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Asymptotic behavior of solutions of mixed type impulsive neutral differential equations

Jessada Tariboon[1*], Sotiris K Ntouyas[2,3] and Chatthai Thaiprayoon[1]

*Correspondence: jessadat@kmutnb.ac.th
[1]Nonlinear Dynamic Analysis Research Center, Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok, 10800, Thailand
Full list of author information is available at the end of the article

Abstract

This paper investigates the asymptotic behavior of solutions of the mixed type neutral differential equation with impulsive perturbations $[x(t)+C(t)x(t-\tau)-D(t)x(\alpha t)]'+P(t)f(x(t-\delta))+\frac{Q(t)}{t}x(\beta t)=0$, $0<t_0\le t$, $t\ne t_k$, $x(t_k)=b_kx(t_k^-)+(1-b_k)(\int_{t_k-\delta}^{t_k}P(s+\delta)f(x(s))\,ds+\int_{\beta t_k}^{t_k}\frac{Q(s/\beta)}{s}x(s)\,ds)$, $k=1,2,3,\ldots$. Sufficient conditions are obtained to guarantee that every solution tends to a constant as $t\to\infty$. Examples illustrating the abstract results are also presented.

MSC: 34K25; 34K45

Keywords: asymptotic behavior; nonlinear neutral delay differential equation; impulse; Lyapunov functional

1 Introduction

The main purpose of this paper is to investigate the asymptotic behavior of solutions of the following mixed type neutral differential equation with impulsive perturbations:

$$\begin{cases} [x(t)+C(t)x(t-\tau)-D(t)x(\alpha t)]'+P(t)f(x(t-\delta))+\frac{Q(t)}{t}x(\beta t)=0, \\ \quad 0<t_0\le t, t\ne t_k, \\ x(t_k)=b_kx(t_k^-)+(1-b_k)(\int_{t_k-\delta}^{t_k}P(s+\delta)f(x(s))\,ds+\int_{\beta t_k}^{t_k}\frac{Q(s/\beta)}{s}x(s)\,ds), \\ \quad k=1,2,3,\ldots, \end{cases} \tag{1.1}$$

where $\tau,\delta>0$, $0<\alpha,\beta<1$, $C(t),D(t)\in PC([t_0,\infty),\mathbb{R})$, $P(t),Q(t)\in PC([t_0,\infty),\mathbb{R}_0^+)$, $f\in C(\mathbb{R},\mathbb{R})$, $0<t_k<t_{k+1}$ with $\lim_{k\to\infty}t_k=\infty$ and b_k, $k=1,2,3,\ldots$, are given constants. For $J\subset\mathbb{R}$, $PC(J,\mathbb{R})$ denotes the set of all functions $h:J\to\mathbb{R}$ such that h is continuous for $t_k\le t<t_{k+1}$ and $\lim_{t\to t_k^-}h(t)=h(t_k^-)$ exists for all $k=1,2,\ldots$.

The theory of impulsive differential equations appears as a natural description of several real processes subject to certain perturbations whose duration is negligible in comparison with the duration of the process. Differential equations involving impulse effects occurs in many applications: physics, population dynamics, ecology, biological systems, biotechnology, industrial robotic, pharmacokinetics, optimal control, *etc.* The reader may refer, for instance, to the monographs by Bainov and Simeonov [1], Lakshmikantham *et al.* [2], Samoilenko and Perestyuk [3], and Benchohra *et al.* [4]. In recent years, there has been increasing interest in the oscillation, asymptotic behavior, and stability theory of impulsive delay differential equations and many results have been obtained (see [5–20] and the references cited therein).

Let us mention some papers from which are motivation for our work. By the construction of Lyapunov functionals, the authors in [8] studied the asymptotic behavior of solutions of the nonlinear neutral delay differential equation under impulsive perturbations,

$$\begin{cases} [x(t)+C(t)x(t-\tau)]' + P(t)f(x(t-\delta)) = 0, & 0 < t_0 \le t, t \ne t_k, \\ x(t_k) = b_k x(t_k^-) + (1-b_k)\int_{t_k-\delta}^{t_k} P(s+\delta)f(x(s))\,ds, & k = 1,2,3,\ldots. \end{cases} \tag{1.2}$$

A similar method was used in [21] by considering an impulsive Euler type neutral delay differential equation with similar impulsive perturbations

$$\begin{cases} [x(t)-D(t)x(\alpha t)]' + \frac{Q(t)}{t}x(\beta t) = 0, & 0 < t_0 \le t, t \ne t_k, \\ x(t_k) = b_k x(t_k^-) + (1-b_k)\int_{\beta t_k}^{t_k} \frac{Q(s/\beta)}{s}x(s)\,ds, & k = 1,2,3,\ldots. \end{cases} \tag{1.3}$$

In this paper we combine the two papers [8, 21] and we study the mixed type impulsive neutral differential equation problem (1.1). By using a similar method with the help of four Lyapunov functionals, sufficient conditions are obtained to guarantee that every solution of (1.1) tends to a constant as $t \to \infty$. We note that problems (1.2) and (1.3) can be derived from the problem (1.1) as special cases, *i.e.*, if $D(t) \equiv 0$ and $Q(t) \equiv 0$, then (1.1) reduces to (1.2) while if $C(t) \equiv 0$ and $P(t) \equiv 0$, then (1.1) reduces to (1.3). Therefore, the mixed type of nonlinear delay with an Euler form of impulsive neutral differential equations gives more general results than the previous one.

Setting $\eta_1 = \max\{\tau, \delta\}$, $\eta_2 = \min\{\alpha, \beta\}$, and $\eta = \min\{t_0 - \eta_1, \eta_2 t_0\}$, we define an initial function as

$$x(t) = \varphi(t), \quad t \in [\eta, t_0], \tag{1.4}$$

where $\varphi \in PC([\eta, t_0], \mathbb{R}) = \{\varphi : [\eta, t_0] \to \mathbb{R} | \varphi$ is continuous everywhere except at a finite number of point s, and $\varphi(s^-)$ and $\varphi(s^+) = \lim_{s\to s^+} \varphi(s)$ exist with $\varphi(s^+) = \varphi(s)\}$.

A function $x(t)$ is said to be a solution of (1.1) satisfying the initial condition (1.4) if

(i) $x(t) = \varphi(t)$ for $\eta \le t \le t_0$, $x(t)$ is continuous for $t \ge t_0$ and $t \ne t_k$, $k = 1,2,3,\ldots$;

(ii) $x(t) + C(t)x(t-\tau) - D(t)x(\alpha t)$ is continuously differentiable for $t > t_0$, $t \ne t_k$, $k = 1,2,3,\ldots$, and satisfies the first equation of system (1.1);

(iii) $x(t_k^+)$ and $x(t_k^-)$ exist with $x(t_k^+) = x(t_k)$ and satisfy the second equation of system (1.1).

A solution of (1.1) is said to be nonoscillatory if it is eventually positive or eventually negative. Otherwise, it is said to be oscillatory.

2 Main results

We are now in a position to establish our main results.

Theorem 2.1 *Assume that*:

(H_1) *There exists a constant* $M > 0$ *such that*

$$|x| \le |f(x)| \le M|x|, \quad x \in \mathbb{R}, xf(x) > 0, \textit{ for } x \ne 0. \tag{2.1}$$

(H_2) *The functions* C, D *satisfy*

$$\lim_{t\to\infty}|C(t)| = \mu < 1, \qquad \lim_{t\to\infty}|D(t)| = \gamma < 1 \quad \text{with } \mu + \gamma < 1, \tag{2.2}$$

and

$$C(t_k) = b_k C(t_k^-), \qquad D(t_k) = b_k D(t_k^-). \tag{2.3}$$

(H_3) $t_k - \tau$ *and* αt_k *are not impulsive points*, $0 < b_k \le 1$, $k = 1, 2, \ldots$, *and* $\sum_{k=1}^{\infty}(1 - b_k) < \infty$.

