi_i(t,qs) &= t^{\alpha_i-1}(1-qs)^{(\alpha_i-n_i)} - t^{\alpha_i-1}\left(1 - q\frac{s}{t}\right)^{(\alpha_i-1)} \\
&\ge t^{\alpha_i-1}\left((1-qs)^{(\alpha_i-n_i)} - (1-qs)^{(\alpha_i-1)}\right) \ge 0.
\end{aligned}$$

Therefore, $G_i(t,qs) \ge 0$. Moreover, for fixed $s \in [0,1]$,

$$\begin{aligned}
tD_q\varphi_i(t,qs) &= [\alpha_i-1]_q t^{\alpha_i-2}(1-qs)^{(\alpha_i-n_i)} - [\alpha_i-1]_q(t-qs)^{(\alpha_i-2)} \\
&\ge [\alpha_i-1]_q t^{\alpha_i-2}(1-qs)^{(\alpha_i-n_i)} - [\alpha_i-1]_q t^{(\alpha_i-2)}\left(1 - q\frac{s}{t}\right)^{(\alpha_i-2)} \\
&\ge [\alpha_i-1]_q t^{\alpha_i-2}\left((1-qs)^{(\alpha_i-n_i)} - (1-qs)^{(\alpha_i-2)}\right) \ge 0,
\end{aligned}$$

i.e., $\varphi_i(t,qs)$ is an increasing function of t. Obviously, $\psi_i(t,qs)$ is increasing in t, therefore $G_i(t,qs)$ is an increasing function of t for fixed $s \in [0,1]$. This concludes the proof of (a).

Suppose now that $t \ge qs$. Then we have

$$\begin{aligned}
\frac{G_i(t,qs)}{G_i(1,qs)} &= \frac{t^{\alpha_i-1}(1-qs)^{(\alpha_i-n_i)} - (t-qs)^{(\alpha_i-1)}}{(1-qs)^{(\alpha_i-n_i)} - (1-qs)^{(\alpha_i-1)}} \\
&= \frac{t^{\alpha_i-1}\left((1-qs)^{(\alpha_i-n_i)} - (1-q\frac{s}{t})^{(\alpha_i-1)}\right)}{(1-qs)^{(\alpha_i-n_i)} - (1-qs)^{(\alpha_i-1)}} \\
&\ge \frac{t^{\alpha_i-1}\left((1-qs)^{(\alpha_i-n_i)} - (1-qs)^{(\alpha_i-1)}\right)}{(1-qs)^{(\alpha_i-n_i)} - (1-qs)^{(\alpha_i-1)}} = t^{\alpha_i-1}.
\end{aligned}$$

On the other hand, if $t \le qs$, then we have

$$\frac{G_i(t,qs)}{G_i(1,qs)} = t^{\alpha_i-1},$$

and this finishes the proof of (b). □

Now, we are ready to present the main results.

Theorem 3.3 *Suppose that there exist functions $\lambda_{ij}(t) \in C([0,1], \mathbb{R}_+)$, $i,j = 1,2,\dots,N$, such that*

$$\left|f_i(t,u_1,\dots,u_N) - f_i(t,v_1,\dots,v_N)\right| \le \sum_{j=1}^{N} \lambda_{ij}(t)|u_j - v_j| \tag{3.4}$$

for $t \in [0,1]$, $(u_1, u_2, \ldots, u_N)^T, (v_1, v_2, \ldots, v_N)^T \in \mathbb{R}_+^N$. *If*

$$\max_{1 \le i \le N} \int_0^1 G_i(1, qs) \left(\sum_{j=1}^N \lambda_{ij}(s) \right) d_q s < 1, \tag{3.5}$$

then (1.1) *has a unique positive solution.*

Proof Let

$$\Omega = \left\{ u \in E \,|\, u_i(t) \ge 0 \text{ for } t \in [0,1], i = 1, 2, \ldots, N \right\}. \tag{3.6}$$

It is easy to see that Ω is a complete metric space. Define an operator T on Ω by

$$Tu(t) = \int_0^1 G(t, qs) f(s) \, d_q s$$
$$+ \text{diag} \left(\frac{\eta_1 t^{\alpha_1 - 1}}{[\alpha_1 - 1]_q \cdots [\alpha_1 - n_1 + 1]_q}, \ldots, \frac{\eta_N t^{\alpha_N - 1}}{[\alpha_N - 1]_q \cdots [\alpha_N - n_N + 1]_q} \right), \tag{3.7}$$

where $G(t, s) = \text{diag}(G_1(t, s), G_2(t, s), \ldots, G_N(t, s))$ and

$$f(t) = \begin{pmatrix} f_1(t, u_1(\tau_{11}(t)), \ldots, u_N(\tau_{1N}(t))) \\ f_2(t, u_1(\tau_{21}(t)), \ldots, u_N(\tau_{2N}(t))) \\ \vdots \\ f_N(t, u_1(\tau_{N1}(t)), \ldots, u_N(\tau_{NN}(t))) \end{pmatrix}.$$