(H_4) *The functions* P, Q *satisfy*

$$\begin{cases} \limsup_{t\to\infty}[\int_{t-\delta}^{t+\delta} P(s+\delta)\,ds + \int_{\beta t}^{t+\delta} \frac{Q(s/\beta)}{s}\,ds \\ \quad + \mu(1 + \frac{P(t+\tau+\delta)}{P(t+\delta)}) + \gamma(1 + \frac{P((t/\alpha)+\delta)}{\alpha P(t+\delta)})] < \frac{2}{M} \end{cases} \tag{2.4}$$

and

$$\begin{cases} \limsup_{t\to\infty}[\int_{t-\delta}^{t/\beta} P(s+\delta)\,ds + \int_{\beta t}^{t/\beta} \frac{Q(s/\beta)}{s}\,ds \\ \quad + \mu(1 + \frac{tQ((t+\tau)/\beta)}{(t+\tau)Q(t/\beta)}) + \gamma(1 + \frac{Q(t/(\alpha\beta))}{Q(t/\beta)})] < 2. \end{cases} \tag{2.5}$$

Then every solution of (1.1) *tends to a constant as* $t \to \infty$.

Proof Let $x(t)$ be any solution of system (1.1). We will prove that the $\lim_{t\to\infty} x(t)$ exists and is finite. Indeed, the system (1.1) can be written as

$$\begin{aligned} &\left[x(t) + C(t)x(t-\tau) - D(t)x(\alpha t) - \int_{t-\delta}^{t} P(s+\delta)f(x(s))\,ds - \int_{\beta t}^{t} \frac{Q(s/\beta)}{s}x(s)\,ds\right]' \\ &\quad + P(t+\delta)f(x(t)) + \frac{Q(t/\beta)}{t}x(t) = 0, \quad t \ge t_0, t \ne t_k, \end{aligned} \tag{2.6}$$

$$\begin{aligned} x(t_k) = b_k x(t_k^-) + (1 - b_k)\biggl(&\int_{t_k-\delta}^{t_k} P(s+\delta)f(x(s))\,ds \\ &+ \int_{\beta t_k}^{t_k} \frac{Q(s/\beta)}{s}x(s)\,ds\biggr), \quad k = 1, 2, \ldots. \end{aligned} \tag{2.7}$$

From (H_2) and (H_4), we choose constants $\varepsilon, \lambda, \upsilon, \rho > 0$ sufficiently small such that $\mu + \varepsilon < 1$ and $\gamma + \lambda < 1$ and $T > t_0$ sufficiently large, for $t \ge T$,

$$\begin{aligned} &\biggl[\int_{t-\delta}^{t+\delta} P(s+\delta)\,ds + \int_{\beta t}^{t+\delta} \frac{Q(s/\beta)}{s}\,ds + (\mu+\varepsilon)\left(1 + \frac{P(t+\tau+\delta)}{P(t+\delta)}\right) \\ &\quad + (\gamma+\lambda)\left(1 + \frac{P((t/\alpha)+\delta)}{\alpha P(t+\delta)}\right)\biggr] \le \frac{2}{M} - \upsilon, \end{aligned} \tag{2.8}$$

$$\begin{aligned} &\biggl[\int_{t-\delta}^{t/\beta} P(s+\delta)\,ds + \int_{\beta t}^{t/\beta} \frac{Q(s/\beta)}{s}\,ds + (\mu+\varepsilon)\left(1 + \frac{tQ((t+\tau)/\beta)}{(t+\tau)Q(t/\beta)}\right) \\ &\quad + (\gamma+\lambda)\left(1 + \frac{Q(t/(\alpha\beta))}{Q(t/\beta)}\right)\biggr] \le 2 - \rho, \end{aligned} \tag{2.9}$$

and, for $t \ge T$,

$$|C(t)| \le \mu + \varepsilon, \qquad |D(t)| \le \gamma + \lambda. \tag{2.10}$$

From (2.1), (2.10), we have

$$\frac{|C(t)|}{\mu+\varepsilon} \le 1 \le \frac{f^2(x(t-\tau))}{x^2(t-\tau)}, \qquad \frac{|D(t)|}{\gamma+\lambda} \le 1 \le \frac{f^2(x(\alpha t))}{x^2(\alpha t)}, \quad t \ge T,$$

which lead to

$$\begin{aligned} &\bigl|C(t)\bigr|x^2(t-\tau) \le (\mu+\varepsilon)f^2\bigl(x(t-\tau)\bigr), \\ &\bigl|D(t)\bigr|x^2(\alpha t) \le (\gamma+\lambda)f^2\bigl(x(\alpha t)\bigr), \quad t \ge T. \end{aligned} \tag{2.11}$$

In the following, for convenience, the expressions of functional equalities and inequalities will be written without its domain. This means that the relations hold for all sufficiently large t.

Let $V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t)$, where

$$\begin{aligned} V_1(t) &= \left[x(t) + C(t)x(t-\tau) - D(t)x(\alpha t) - \int_{t-\delta}^{t} P(s+\delta)f\bigl(x(s)\bigr)\,ds - \int_{\beta t}^{t} \frac{Q(s/\beta)}{s}x(s)\,ds\right]^2, \\ V_2(t) &= \int_{t-\delta}^{t} P(s+2\delta)\int_{s}^{t} P(u+\delta)f^2\bigl(x(u)\bigr)\,du\,ds \\ &\quad + \int_{\beta t}^{t} \frac{P((s+\beta\delta)/\beta)}{\beta}\int_{s}^{t} \frac{Q(u/\beta)}{u}x^2(u)\,du\,ds, \\ V_3(t) &= \int_{t-\delta}^{t} \frac{Q((s+\delta)/\beta)}{s+\delta}\int_{s}^{t} P(u+\delta)f^2\bigl(x(u)\bigr)\,du\,ds \\ &\quad + \int_{\beta t}^{t} \frac{Q(s/\beta^2)}{s}\int_{s}^{t} \frac{Q(u/\beta)}{u}x^2(u)\,du\,ds, \end{aligned}$$

and

$$\begin{aligned} V_4(t) &= (\mu+\varepsilon)\int_{t-\tau}^{t} P(s+\tau+\delta)f^2\bigl(x(s)\bigr)\,ds + (\mu+\varepsilon)\int_{t-\tau}^{t} \frac{Q((s+\tau)/\beta)}{s+\tau}x^2(s)\,ds \\ &\quad + (\gamma+\lambda)\int_{\alpha t}^{t} \frac{Q(s/(\alpha\beta))}{s}x^2(s)\,ds + \frac{\gamma+\lambda}{\alpha}\int_{\alpha t}^{t} P\bigl((s/\alpha)+\delta\bigr)f^2\bigl(x(s)\bigr)\,ds. \end{aligned}$$

Computing dV_1/dt along the solution of (1.1) and using the inequality $2ab \le a^2 + b^2$, we have

$$\begin{aligned} \frac{dV_1}{dt} &= -2\biggl[x(t) + C(t)x(t-\tau) - D(t)x(\alpha t) \\ &\qquad - \int_{t-\delta}^{t} P(s+\delta)f\bigl(x(s)\bigr)\,ds - \int_{\beta t}^{t} \frac{Q(s/\beta)}{s}x(s)\,ds\biggr] \\ &\qquad \times \left(P(t+\delta)f\bigl(x(t)\bigr) + \frac{Q(t/\beta)}{t}x(t)\right) \\ &\le -P(t+\delta)\biggl[2x(t)f\bigl(x(t)\bigr) - \bigl|C(t)\bigr|x^2(t-\tau) - \bigl|C(t)\bigr|f^2\bigl(x(t)\bigr) - \bigl|D(t)\bigr|x^2(\alpha t) \\ &\qquad - \bigl|D(t)\bigr|f^2\bigl(x(t)\bigr) - \int_{t-\delta}^{t} P(s+\delta)f^2\bigl(x(s)\bigr)\,ds - f^2\bigl(x(t)\bigr)\int_{t-\delta}^{t} P(s+\delta)\,ds \end{aligned}$$

$$
\begin{aligned}
&- \int_{\beta t}^{t} \frac{Q(s/\beta)}{s} x^2(s)\,ds - f^2\big(x(t)\big) \int_{\beta t}^{t} \frac{Q(s/\beta)}{s}\,ds \Bigg] \\
&- \frac{Q(t/\beta)}{t} \Bigg[2x^2(t) - \big|C(t)\big| x^2(t) - \big|C(t)\big| x^2(t-\tau) - \big|D(t)\big| x^2(t) \\
&- \big|D(t)\big| x^2(\alpha t) - \int_{t-\delta}^{t} P(s+\delta) f^2\big(x(s)\big)\,ds - x^2(t) \int_{t-\delta}^{t} P(s+\delta)\,ds \\
&- \int_{\beta t}^{t} \frac{Q(s/\beta)}{s} x^2(s)\,ds - x^2(t) \int_{\beta t}^{t} \frac{Q(s/\beta)}{s}\,ds \Bigg].
\end{aligned}
$$

Calculating directly for dV_i/dt, $i = 2,3,4$, $t \neq t_k$, we have

$$
\begin{aligned}
\frac{dV_2}{dt} &= P(t+\delta) f^2\big(x(t)\big) \int_{t-\delta}^{t} P(s+2\delta)\,ds - P(t+\delta) \int_{t-\delta}^{t} P(s+\delta) f^2\big(x(s)\big)\,ds \\
&\quad + \frac{Q(t/\beta)}{\beta t} x^2(t) \int_{\beta t}^{t} P\big((s+\beta\delta)/\beta\big)\,ds - P(t+\delta) \int_{\beta t}^{t} \frac{Q(s/\beta)}{s} x^2(s)\,ds, \\
\frac{dV_3}{dt} &= P(t+\delta) f^2\big(x(t)\big) \int_{t-\delta}^{t} \frac{Q((s+\delta)/\beta)}{s+\delta}\,ds - \frac{Q(t/\beta)}{t} \int_{t-\delta}^{t} P(s+\delta) f^2\big(x(s)\big)\,ds \\
&\quad + \frac{Q(t/\beta)}{t} x^2(t) \int_{\beta t}^{t} \frac{Q(s/\beta^2)}{s}\,ds - \frac{Q(t/\beta)}{t} \int_{\beta t}^{t} \frac{Q(s/\beta)}{s} x^2(s)\,ds,
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{dV_4}{dt} &= (\mu+\varepsilon) P(t+\tau+\delta) f^2\big(x(t)\big) - (\mu+\varepsilon) P(t+\delta) f^2\big(x(t-\tau)\big) \\
&\quad + \frac{(\mu+\varepsilon)}{(t+\tau)} Q\big((t+\tau)/\beta\big) x^2(t) - \frac{(\mu+\varepsilon)}{t} Q(t/\beta) x^2(t-\tau) \\
&\quad + (\gamma+\lambda) \frac{Q(t/(\alpha\beta))}{t} x^2(t) - (\gamma+\lambda) \frac{Q(t/\beta)}{t} x^2(\alpha t) \\
&\quad + \frac{(\gamma+\lambda)}{\alpha} P\big((t/\alpha)+\delta\big) f^2\big(x(t)\big) - (\gamma+\lambda) P(t+\delta) f^2\big(x(\alpha t)\big).
\end{aligned}
$$

Summing for dV_i/dt, $i = 1,2,3$, we obtain

$$
\begin{aligned}
&\frac{dV_1}{dt} + \frac{dV_2}{dt} + \frac{dV_3}{dt} \\
&\quad \leq -P(t+\delta) \Bigg[2x(t) f\big(x(t)\big) - \big|C(t)\big| x^2(t-\tau) - \big|C(t)\big| f^2\big(x(t)\big) - \big|D(t)\big| x^2(\alpha t) \\
&\quad - \big|D(t)\big| f^2\big(x(t)\big) - f^2\big(x(t)\big) \int_{t-\delta}^{t} P(s+\delta)\,ds - f^2\big(x(t)\big) \int_{\beta t}^{t} \frac{Q(s/\beta)}{s}\,ds \\
&\quad - f^2\big(x(t)\big) \int_{t-\delta}^{t} P(s+2\delta)\,ds - f^2\big(x(t)\big) \int_{t-\delta}^{t} \frac{Q((s+\delta)/\beta)}{s+\delta}\,ds \Bigg] \\
&\quad - \frac{Q(t/\beta)}{t} \Bigg[2x^2(t) - \big|C(t)\big| x^2(t) - \big|C(t)\big| x^2(t-\tau) - \big|D(t)\big| x^2(t) \\
&\quad - \big|D(t)\big| x^2(\alpha t) - x^2(t) \int_{t-\delta}^{t} P(s+\delta)\,ds - x^2(t) \int_{\beta t}^{t} \frac{Q(s/\beta)}{s}\,ds \\
&\quad - \frac{x^2(t)}{\beta} \int_{\beta t}^{t} P\big((s+\beta\delta)/\beta\big)\,ds - x^2(t) \int_{\beta t}^{t} \frac{Q(s/\beta^2)}{s}\,ds \Bigg].
\end{aligned}
$$

Since

$$\int_{t-\delta}^{t} P(s+2\delta)\,ds = \int_{t}^{t+\delta} P(s+\delta)\,ds, \qquad \int_{\beta t}^{t} \frac{Q(s/\beta^2)}{s}\,ds = \int_{t}^{t/\beta} \frac{Q(s/\beta)}{s}\,ds,$$

$$\int_{t-\delta}^{t} \frac{Q((s+\delta)/\beta)}{s+\delta}\,ds = \int_{t}^{t+\delta} \frac{Q(s/\beta)}{s}\,ds, \qquad \frac{1}{\beta}\int_{\beta t}^{t} P\big((s+\beta\delta)/\beta\big)\,ds = \int_{t}^{t/\beta} P(s+\delta)\,ds,$$

it follows that

$$\begin{aligned}
&\frac{dV_1}{dt} + \frac{dV_2}{dt} + \frac{dV_3}{dt} \\
&\quad \le -P(t+\delta)\bigg[2x(t)f\big(x(t)\big) - \big|C(t)\big|x^2(t-\tau) - \big|C(t)\big|f^2\big(x(t)\big) - \big|D(t)\big|x^2(\alpha t) \\
&\qquad - \big|D(t)\big|f^2\big(x(t)\big) - f^2\big(x(t)\big)\int_{t-\delta}^{t+\delta} P(s+\delta)\,ds - f^2\big(x(t)\big)\int_{\beta t}^{t+\delta} \frac{Q(s/\beta)}{s}\,ds\bigg] \\
&\qquad - \frac{Q(t/\beta)}{t}\bigg[2x^2(t) - \big|C(t)\big|x^2(t) - \big|C(t)\big|x^2(t-\tau) - \big|D(t)\big|x^2(t) \\
&\qquad - \big|D(t)\big|x^2(\alpha t) - x^2(t)\int_{t-\delta}^{t/\beta} P(s+\delta)\,ds - x^2(t)\int_{\beta t}^{t/\beta} \frac{Q(s/\beta)}{s}\,ds\bigg].
\end{aligned}$$

Adding the above inequality with dV_4/dt and using condition (2.11), we have

$$\begin{aligned}
&\frac{dV_1}{dt} + \frac{dV_2}{dt} + \frac{dV_3}{dt} + \frac{dV_4}{dt} \\
&\quad \le -P(t+\delta)\bigg[2x(t)f\big(x(t)\big) - \big|C(t)\big|f^2\big(x(t)\big) - \big|D(t)\big|f^2\big(x(t)\big) \\
&\qquad - f^2\big(x(t)\big)\int_{t-\delta}^{t+\delta} P(s+\delta)\,ds - f^2\big(x(t)\big)\int_{\beta t}^{t+\delta} \frac{Q(s/\beta)}{s}\,ds\bigg] \\
&\qquad - \frac{Q(t/\beta)}{t}\bigg[2x^2(t) - \big|C(t)\big|x^2(t) - \big|D(t)\big|x^2(t) \\
&\qquad - x^2(t)\int_{t-\delta}^{t/\beta} P(s+\delta)\,ds - x^2(t)\int_{\beta t}^{t/\beta} \frac{Q(s/\beta)}{s}\,ds\bigg] \\
&\qquad + (\mu+\varepsilon)P(t+\tau+\delta)f^2\big(x(t)\big) + \frac{(\mu+\varepsilon)}{t+\tau}Q\big((t+\tau)/\beta\big)x^2(t) \\
&\qquad + (\gamma+\lambda)\frac{Q(t/(\alpha\beta))}{t}x^2(t) + \frac{(\gamma+\lambda)}{\alpha}P\big((t/\alpha)+\delta\big)f^2\big(x(t)\big).
\end{aligned}$$