Because of the continuity of G and f, it follows easily from Lemma 3.2 that T maps Ω into itself. To finish the proof, we only need to show that T is a contraction. Indeed, for $u, v \in \Omega$, by (3.4), we have

$$\left| \left(Tu(t) \right)_i - \left(Tv(t) \right)_i \right|$$
$$= \left| \int_0^1 G_i(t, qs) \left(f_i \left(s, u_i(\tau_{i1}(s)), \ldots, u_N(\tau_{iN}(s)) \right) \, d_q s \right. \right.$$
$$\left. \left. - f_i \left(s, v_1(\tau_{i1}(s)), \ldots, v_N(\tau_{iN}(s)) \right) \right) d_q s \right|$$
$$\le \int_0^1 G_i(t, qs) \left| f_i \left(s, u_i(\tau_{i1}(s)), \ldots, u_N(\tau_{iN}(s)) \right) \right.$$
$$\left. - f_i \left(t, v_1(\tau_{i1}(t)), \ldots, v_N(\tau_{iN}(t)) \right) \right| d_q s$$
$$\le \int_0^1 G_i(t, qs) \left(\sum_{j=1}^N \lambda_{ij}(t) \left| u_j(\tau_{ij}(s)) - v_j(\tau_{ij}(s)) \right| \right) d_q s.$$

This, combined with Theorem 2.6 and (3.5), immediately implies that $T : \Omega \to \Omega$ is a contraction. Therefore, the proof is complete with the help of Lemma 3.1 and Theorem 2.6.

$\square$

The following result can be proved in the same spirit as that for Theorem 3.3.

Theorem 3.4 *Suppose that there exist functions* $\lambda_i(t) \in C([0,1], \mathbb{R}_+)$, $i, j = 1, 2, \ldots, N$, *and nonnegative constants* $p_{i1}, p_{i2}, \ldots, p_{iN}$ *such that* $\sum_{j=1}^N p_{ij} = 1$ *and*

$$\left| f_i(t, u_1, \ldots, u_N) - f_i(t, v_1, \ldots, v_N) \right| \le \lambda_i(t) \prod_{j=1}^N |u_j - v_j|^{p_{ij}}$$

for $t \in [0,1]$, $(u_1, u_2, \ldots, u_N)^T, (v_1, v_2, \ldots, v_N)^T \in \mathbb{R}_+^N$. *If*

$$\max_{1 \le i \le N} \int_0^1 G_i(1, qs) \lambda_i(s) \, d_q s < 1,$$

then (1.1) *has a unique positive solution.*

Theorem 3.5 *Suppose that there exist nonnegative real-valued functions* $m_i, n_{i1}, \ldots, n_{iN} \in L[0,1]$, $i, j = 1, 2, \ldots, N$, *such that*

$$f_i(t, u_1, \ldots, u_N) \le m_i(t) + \sum_{j=1}^N n_{ij}(t) u_j \tag{3.8}$$

for almost every $t \in [0,1]$ *and all* $(u_1, u_2, \ldots, u_N)^T \in \mathbb{R}_+^N$. *If*

$$\max_{1 \le i \le N} \int_0^1 G_i(1, qs) \left(\sum_{j=1}^N n_{ij}(s) \right) d_q s < 1,$$

then (1.1) *has at least one positive solution.*

Proof Let Ω and $T : \Omega \to \Omega$ be defined by (3.6) and (3.7), respectively. We first show that T is completely continuous through the following three steps.