Applying (2.8), (2.9), and (2.10), it follows that

$$\begin{aligned}
\frac{dV}{dt} &= \frac{dV_1}{dt} + \frac{dV_2}{dt} + \frac{dV_3}{dt} + \frac{dV_4}{dt} \\
&\le -P(t+\delta)f^2\big(x(t)\big)\bigg[\frac{2x(t)}{f(x(t))} - \big|C(t)\big| - \big|D(t)\big| - \int_{t-\delta}^{t+\delta} P(s+\delta)\,ds \\
&\qquad - \int_{\beta t}^{t+\delta} \frac{Q(s/\beta)}{s}\,ds - (\mu+\varepsilon)\frac{P(t+\tau+\delta)}{P(t+\delta)} - \frac{(\gamma+\lambda)}{\alpha}\frac{P((t/\alpha)+\delta)}{P(t+\delta)}\bigg] \\
&\qquad - \frac{Q(t/\beta)}{t}x^2(t)\bigg[2 - \big|C(t)\big| - \big|D(t)\big| - \int_{t-\delta}^{t/\beta} P(s+\delta)\,ds
\end{aligned}$$

$$
\begin{aligned}
&- \int_{\beta t}^{t/\beta} \frac{Q(s/\beta)}{s}\,ds - \frac{(\mu+\varepsilon)t}{t+\tau}\frac{Q((t+\tau)/\beta)}{Q(t/\beta)} - (\gamma+\lambda)\frac{Q(t/(\alpha\beta))}{Q(t/\beta)}\Bigg] \\
&\leq -P(t+\delta)f^2\big(x(t)\big)\Bigg[\frac{2}{M} - \int_{t-\delta}^{t+\delta} P(s+\delta)\,ds - \int_{\beta t}^{t+\delta} \frac{Q(s/\beta)}{s}\,ds \\
&\quad - (\mu+\varepsilon)\left(1 + \frac{P(t+\tau+\delta)}{P(t+\delta)}\right) - (\gamma+\lambda)\left(1 + \frac{P((t/\alpha)+\delta)}{\alpha P(t+\delta)}\right)\Bigg] \\
&\quad - \frac{Q(t/\beta)}{t}x^2(t)\Bigg[2 - \int_{t-\delta}^{t/\beta} P(s+\delta)\,ds - \int_{\beta t}^{t/\beta} \frac{Q(s/\beta)}{s}\,ds \\
&\quad - (\mu+\varepsilon)\left(1 + \frac{t}{t+\tau}\frac{Q((t+\tau)/\beta)}{Q(t/\beta)}\right) - (\gamma+\lambda)\left(1 + \frac{Q(t/(\alpha\beta))}{Q(t/\beta)}\right)\Bigg] \\
&\leq -P(t+\delta)f^2\big(x(t)\big)\upsilon - \frac{Q(t/\beta)}{t}x^2(t)\rho.
\end{aligned}
\tag{2.12}
$$

For $t = t_k$, we have

$$
\begin{aligned}
V_1(t_k) &= \Bigg[x(t_k) + C(t_k)x(t_k-\tau) - D(t_k)x(\alpha t_k) \\
&\quad - \int_{t_k-\delta}^{t_k} P(s+\delta)f\big(x(s)\big)\,ds - \int_{\beta t_k}^{t_k} \frac{Q(s/\beta)}{s}x(s)\,ds\Bigg]^2 \\
&= \Bigg[b_k x\big(t_k^-\big) + b_k C\big(t_k^-\big)x\big(t_k^- - \tau\big) - b_k D\big(t_k^-\big)x\big(\alpha t_k^-\big) \\
&\quad - b_k\left(\int_{t_k-\delta}^{t_k} P(s+\delta)f\big(x(s)\big)\,ds + \int_{\beta t_k}^{t_k} \frac{Q(s/\beta)}{s}x(s)\,ds\right)\Bigg]^2 \\
&= b_k^2 V_1\big(t_k^-\big).
\end{aligned}
$$

It is easy to see that $V_2(t_k) = V_2(t_k^-)$, $V_3(t_k) = V_3(t_k^-)$, and $V_4(t_k) = V_4(t_k^-)$.

Therefore,

$$
\begin{aligned}
V(t_k) &= V_1(t_k) + V_2(t_k) + V_3(t_k) + V_4(t_k) \\
&= b_k^2 V_1\big(t_k^-\big) + V_2\big(t_k^-\big) + V_3\big(t_k^-\big) + V_4\big(t_k^-\big) \\
&\leq V_1\big(t_k^-\big) + V_2\big(t_k^-\big) + V_3\big(t_k^-\big) + V_4\big(t_k^-\big) \\
&= V\big(t_k^-\big).
\end{aligned}
\tag{2.13}
$$

From (2.12) and (2.13), we conclude that $V(t)$ is decreasing. In view of the fact that $V(t) \geq 0$, we have $\lim_{t\to\infty} V(t) = \psi$ exist and $\psi \geq 0$.

By using (2.8), (2.9), (2.12), and (2.13), we have

$$
\upsilon \int_T^\infty P(t+\delta)f^2\big(x(t)\big)\,dt + \rho \int_T^\infty \frac{Q(t/\beta)}{t}x^2(t)\,dt \leq V(T),
$$

which yields

$$
P(t+\delta)f^2\big(x(t)\big), \frac{Q(t/\beta)}{t}x^2(t) \in L^1(t_0,\infty).
$$

Hence, for any $\phi > 0$ and $\xi \in (0,1)$, we get

$$\lim_{t\to\infty}\int_{t-\phi}^{t} P(s+\delta)f^2\bigl(x(s)\bigr)\,ds = 0, \qquad \lim_{t\to\infty}\int_{\xi t}^{t} \frac{Q(s/\beta)}{s}x^2(s)\,ds = 0.$$

Thus, it follows from (2.4) and (2.5) that

$$\begin{aligned}
&\int_{t-\delta}^{t} P(s+2\delta)\int_{s}^{t} P(u+\delta)f^2\bigl(x(u)\bigr)\,du\,ds \\
&\quad + \int_{\beta t}^{t} \frac{P((s+\beta\delta)/\beta)}{\beta}\int_{s}^{t} \frac{Q(u/\beta)}{u}x^2(u)\,du\,ds \\
&\quad \le \int_{t-\delta}^{t+\delta} P(s+\delta)\,ds\int_{t-\delta}^{t} P(u+\delta)f^2\bigl(x(u)\bigr)\,du \\
&\quad + \int_{t-\delta}^{t/\beta} P(s+\delta)\,ds\int_{\beta t}^{t} \frac{Q(u/\beta)}{u}x^2(u)\,du \\
&\quad \le \frac{2}{M}\int_{t-\delta}^{t} P(u+\delta)f^2\bigl(x(u)\bigr)\,du\,ds \\
&\quad + \frac{2}{M}\int_{\beta t}^{t} \frac{Q(u/\beta)}{u}x^2(u)\,du \to 0, \quad \text{as } t\to\infty, \\
&\int_{t-\delta}^{t} \frac{Q((s+\delta)/\beta)}{s+\delta}\int_{s}^{t} P(u+\delta)f^2\bigl(x(u)\bigr)\,du\,ds \\
&\quad + \int_{\beta t}^{t} \frac{Q(s/\beta^2)}{s}\int_{s}^{t} \frac{Q(u/\beta)}{u}x^2(u)\,du\,ds \\
&\quad \le \int_{\beta t}^{t+\delta} \frac{Q(s/\beta)}{s}\,ds\int_{t-\delta}^{t} P(u+\delta)f^2\bigl(x(u)\bigr)\,du \\
&\quad + \int_{\beta t}^{t/\beta} \frac{Q(s/\beta)}{s}\,ds\int_{\beta t}^{t} \frac{Q(u/\beta)}{u}x^2(u)\,du \\
&\quad \le \frac{2}{M}\int_{t-\delta}^{t} P(u+\delta)f^2\bigl(x(u)\bigr)\,du \\
&\quad + 2\int_{\beta t}^{t} \frac{Q(u/\beta)}{u}x^2(u)\,du \to 0, \quad \text{as } t\to\infty,
\end{aligned}$$