Step 1. Show that $T : \Omega \to \Omega$ is continuous. Let $\{u^k(t)\}$ be a sequence in Ω such that $u^k(t) \to u(t) \in \Omega$. Then $\Omega_0 = [0,1] \times \{u(t) | u^k(t) \in \Omega, t \in [0,1], k \ge 1\}$ is bounded in $[0,1] \times \mathbb{R}_+^N$. Since f is continuous, it is uniformly continuous on any compact set. In particular, for any $\varepsilon > 0$, there exists a positive integer K_0 such that

$$\left| f_i\bigl(t, u_1^k(\tau_{i1}(t)), \ldots, u_N^k(\tau_{iN}(t))\bigr) - f_i\bigl(t, u_1(\tau_{i1}(t)), \ldots, u_N(\tau_{iN}(t))\bigr) \right|$$
$$< \frac{\varepsilon}{\max_{1 \le i \le N} \max_{t \in [0,1]} \int_0^1 G_i(t, qs) \, d_q s},$$

for $t \in [0,1]$ and $k \ge K_0$, $i = 1, 2, \ldots, N$. Then, for $t \in [0,1]$ and $k \ge K_0$, $i = 1, 2, \ldots, N$, we have

$$\left| \bigl(Tu^k(t)\bigr)_i - \bigl(Tu(t)\bigr)_i \right|$$
$$= \left| \int_0^1 G_i(t, qs) \bigl(f_i\bigl(s, u_1^k(\tau_{i1}(s)), \ldots, u_N^k(\tau_{iN}(s))\bigr) \right.$$
$$\left. - f_i\bigl(s, u_1(\tau_{i1}(s)), \ldots, u_N(\tau_{iN}(s))\bigr) \bigr) \, d_q s \right|$$
$$\le \int_0^1 G_i(t, qs) \left| f_i\bigl(s, u_1^k(\tau_{i1}(s)), \ldots, u_N^k(\tau_{iN}(s))\bigr) \right.$$

$$-f_i\big(s, u_1\big(\tau_{i1}(s)\big), \ldots, u_N\big(\tau_{iN}(s)\big)\big)\big|\, d_q s$$

$$\leq \frac{\varepsilon}{\max_{1 \leq i \leq N} \max_{t \in [0,1]} \int_0^1 G_i(t, qs)\, d_q s} \int_0^1 G_i(t, qs)\, d_q s \leq \varepsilon.$$

Therefore, $\|Tu^k(t) - Tu(t)\| \leq \varepsilon$ for $k \geq K_0$, which implies that T is continuous.

Step 2. Show that T maps bounded sets of Ω into bounded sets. Let A be a bounded subset of Ω. Then $[0,1] \times \{u(t) | t \in [0,1], u \in A\} \subseteq [0,1] \times \mathbb{R}_+^N$ is bounded. Since f is continuous, there exists an $M > 0$ such that

$$f_i\big(t, u_1\big(\tau_{i1}(t)\big), \ldots, u_N\big(\tau_{iN}(t)\big)\big) \leq M, \quad \text{for } u \in A, t \in [0,1], 1 \leq i \leq N.$$

It follows that, for $u \in A$, $t \in [0,1]$ and $1 \leq i \leq N$,

$$\big(Tu(t)\big)_i = \int_0^1 G_i(t, qs) f_i\big(s, u_1\big(\tau_{i1}(s)\big), \ldots, u_N\big(\tau_{iN}(s)\big)\big)\, d_q s + \frac{\eta_i t^{\alpha_i - 1}}{[\alpha_i - 1]_q \cdots [\alpha_i - n_i + 1]_q}$$

$$\leq M \int_0^1 G_i(t, qs)\, d_q s + \frac{\eta_i}{[\alpha_i - 1]_q \cdots [\alpha_i - n_i + 1]_q}$$

$$\leq \max_{1 \leq i \leq M} \left(M \max_{t \in [0,1]} \int_0^1 G_i(t, qs)\, d_q s + \frac{\eta_i}{[\alpha_i - 1]_q \cdots [\alpha_i - n_i + 1]_q} \right).$$

Immediately, we can easily see that TA is a bounded subset of Ω.

Step 3. Show that T maps bounded sets of Ω into equicontinuous sets. Let B be a bounded subset of Ω. Similarly as in Step 2, there exists $L > 0$ such that

$$f_i\big(t, u_1\big(\tau_{i1}(t)\big), \ldots, u_N\big(\tau_{iN}(t)\big)\big) \leq L, \quad \text{for } u \in B, t \in [0,1], 1 \leq i \leq N.$$