and

$$\begin{aligned}
&(\mu+\varepsilon)\int_{t-\tau}^{t} P(s+\tau+\delta)f^2\bigl(x(s)\bigr)\,ds + (\mu+\varepsilon)\int_{t-\tau}^{t} \frac{Q((s+\tau)/\beta)}{s+\tau}x^2(s)\,ds \\
&\qquad + (\gamma+\lambda)\int_{\alpha t}^{t} \frac{Q(s/(\alpha\beta))}{s}x^2(s)\,ds + \frac{\gamma+\lambda}{\alpha}\int_{\alpha t}^{t} P\bigl((s/\alpha)+\delta\bigr)f^2\bigl(x(s)\bigr)\,ds \\
&\quad = (\mu+\varepsilon)\int_{t-\tau}^{t} \frac{P(s+\tau+\delta)}{P(s+\delta)}P(s+\delta)f^2\bigl(x(s)\bigr)\,ds \\
&\qquad + (\mu+\varepsilon)\int_{t-\tau}^{t} \frac{sQ((s+\tau)/\beta)}{Q(s/\beta)(s+\tau)}\cdot\frac{Q(s/\beta)}{s}x^2(s)\,ds \\
&\qquad + (\gamma+\lambda)\int_{\alpha t}^{t} \frac{Q(s/(\alpha\beta))}{Q(s/\beta)}\cdot\frac{Q(s/\beta)}{s}x^2(s)\,ds
\end{aligned}$$

$$\begin{aligned}&+\frac{\gamma+\lambda}{\alpha}\int_{\alpha t}^{t}\frac{P((s/\alpha)+\delta)}{P(s+\delta)}P(s+\delta)f^2\bigl(x(s)\bigr)\,ds\\&\leq\frac{2}{M}\int_{t-\tau}^{t}P(s+\delta)f^2\bigl(x(s)\bigr)\,ds+2\int_{t-\tau}^{t}\frac{Q(s/\beta)}{s}x^2(s)\,ds\\&+2\int_{\alpha t}^{t}\frac{Q(s/\beta)}{s}x^2(s)\,ds+\frac{2}{M}\int_{\alpha t}^{t}P(s+\delta)f^2\bigl(x(s)\bigr)\,ds\to 0,\quad\text{as }t\to\infty.\end{aligned}$$

Therefore, from the above estimations, we have $\lim_{t\to\infty}V_2(t)=0$, $\lim_{t\to\infty}V_3(t)=0$, and $\lim_{t\to\infty}V_4(t)=0$, respectively.

Thus, $\lim_{t\to\infty}V_1(t)=\lim_{t\to\infty}V(t)=\psi$, that is,

$$\begin{aligned}&\lim_{t\to\infty}\biggl[x(t)+C(t)x(t-\tau)-D(t)x(\alpha t)\\&-\int_{t-\delta}^{t}P(s+\delta)f\bigl(x(s)\bigr)\,ds-\int_{\beta t}^{t}\frac{Q(s/\beta)}{s}x(s)\,ds\biggr]^2=\psi.\end{aligned}\tag{2.14}$$

Now, we will prove that the limit

$$\begin{aligned}&\lim_{t\to\infty}\biggl[x(t)+C(t)x(t-\tau)-D(t)x(\alpha t)\\&-\int_{t-\delta}^{t}P(s+\delta)f\bigl(x(s)\bigr)\,ds-\int_{\beta t}^{t}\frac{Q(s/\beta)}{s}x(s)\,ds\biggr]\end{aligned}\tag{2.15}$$

exists and is finite. Setting

$$\begin{aligned}y(t)&=x(t)+C(t)x(t-\tau)-D(t)x(\alpha t)\\&\quad-\int_{t-\delta}^{t}P(s+\delta)f\bigl(x(s)\bigr)\,ds-\int_{\beta t}^{t}\frac{Q(s/\beta)}{s}x(s)\,ds,\end{aligned}\tag{2.16}$$

and using (1.1) and condition (H_3), we have

$$\begin{aligned}y(t_k)&=x(t_k)+C(t_k)x(t_k-\tau)-D(t_k)x(\alpha t_k)\\&\quad-\int_{t_k-\delta}^{t_k}P(s+\delta)f\bigl(x(s)\bigr)\,ds-\int_{\beta t_k}^{t_k}\frac{Q(s/\beta)}{s}x(s)\,ds\\&=b_k\biggl[x\bigl(t_k^-\bigr)+C\bigl(t_k^-\bigr)x\bigl(t_k^--\tau\bigr)-D\bigl(t_k^-\bigr)x\bigl(\alpha t_k^-\bigr)\\&\quad-\int_{t_k-\delta}^{t_k}P(s+\delta)f\bigl(x(s)\bigr)\,ds-\int_{\beta t_k}^{t_k}\frac{Q(s/\beta)}{s}x(s)\,ds\biggr]\\&=b_ky\bigl(t_k^-\bigr).\end{aligned}\tag{2.17}$$

In view of (2.14), it follows that

$$\lim_{t\to\infty}y^2(t)=\psi.$$

In addition, from (2.16) and (2.17), system (2.6)-(2.7) can be written as

$$\begin{cases}y'(t)+P(t+\delta)f(x(t))+\frac{Q(t/\beta)}{t}x(t)=0,\quad 0<t_0\leq t,t\neq t_k,\\y(t_k)=b_ky(t_k^-),\quad k=1,2,3,\ldots.\end{cases}\tag{2.18}$$

If $\psi = 0$, then $\lim_{t\to\infty} y(t) = 0$. If $\psi > 0$, then there exists a sufficiently large T^* such that $y(t) \neq 0$ for any $t > T^*$. Otherwise, there is a sequence $\{a_k\}$ with $\lim_{k\to\infty} a_k = \infty$ such that $y(a_k) = 0$, and so $y^2(a_k) \to 0$ as $k \to \infty$. This contradicts $\psi > 0$. Therefore, for any $t_k > T^*$, and $t \in [t_k, t_{k+1})$, we have $y(t) > 0$ or $y(t) < 0$ from the continuity of y on $[t_k, t_{k+1})$. Without loss of generality, we assume that $y(t) > 0$ on $[t_k, t_{k+1})$. It follows from (H_3) that $y(t_{k+1}) = b_k y(t_{k+1}^-) > 0$, and thus $y(t) > 0$ on $[t_{k+1}, t_{k+2})$. By using mathematical induction, we deduce that $y(t) > 0$ on $[t_k, \infty)$. Therefore, from (2.14), we have

$$\begin{aligned}\lim_{t\to\infty} y(t) = \lim_{t\to\infty}\bigg[&x(t) + C(t)x(t-\tau) - D(t)x(\alpha t)\\ &- \int_{t-\delta}^{t} P(s+\delta) f\big(x(s)\big)\,ds - \int_{\beta t}^{t} \frac{Q(s/\beta)}{s} x(s)\,ds\bigg] = \kappa,\end{aligned} \tag{2.19}$$

where $\kappa = \sqrt{\psi}$ and is finite. In view of (2.18), for sufficient large t, we have

$$\begin{aligned}&\int_{\beta t-\delta}^{t} P(s+\delta) f\big(x(s)\big)\,ds + \int_{\beta t-\delta}^{t} \frac{Q(s/\beta)}{s} x(s)\,ds\\ &\quad = y(\beta t-\delta) - y(t) - \sum_{\beta t-\delta<t_k<t} \big[y(t_k) - y(t_k^-)\big]\\ &\quad = y(\beta t-\delta) - y(t) - \sum_{\beta t-\delta<t_k<t} (1-b_k) y\big(t_k^-\big).\end{aligned}$$