Then, for any $u \in B$ and $t_1, t_2 \in [0,1]$ and $1 \leq i \leq N$,

$$\big|\big(Tu(t_2)\big)_i - \big(Tu(t_1)\big)_i\big|$$

$$= \left| \int_0^1 \big(G_i(t_2, qs) - G_i(t_1, qs)\big) f_i\big(s, u_1\big(\tau_{i1}(s)\big), \ldots, u_N\big(\tau_{iN}(s)\big)\big)\, d_q s \right.$$

$$\left. + \frac{\eta_i(t_2^{\alpha_i - 1} - t_1^{\alpha_i - 1})}{[\alpha_i - 1]_q \cdots [\alpha_i - n_i + 1]_q} \right|$$

$$\leq \int_0^1 \big|G_i(t_2, qs) - G_i(t_1, qs)\big| L\, d_q s + \frac{\eta_i |t_2^{\alpha_i - 1} - t_1^{\alpha_i - 1}|}{[\alpha_i - 1]_q \cdots [\alpha_i - n_i + 1]_q}$$

$$\leq \max_{s \in [0,1]} \big|G_i(t_2, qs) - G_i(t_1, qs)\big| L + \frac{\eta_i |t_2^{\alpha_i - 1} - t_1^{\alpha_i - 1}|}{[\alpha_i - 1]_q \cdots [\alpha_i - n_i + 1]_q}.$$

Now the equicontinuity of T on B follows easily from the fact that G_i is continuous and hence uniformly continuous on $[0,1] \times [0,1]$.

Now we have shown that T is completely continuous. To apply Theorem 2.7, let

$$\mu = \frac{\max_{1 \leq i \leq N} \{ \int_0^1 G_i(1, qs) m_i(s)\, d_q s + \eta_i / [\alpha_i - 1]_q \cdots [\alpha_i - n_i + 1]_q \}}{1 - \max_{1 \leq i \leq N} \{ \int_0^1 G_i(1, qs)(\sum_{j=1}^N n_{ij}(s))\, d_q s \}}.$$

Fix $r > \mu$ and define $U = \{u \in \Omega : \|u\| < r\}$. We claim that there is no $u \in U$ such that $u = \lambda Tu$ for some $\lambda \in (0,1)$. Otherwise, assume that there exist $\lambda \in (0,1)$ and $u \in \partial U$ such that $u = \lambda Tu$. Then

$$
\begin{aligned}
|u_i(t)| &= |\lambda(Tu(t))_i| \le |(Tu(t))_i| \\
&= \int_0^1 G_i(t,qs) f_i\big(s, u_1(\tau_{i1}(s)), \ldots, u_N(\tau_{iN}(s))\big) d_q s + \frac{\eta_i t^{\alpha_i - 1}}{[\alpha_i - 1]_q \cdots [\alpha_i - n_i + 1]_q} \\
&\le \int_0^1 G_i(t,qs)\left(m_i(s) + \sum_{j=1}^N n_{ij}(s) u_j(\tau_{ij}(s))\right) d_q s + \frac{\eta_i}{[\alpha_i - 1]_q \cdots [\alpha_i - n_i + 1]_q} \\
&\le \int_0^1 G_i(t,qs) m_i(s)\, d_q s + r\int_0^1 G_i(t,qs)\left(\sum_{j=1}^N n_{ij}(s)\right) d_q s \\
&\quad + \frac{\eta_i}{[\alpha_i - 1]_q \cdots [\alpha_i - n_i + 1]_q} \\
&< \int_0^1 G_i(1,qs) m_i(s)\, d_q s + \frac{\eta_i}{[\alpha_i - 1]_q \cdots [\alpha_i - n_i + 1]_q} \\
&\quad + r\int_0^1 G_i(1,qs)\left(\sum_{j=1}^N n_{ij}(s)\right) d_q s \\
&\le \mu\left(1 - \max_{1 \le i \le N}\left\{\int_0^1 G_i(1,qs)\left(\sum_{j=1}^N n_{ij}(s)\right) d_q s\right\}\right) + r\int_0^1 G_i(1,qs)\left(\sum_{j=1}^N n_{ij}(s)\right) d_q s \\
&< r\left(1 - \max_{1 \le i \le N}\left\{\int_0^1 G_i(1,qs)\left(\sum_{j=1}^N n_{ij}(s)\right) d_q s\right\}\right) \\
&\quad + r\int_0^1 G_i(1,qs)\left(\sum_{j=1}^N n_{ij}(s)\right) d_q s \le r.
\end{aligned}
$$

Therefore, $\|u\| < r$, a contradiction to $u \in \partial U$. This proves the claim. Applying Theorem 2.7, we know that T has a fixed point in $\overline{U}$, which is a positive solution to (1.1) by Lemma 3.1. Therefore, the proof is complete. $\qquad\square$

Corollary 3.6 *If all f_i, $i,j = 1, 2, \ldots, N$, are bounded, then* (1.1) *has at least one positive solution.*

To state the last result of this section, we introduce

$$
M_1 = \left(\max_{1 \le i \le N} \int_0^1 G_i(1,qs)\, d_q s\right)^{-1}.
$$

Theorem 3.7 *Suppose that there exist $M_2 \in (0, M_1)$ and positive constants $0 < r_1 < r_2$ with $r_2 \ge \max_{1 \le i \le N}\{\eta_i/[\alpha_i - 1]_q \cdots [\alpha_i - n_i + 1]_q\}/(1 - M_2/M_1)$ such that*
  (a) *$f_i(t, u_1, \ldots, u_N) \le M_2 r_2$ for $(t, u_1, \ldots, u_N) \in [0,1] \times B_{r_2}$, $i = 1, 2, \ldots, N$, and*
  (b) *$f_i(t, u_1, \ldots, u_N) \ge M_1 r_1$ for $(t, u_1, \ldots, u_N) \in [0,1] \times B_{r_1}$, $i = 1, 2, \ldots, N$,*
where $B_{r_i} = \{u = (u_1, \ldots, u_N)^T \in \mathbb{R}_+^N \mid \max_{1 \le i \le N} u_i \le r_i\}$, $i = 1, 2$. Then (1.1) *has at least a positive solution.*

Proof Let Ω be defined by (3.5) and $\Omega_i = \{u \in E | \|u\| < r_i\}$, $i = 1, 2$. Obviously, Ω is a cone in E. From the proof of Theorem 3.5, we know that the operator T defined by (3.6) is completely continuous on Ω. For any $u \in \Omega \cap \partial\Omega_1$, it follows from Theorem 2.8 and condition (b) that

$$
\begin{aligned}
\|Tu\|_E &= \max_{1 \le i \le N} \max_{0 \le t \le 1} \big(Tu(t)\big)_i \ge \max_{1 \le i \le N} \big(Tu(1)\big)_i \\
&= \max_{1 \le i \le N} \left\{ \int_0^1 G_i(1, qs) f_i\big(s, u_1(\tau_{i1}(s)), \dots, u_N(\tau_{iN}(s))\big) \, d_q s \right. \\
&\quad \left. + \frac{\eta_i}{[\alpha_i - 1]_q \cdots [\alpha_i - n_i + 1]_q} \right\} \\
&\ge \max_{1 \le i \le N} \left\{ \int_0^1 G_i(1, qs) \, d_q s M_1 r_1 + \frac{\eta_i}{[\alpha_i - 1]_q \cdots [\alpha_i - n_i + 1]_q} \right\} \ge r_1 = \|u\|_E,
\end{aligned}
$$

that is, $\|Tu\|_E \ge \|u\|_E$ for $u \in \Omega \cap \partial\Omega_1$.

On the other hand, for any $u \in \Omega \cap \partial\Omega_2$, it follows from Lemma 3.2 and condition (a) that, for $t \in [0, 1]$,

$$
\begin{aligned}
\big(Tu(t)\big)_i &\le \max_{1 \le i \le N} \int_0^1 G_i(1, qs) \, d_q s M_2 r_2 + \max_{1 \le i \le N} \frac{\eta_i}{[\alpha_i - 1]_q \cdots [\alpha_i - n_i + 1]_q} \\
&\le \max_{1 \le i \le N} \int_0^1 G_i(1, qs) \, d_q s M_2 r_2 + (1 - M_2/M_1) r_2 \le r_2 = \|u\|_E,
\end{aligned}
$$

that is, $\|Tu\|_E \le \|u\|_E$ for $u \in \Omega \cap \partial\Omega_2$. Therefore, we have verified condition (b) of Theorem 2.8. It follows that T has a fixed point in $\Omega \cap (\overline{\Omega}_2 \setminus \Omega_1)$, which is a positive solution to (1.1). This completes the proof. □

4 Some examples

In this section, we demonstrate the feasibility of some of the results obtained in Section 3.