Taking $t \to \infty$ and using (H_3), we have

$$\lim_{t\to\infty}\bigg[\int_{\beta t-\delta}^{t} P(s+\delta) f\big(x(s)\big)\,ds + \int_{\beta t-\delta}^{t} \frac{Q(s/\beta)}{s} x(s)\,ds\bigg] = 0,$$

which leads to

$$\lim_{t\to\infty}\int_{t-\delta}^{t} P(s+\delta) f\big(x(s)\big)\,ds = 0 \quad \text{and} \quad \lim_{t\to\infty}\int_{\beta t}^{t} \frac{Q(s/\beta)}{s} x(s)\,ds = 0.$$

This implies that

$$\lim_{t\to\infty}\big[x(t) + C(t)x(t-\tau) - D(t)x(\alpha t)\big] = \kappa. \tag{2.20}$$

Next, we shall prove that

$$\lim_{t\to\infty} x(t) \text{ exists and is finite.} \tag{2.21}$$

Further, we first show that $|x(t)|$ is bounded. Actually, if $|x(t)|$ is unbounded, then there exists a sequence $\{z_n\}$ such that $z_n \to \infty$, $|x(z_n^-)| \to \infty$, as $n \to \infty$ and

$$\big|x\big(z_n^-\big)\big| = \sup_{t_0\le t\le z_n} \big|x(t)\big|, \tag{2.22}$$

where, if z_n is not an impulsive point, then $x(z_n^-) = x(z_n)$. Thus, we have

$$\begin{aligned}&\left|x(z_n^-) + C(z_n^-)x(z_n^- - \tau) - D(z_n^-)x(\alpha z_n^-)\right|\\&\quad\geq \left|x(z_n^-)\right| - \left|C(z_n^-)\right|\left|x(z_n^- - \tau)\right| - \left|D(z_n^-)\right|\left|x(\alpha z_n^-)\right|\\&\quad\geq \left|x(z_n^-)\right|[1 - \mu - \varepsilon - \gamma - \lambda] \to \infty,\end{aligned}$$

as $n \to \infty$, which contradicts (2.20). Therefore, $|x(t)|$ is bounded.

If $\mu = 0$ and $\gamma = 0$, then $\lim_{t\to\infty} x(t) = \kappa$, which implies that (2.21) holds. If $0 < \mu < 1$ and $0 < \gamma < 1$, then we deduce that $C(t)$ and $D(t)$ are eventually positive or eventually negative. Otherwise, there are two sequences $\{w_k\}$ and $\{w_j^*\}$ with $\lim_{k\to\infty} w_k = \infty$ and $\lim_{j\to\infty} w_j^* = \infty$ such that $C(w_k) = 0$ and $D(w_j^*) = 0$. Therefore, $C(w_k) \to 0$ and $D(w_j^*) \to 0$ as $k, j \to \infty$. It is a contradiction to $\mu > 0$ and $\gamma > 0$.

Now, we will show that (2.21) holds. By condition (H_2), we can find a sufficiently large T_1 such that for $t > T_1$, $|C(t)| + |D(t)| < 1$. Set

$$\omega = \liminf_{t\to\infty} x(t), \qquad \theta = \limsup_{t\to\infty} x(t).$$

Then we can choose two sequences $\{u_n\}$ and $\{v_n\}$ such that $u_n \to \infty$, $v_n \to \infty$ as $n \to \infty$, and

$$\lim_{n\to\infty} x(u_n) = \omega, \qquad \lim_{n\to\infty} x(v_n) = \theta.$$

For $t > T_1$, we consider the following eight possible cases.

Case 1. When $\lim_{t\to\infty} C(t) = 0$ and $-1 < D(t) < 0$ for $t > T_1$, we have

$$\kappa = \lim_{n\to\infty}\left[x(u_n) - D(u_n)x(\alpha u_n)\right] \leq \omega + \gamma\theta,$$

and

$$\kappa = \lim_{n\to\infty}\left[x(v_n) - D(v_n)x(\alpha v_n)\right] \geq \theta + \gamma\omega.$$

Thus, we obtain

$$\omega + \gamma\theta \geq \theta + \gamma\omega,$$

that is,

$$\omega(1-\gamma) \geq \theta(1-\gamma).$$

Since $0 < \gamma < 1$ and $\theta \geq \omega$, it follows that $\theta = \omega$. By (2.20), we obtain

$$\theta = \omega = \frac{\kappa}{1-\gamma},$$

which shows that (2.21) holds.

Case 2. When $\lim_{t\to\infty} D(t) = 0$ and $-1 < C(t) < 0$ for $t > T_1$, we get

$$\kappa = \lim_{n\to\infty}\bigl[x(u_n) + C(u_n)x(u_n - \tau)\bigr] \le \omega - \mu\omega$$

and

$$\kappa = \lim_{n\to\infty}\bigl[x(v_n) + C(v_n)x(v_n - \tau)\bigr] \ge \theta - \mu\theta,$$

which leads to

$$\omega(1-\mu) \ge \theta(1-\mu).$$

Since $0 < \mu < 1$ and $\theta \ge \omega$, we conclude that

$$\theta = \omega = \frac{\kappa}{1-\mu},$$

which implies that (2.21) holds.

Case 3. $\lim_{t\to\infty} C(t) = 0$, $0 < D(t) < 1$ for $t > T_1$. The method of proof is similar to the above two cases. Therefore, we omit it.

Case 4. $\lim_{t\to\infty} D(t) = 0$, $0 < C(t) < 1$ for $t > T_1$. The method of proof is similar to the above two first cases. Therefore, we omit it.

Case 5. When $-1 < D(t) < 0$ and $0 < C(t) < 1$ for $t > T_1$, we have

$$\kappa = \lim_{n\to\infty}\bigl[x(u_n) + C(u_n)x(u_n - \tau) - D(u_n)x(\alpha u_n)\bigr] \le \omega + \mu\theta + \gamma\theta$$

and

$$\kappa = \lim_{n\to\infty}\bigl[x(v_n) + C(v_n)x(v_n - \tau) - D(v_n)x(\alpha v_n)\bigr] \ge \theta + \mu\omega + \gamma\omega,$$

which yields

$$\omega(1-\mu-\gamma) \ge \theta(1-\mu-\gamma).$$

Since $0 < \mu + \gamma < 1$ and $\theta \ge \omega$, we have $\theta = \omega$. Thus

$$\theta = \omega = \frac{\kappa}{1-\mu-\gamma},$$

and so (2.21) holds.

Using similar arguments, we can prove that (2.21) also holds for the following cases:

Case 6. $-1 < C(t) < 0$, $0 < D(t) < 1$.

Case 7. $-1 < C(t) < 0$, $-1 < D(t) < 0$.

Case 8. $0 < C(t) < 1$, $0 < D(t) < 1$.

Summarizing the above investigation, we conclude that (2.21) holds and so the proof is completed. □

Theorem 2.2 *Let conditions* (H_1)-(H_4) *of Theorem* 2.1 *hold. Then every oscillatory solution of* (1.1) *tends to zero as* $t \to \infty$.