Example 4.1 Consider the following fractional q-difference system:

$$
\begin{aligned}
&\big(D_{0.5}^{2.5} x_1\big)(t) + \frac{e^{-t}(x_1(t/2) + x_2(\sin t))}{(1 + e^t)(1 + x_1(t/2) + x_2(\sin t))} = 0, \quad t \in (0, 1), \\
&\big(D_{0.5}^{2.5} x_2\big)(t) + \frac{e^t(x_1(t^2) + x_2(\sin t))}{2(1 + x_1(t^2) + x_2(\sin t))} = 0, \quad t \in (0, 1), \\
&x_1(0) = x_2(0) = (D_{0.5} x_1)(0) = (D_{0.5} x_2)(0) = 0, \qquad \big(D_{0.5}^2 x_1\big)(1) = \big(D_{0.5}^2 x_2\big)(1) = \frac{1}{2}.
\end{aligned}
$$
(4.1)

Here $n_1 = n_2 = 3$, $\alpha_1 = \alpha_2 = 2.5$, $q = \eta_1 = \eta_2 = 0.5$, $\tau_{11}(t) = t/2$, $\tau_{12}(t) = \tau_{22}(t) = \sin t$, $\tau_{21}(t) = t^2$,

$$
f_1(t, x_1, x_2) = \frac{e^{-t}(x_1 + x_2)}{(1 + e^t)(1 + x_1 + x_2)}, \qquad f_1(t, x_1, x_2) = \frac{e^t(x_1 + x_2)}{2(1 + x_1 + x_2)}.
$$

One can easily see that (3.4) is satisfied with

$$
\lambda_{11}(t) = \lambda_{12}(t) = \frac{e^{-t}}{1 + e^t}, \qquad \lambda_{21}(t) = \lambda_{22}(t) = \frac{t^2}{2}.
$$

Moreover,

$$G_1(1,s) = G_2(1,s) = \frac{(1-s)^{(-0.5)} - (1-s)^{(1.5)}}{\Gamma_{0.5}(2.5)}, \quad 0 \le s \le 1$$

and hence [27]

$$\max_{1 \le i \le 2} \int_0^1 G_i(1, qs) \left(\sum_{j=1}^2 \lambda_{ij}(s) \right) d_q s \le \int_0^1 G_i(1, qs) \max_{0 \le s \le 1} \left\{ s^2, \frac{2}{e^s(1 + e^s)} \right\} d_q s$$

$$\le \int_0^1 G_i(1, qs) \, d_q s \le 0.48636.$$

It follows from Theorem 3.3 that (4.1) has a unique positive solution on $[0, 1]$.

Example 4.2 Consider the following fractional q-difference system:

$$\left(D_{0.5}^{2.5} x_1 \right)(t) + \frac{tx_1(t)}{2} + \frac{x_2(t)}{2} + \frac{t}{3} + \frac{1}{3} = 0, \quad t \in (0, 1),$$

$$\left(D_{0.5}^{2.5} x_2 \right)(t) + \frac{x_1(t)}{2} + \frac{tx_2(t)}{2} + \frac{t^2}{3} + \frac{1}{3} = 0, \quad t \in (0, 1), \tag{4.2}$$

$$x_1(0) = x_2(0) = (D_{0.5} x_1)(0) = (D_{0.5} x_2)(0) = 0, \qquad \left(D_{0.5}^2 x_1 \right)(1) = \left(D_{0.5}^2 x_2 \right)(1) = \frac{1}{2}.$$

Here $n_1 = n_2 = 3$, $\alpha_1 = \alpha_2 = 2.5$, $q = \eta_1 = \eta_2 = 0.5$, $\eta_1 = \eta_2 = 1/2$,

$$f_i(t, x_1, x_2) = m_i(t) + \sum_{j=1}^2 n_{ij} x_j, \quad i = 1, 2,$$

where

$$m_1(t) = \frac{t}{3} + \frac{1}{3}, \qquad m_2(t) = \frac{t^2}{3} + \frac{1}{3}, \qquad n_{11}(t) = n_{22}(t) = \frac{t}{2}, \qquad n_{12}(t) = n_{21}(t) = \frac{1}{2}.$$

One can easily see that (3.8) is satisfied. Moreover,

$$G_1(1,s) = G_2(1,s) = \frac{(1-s)^{(-0.5)} - (1-s)^{(1.5)}}{\Gamma_{0.5}(2.5)}, \quad 0 \le s \le 1$$

and hence [27]

$$\max_{1 \le i \le 2} \int_0^1 G_i(1, qs) \left(\sum_{j=1}^2 n_{ij}(s) \right) d_q s \le \int_0^1 G_i(1, qs) \max_{0 \le s \le 1} \left\{ \frac{s+1}{2} \right\} d_q s$$

$$\le \int_0^1 G_i(1, qs) \, d_q s \le 0.48636.$$

It follows from Theorem 3.5 that (4.2) has at least one positive solution on $[0, 1]$.

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