Corollary 2.1 *Assume that* (H_3) *holds and*

$$\limsup_{t\to\infty}\left[\int_{t-\delta}^{t+\delta} P(s+\delta)\,ds + \int_{\beta t}^{t+\delta}\frac{Q(s/\beta)}{s}\,ds\right] < 2 \tag{2.23}$$

and

$$\limsup_{t\to\infty}\left[\int_{t-\delta}^{t/\beta} P(s+\delta)\,ds + \int_{\beta t}^{t/\beta}\frac{Q(s/\beta)}{s}\,ds\right] < 2. \tag{2.24}$$

Then every solution of the equation

$$\begin{cases} x'(t) + P(t)x(t-\delta) + \frac{Q(t)}{t}x(\beta t) = 0, & 0 < t_0 \le t, t \ne t_k, \\ x(t_k) = b_k x(t_k^-) + (1-b_k)(\int_{t_k-\delta}^{t_k} P(s+\delta)x(s)\,ds \\ \qquad + \int_{\beta t_k}^{t_k}\frac{Q(s/\beta)}{s}x(s)\,ds), & k = 1,2,3,\ldots, \end{cases} \tag{2.25}$$

tends to a constant as $t \to \infty$.

Corollary 2.2 *The conditions* (2.23) *and* (2.24) *imply that every solution of the equation*

$$x'(t) + P(t)x(t-\delta) + \frac{Q(t)}{t}x(\beta t) = 0, \quad 0 < t_0 \le t, \tag{2.26}$$

tends to a constant as $t \to \infty$.

Theorem 2.3 *The conditions* (H_1)-(H_4) *of Theorem* 2.1 *together with*

$$\int_{t_0}^{\infty} P(s+\delta)\,ds = \infty, \qquad \int_{t_0}^{\infty}\frac{Q(s/\beta)}{s}\,ds = \infty, \tag{2.27}$$

imply that every solution of (1.1) *tends to zero as* $t \to \infty$.

Proof From Theorem 2.2, we only have to prove that every nonoscillatory solution of (1.1) tends to zero as $t \to \infty$. Without loss of generality, we assume that $x(t)$ is an eventually positive solution of (1.1). As in the proof of Theorem 2.1, (1.1) can be written as in the form (2.18). Integrating from t_0 to t both sides of the first equation of (2.18), one has

$$\int_{t_0}^{t} P(s+\delta)f\big(x(s)\big)\,ds + \int_{t_0}^{t}\frac{Q(s/\beta)}{s}x(s)\,ds = y(t_0) - y(t) - \sum_{t_0<t_k<t}(1-b_k)y\big(t_k^-\big).$$

Applying (2.19) and (H_3), we have

$$\int_{t_0}^{\infty} P(s+\delta)f\big(x(s)\big)\,ds < \infty \quad \text{and} \quad \int_{t_0}^{\infty}\frac{Q(s/\beta)}{s}x(s)\,ds < \infty.$$

This, together with (2.27), implies that $\liminf_{t\to\infty} f(x(t)) = 0$ and $\liminf_{t\to\infty} x(t) = 0$. By Theorem 2.1, $\lim_{t\to\infty} x(t) = 0$. This completes the proof. □

Corollary 2.3 *Assume that* (2.1), (2.2), (2.4), (2.5), *and* (2.27) *hold. Then every solution of the equation*

$$\big[x(t) + C(t)x(t-\tau) - D(t)x(\alpha t)\big]' + P(t)f\big(x(t-\delta)\big) + \frac{Q(t)}{t}x(\beta t) = 0, \tag{2.28}$$

$0 < t_0 \leq t$, *tends to zero as* $t \to \infty$.

3 Examples

In this section, we present two examples to illustrate our results.

Example 3.1 Consider the following mixed type neutral differential equation with impulsive perturbations:

$$\begin{cases} [x(t) + \frac{(k+3)\lfloor t\rfloor}{3k^2+3k-6}x(t-\frac{1}{2}) - \frac{(3k+9)\lfloor t\rfloor}{8k^2+8k-16}x(\frac{t}{e})]' \\ \quad + (\frac{2+t}{5t^2})(1+\frac{1}{4}\cos^2 x(t-\pi))x(t-\pi) + \frac{1}{t(\ln t+2)}x(\frac{t}{e^2}) = 0, \quad t \geq 1, \\ x(k) = \frac{k^2+6k+8}{(k+3)^2}x(k^-) + (1-\frac{k^2+6k+8}{(k+3)^2})(\int_{t_k-\pi}^{t_k}\frac{2+s+\pi}{5(s+\pi)^2} \\ \qquad \times (1+\frac{1}{4}\cos^2 x(s))x(s)\,ds + \int_{\frac{t_k}{e^2}}^{t_k}\frac{1}{s(\ln(e^2 s)+2)}x(s)\,ds), \quad k = 2,3,4,\ldots. \end{cases} \tag{3.1}$$

Here $C(t) = ((k+3)\lfloor t\rfloor)/(3k^2+3k-6)$, $D(t) = ((3k+9)\lfloor t\rfloor)/(8k^2+8k-16)$, $P(t) = (2+t)/(5t^2)$, $Q(t) = 1/(\ln t + 2)$, $t \in [k-1,k)$, $b_k = (k^2+6k+8)/((k+3)^2)$, $t_0 = 1$, $k = 2,3,4,\ldots$, $f(x) = x(1+((1/4)(\cos^2 x)))$, $\tau = 1/2$, $\delta = \pi$, $\alpha = 1/e$, and $\beta = 1/e^2$. We can find that

(i) $|x| \leq |(1+\frac{1}{4}\cos^2 x)x| \leq \frac{5}{4}|x|$, $x \in \mathbb{R}$, $(1+\frac{1}{4}\cos^2 x)x^2 > 0$ for $x \neq 0$;

(ii) $\lim_{t\to\infty}|C(t)| = \frac{1}{3} = \mu < 1$, $\lim_{t\to\infty}|D(t)| = \frac{3}{8} = \gamma < 1$ with $\mu + \gamma = \frac{17}{24} < 1$, and $C(k) = \frac{k^2+6k+8}{(k+3)^2}C(k^-)$, $D(k) = \frac{k^2+6k+8}{(k+3)^2}D(k^-)$;

(iii) $t_k - (1/2)$ and $(1/e)t_k$ are not impulsive points, $0 < (k^2+6k+8)/((k+3)^2) \leq 1$ for $k = 1,2,\ldots$, and

$$\sum_{k=1}^{\infty}\left(1 - \frac{k^2+6k+8}{(k+3)^2}\right) = \sum_{k=1}^{\infty}\frac{1}{(k+3)^2} < \infty;$$

(iv)

$$\begin{cases} \limsup_{t\to\infty}[\int_{t-\delta}^{t+\delta}P(s+\delta)\,ds + \int_{\beta t}^{t+\delta}\frac{Q(s/\beta)}{s}\,ds \\ \quad + \mu(1+\frac{P(t+\tau+\delta)}{P(t+\delta)}) + \gamma(1+\frac{P((t/\alpha)+\delta)}{\alpha P(t+\delta)})] = \frac{17}{12} < \frac{8}{5} \end{cases}$$

and

$$\begin{cases} \limsup_{t\to\infty}[\int_{t-\delta}^{t/\beta}P(s+\delta)\,ds + \int_{\beta t}^{t/\beta}\frac{Q(s/\beta)}{s}\,ds \\ \quad + \mu(1+\frac{tQ((t+\tau)/\beta)}{(t+\tau)Q(t/\beta)}) + \gamma(1+\frac{Q(t/(\alpha\beta))}{Q(t/\beta)})] = \frac{109}{60} < 2. \end{cases}$$

Hence, by (i)-(iv) all assumptions of Theorem 2.1 are satisfied. Therefore, we conclude that every solution of (3.1) tends to a constant as $t \to \infty$.

Example 3.2 Consider the following mixed type neutral differential equation with impulsive perturbations

$$\begin{cases} [x(t) + \frac{(12k+16)\lfloor t\rfloor}{54k^2+27k-27}x(t-\frac{2}{3}) - \frac{(12k+16)\lfloor t\rfloor}{42k^2+21k-21}x(\frac{t}{2e^3})]' \\ \quad + (\frac{2t+1}{(4t+3)^2})(1+\frac{2}{5}\sin^2 x(t-\frac{\pi}{2}))x(t-\frac{\pi}{2}) + \frac{4}{t(2\ln t+3)}x(\frac{t}{3e}) = 0, \quad t \geq 1, \\ x(k) = \frac{6k^2+17k+7}{6k^2+17k+12}x(k^-) + (1-\frac{6k^2+17k+7}{6k^2+17k+12})(\int_{t_k-\frac{\pi}{2}}^{t_k}\frac{2s+1+\pi}{(4s+3+2\pi)^2} \\ \quad \times (1+\frac{2}{5}\sin^2 x(s))x(s)\,ds + \int_{\frac{t_k}{3e}}^{t_k}\frac{4}{s(2\ln(3es)+3)}x(s)\,ds), \quad k = 2,3,4,\ldots. \end{cases} \tag{3.2}$$

Here $C(t) = ((12k+16)\lfloor t\rfloor)/(54k^2+27k-27)$, $D(t) = ((12k+16)\lfloor t\rfloor)/(42k^2+21k-21)$, $P(t) = (2t+1)/((4t+3)^2)$, $Q(t) = 4/(2\ln t+3)$, $t \in [k-1,k)$, $b_k = (6k^2+17k+7)/(6k^2+17k+12)$, $t_0 = 1$, $k = 2,3,4,\ldots$, $f(x) = x(1+((2/5)\sin^2 x))$, $\tau = 2/3$, $\delta = \pi/2$, $\alpha = 1/(2e^3)$, and $\beta = 1/(3e)$. We can show that

(i) $|x| \leq |(1+\frac{2}{5}\sin^2 x)x| \leq \frac{7}{5}|x|$, $x \in \mathbb{R}$, $(1+\frac{2}{5}\sin^2 x)x^2 > 0$ for $x \neq 0$;

(ii) $\lim_{t\to\infty}|C(t)| = \frac{2}{9} = \mu < 1$, $\lim_{t\to\infty}|D(t)| = \frac{2}{7} = \gamma < 1$ with $\mu+\gamma = \frac{32}{63} < 1$, and $C(k) = \frac{6k^2+17k+7}{6k^2+17k+12}C(k^-)$, $D(k) = \frac{6k^2+17k+7}{6k^2+17k+12}D(k^-)$;

(iii) $t_k-(2/3)$ and $(1/(2e^3))t_k$ are not impulsive points, $0 < (6k^2+17k+7)/(6k^2+17k+12) \leq 1$ for $k = 1,2,\ldots$, and

$$\sum_{k=1}^{\infty}\left(1-\frac{6k^2+17k+7}{6k^2+17k+12}\right) = \sum_{k=1}^{\infty}\frac{5}{6k^2+17k+12} < \infty;$$

(iv)

$$\begin{cases} \limsup_{t\to\infty}[\int_{t-\delta}^{t+\delta}P(s+\delta)\,ds + \int_{\beta t}^{t+\delta}\frac{Q(s/\beta)}{s}\,ds \\ \quad + \mu(1+\frac{P(t+\tau+\delta)}{P(t+\delta)}) + \gamma(1+\frac{P((t/\alpha)+\delta)}{\alpha P(t+\delta)})] = \frac{64}{63} < \frac{10}{7} \end{cases}$$

and

$$\begin{cases} \limsup_{t\to\infty}[\int_{t-\delta}^{t/\beta}P(s+\delta)\,ds + \int_{\beta t}^{t/\beta}\frac{Q(s/\beta)}{s}\,ds \\ \quad + \mu(1+\frac{tQ((t+\tau)/\beta)}{(t+\tau)Q(t/\beta)}) + \gamma(1+\frac{Q(t/(\alpha\beta))}{Q(t/\beta)})] = 1.2781996 < 2; \end{cases}$$

(v)

$$\int_1^{\infty}P(s+\delta)\,ds = \int_1^{\infty}\frac{2s+1+\pi}{(4s+3+2\pi)^2}\,ds = \infty$$

and

$$\int_1^{\infty}\frac{Q(s/\beta)}{s}\,ds = \int_1^{\infty}\frac{4}{s(2\ln(3es)+3)}\,ds = \infty.$$

Hence, all assumptions of Theorem 2.3 are satisfied and therefore every solution of (3.2) tends to zero as $t \to \infty$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally in this article. They read and approved the final manuscript.

Author details

[1]Nonlinear Dynamic Analysis Research Center, Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok, 10800, Thailand. [2]Department of Mathematics, University of Ioannina, Ioannina, 451 10, Greece. [3]Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia.

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References

1. Bainov, DD, Simeonov, PS: Systems with Impulse Effect. Ellis Horwood, Chichester (1989)
2. Lakshmikantham, V, Bainov, DD, Simeonov, PS: Theory of Impulsive Differential Equations. World Scientific, Singapore (1989)
3. Samoilenko, AM, Perestyuk, NA: Impulsive Differential Equations. World Scientific, Singapore (1995)
4. Benchohra, M, Henderson, J, Ntouyas, SK: Impulsive Differential Equations and Inclusions, vol. 2. Hindawi Publishing Corporation, New York (2006)
5. Bainov, DD, Dinitrova, MB, Dishliev, AB: Oscillation of the solutions of impulsive differential equations and inequalities with a retarded argument. Rocky Mt. J. Math. **28**, 25-40 (1998)
6. Luo, Z, Shen, J: Stability and boundedness for impulsive differential equations with infinite delays. Nonlinear Anal. **46**, 475-493 (2001)
7. Liu, X, Shen, J: Asymptotic behavior of solutions of impulsive neutral differential equations. Appl. Math. Lett. **12**, 51-58 (1999)
8. Shen, J, Liu, Y, Li, J: Asymptotic behavior of solutions of nonlinear neutral differential equations with impulses. J. Math. Anal. Appl. **332**, 179-189 (2007)
9. Shen, J, Liu, Y: Asymptotic behavior of solutions for nonlinear delay differential equation with impulses. J. Appl. Math. Comput. **213**, 449-454 (2009)
10. Wei, G, Shen, J: Asymptotic behavior of solutions of nonlinear impulsive delay differential equations with positive and negative coefficients. Math. Comput. Model. **44**, 1089-1096 (2006)
11. Luo, J, Debnath, L: Asymptotic behavior of solutions of forced nonlinear neutral delay differential equations with impulses. J. Appl. Math. Comput. **12**, 39-47 (2003)
12. Jiang, F, Sun, J: Asymptotic behavior of neutral delay differential equation of Euler form with constant impulsive jumps. Appl. Math. Comput. **219**, 9906-9913 (2013)
13. Pandian, S, Balachandran, Y: Asymptotic behavior results for nonlinear impulsive neutral differential equations with positive and negative coefficients. Bonfring Int. J. Data Min. **2**, 13-21 (2012)
14. Wang, QR: Oscillation criteria for first-order neutral differential equations. Appl. Math. Lett. **8**, 1025-1033 (2002)
15. Tariboon, J, Thiramanus, P: Oscillation of a class of second-order linear impulsive differential equations. Adv. Differ. Equ. **2012**, 205 (2012)
16. Jiang, F, Shen, J: Asymptotic behavior of solutions for a nonlinear differential equation with constant impulsive jumps. Acta Math. Hung. **138**, 1-14 (2013)
17. Jiang, F, Shen, J: Asymptotic behaviors of nonlinear neutral impulsive delay differential equations with forced term. Kodai Math. J. **35**, 126-137 (2012)
18. Gunasekar, T, Samuel, FP, Arjunan, MM: Existence results for impulsive neutral functional integrodifferential equation with infinite delay. J. Nonlinear Sci. Appl. **6**, 234-243 (2013)
19. Kumar, P, Pandey, DN, Bahuguna, D: On a new class of abstract impulsive functional differential equations of fractional order. J. Nonlinear Sci. Appl. **7**, 102-114 (2014)
20. Samuel, FP, Balachandran, K: Existence of solutions for quasi-linear impulsive functional integrodifferential equations in Banach spaces. J. Nonlinear Sci. Appl. **7**, 115-125 (2014)
21. Guan, K, Shen, J: Asymptotic behavior of solutions of a first-order impulsive neutral differential equation in Euler form. Appl. Math. Lett. **24**, 1218-1224 (2011)

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