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Approximate controllability and optimal controls of fractional evolution systems in abstract spaces

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Abstract

In this paper, under the assumption that the corresponding linear system is approximately controllable, we obtain the approximate controllability of semilinear fractional evolution systems in Hilbert spaces. The approximate controllability results are proved by means of the Hölder inequality, the Banach contraction mapping principle, and the Schauder fixed point theorem. We also discuss the existence of optimal controls for semilinear fractional controlled systems. Finally, an example is also given to illustrate the applications of the main results.

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Keywords: fractional evolution systems; approximate controllability; optimal controls; semigroup theory; fixed point theorem

1 Introduction

During the past few decades, fractional differential equations have proved to be valuable tools in the modeling of many phenomena in viscoelasticity, electrochemistry, control, porous media, and electromagnetism, *etc.* Due to its tremendous scopes and applications, several monographs have been devoted to the study of fractional differential equations; see the monographs [1–5]. Controllability is a mathematical problem. Since approximately controllable systems are considered to be more prevalent and very often approximate controllability is completely adequate in applications, a considerable interest has been shown in approximate controllability of control systems consisting of a linear and a nonlinear part [6–10]. In addition, the problems associated with optimal controls for fractional systems in abstract spaces have been widely discussed [10–22]. Wang and Wei [23] obtained the existence and uniqueness of the PC-mild solution for one order nonlinear integro-differential impulsive differential equations with nonlocal conditions. Bragdi [24] established exact controllability results for a class of nonlocal quasilinear differential inclusions of fractional order in a Banach space. Machado *et al.* [25] considered the exact controllability for one order abstract impulsive mixed point-type functional integro-differential equations with finite delay in a Banach space. Approximate controllability for one order nonlinear evolution equations with monotone operators was attained in [26]. By the well-known monotone iterative technique, Mu and Li [27] obtained existence and uniqueness results for fractional evolution equations without mixed type operators in nonlinearity.

Wang and Zhou [20] studied a class of fractional evolution equations of the following type:

$$\begin{cases} D^q x(t) = -Ax(t) + f(t, x(t)), & t \in [0, T], q \in (0,1), \\ x(0) = x_0, \end{cases}$$

where D^q is the Caputo fractional derivative of order $0 < q < 1$, $-A$ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators. A suitable α-mild solution of the semilinear fractional evolution equations is given, and the existence and uniqueness of α-mild solutions are also proved. Then by inducing a control term, the existence of an optimal pair of systems governed by a class of fractional evolution equations is also presented.

Mahmudov and Zorlu [6] considered the following semilinear fractional evolution system:

$$\begin{cases} {}^C D_t^q x(t) = -Ax(t) + Bu(t) + f(t, x(t), (Gx)(t)), & t \in [0, T], \\ x(0) = x_0, \end{cases}$$

where ${}^C D_t^q$ is the Caputo fractional derivative of order $0 < q < 1$, the state variable x takes values in a Hilbert space $\mathbb{X}$, A is the infinitesimal generator of a C_0-semigroup of bounded operators on the Hilbert space $\mathbb{X}$, the control function u is given in $L^2([0, T], U)$, U is a Hilbert space, B is a bounded linear operator from U into $\mathbb{X}_\alpha$. $(Gx)(t) := \int_0^t K(t,s)x(s)\,ds$ is a Volterra integral operator. They studied the approximate controllability of the above controlled system described by semilinear fractional integro-differential evolution equation by the Schauder fixed point theorem. Very recently, Wang *et al.* [15] researched nonlocal problems for fractional integro-differential equations via fractional operators and optimal controls, and they obtained the existence of mild solutions and the existence of optimal pairs of systems governed by fractional integro-differential equations with nonlocal conditions. Subsequently, Ganesh *et al.* [7] presented the approximate controllability results for fractional integro-differential equations studied in [15].

In this paper, we concern the following fractional semilinear integro-differential evolution equation with nonlocal initial conditions:

$$\begin{cases} {}^C D^q x(t) = -Ax(t) + a(t) f(t, x(t), (Hx)(t)) + Bu(t), & t \in I = [0, b], \\ x(0) = g(x) + x_0 \in \mathbb{X}_\alpha, \end{cases} \tag{1.1}$$

where ${}^C D^q$ denotes the Caputo derivative, $0 < q < 1$, the state variable x takes values in a Hilbert space $\mathbb{X}$ with the norm $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$, $-A : D(A) \to \mathbb{X}$ is the infinitesimal generator of a C_0-semigroup of uniformly bounded linear operators, that is, there exists $M > 1$ such that $\|T(t)\| \le M$ for all $t \ge 0$, $a \in L^{p_1}([0, b], \mathbb{R}^+)$, $p_1 > 1$. We denote by $\mathbb{X}_\alpha$ a Hilbert space of $D(A^\alpha)$ equipped with norm $\|x\|_\alpha = \|A^\alpha x\| = \sqrt{\langle A^\alpha x, A^\alpha x \rangle}$ for all $x \in D(A^\alpha)$, which is equivalent to the graph norm of A^α, $0 < \alpha < 1$. The control function u is given in $L^2([0, b], U)$, U is a Hilbert space, B is a bounded linear operator from U into $\mathbb{X}_\alpha$. The Volterra integral operator H is defined by $(Hx)(t) = \int_0^t h(t, s, x(s))\,ds$. The nonlinear term f and the nonlocal term g will be specified later.

Here, it should be emphasized that no one has investigated the approximate controllability and further the existence of optimal controls for the fractional evolution system (1.1) in a Hilbert space, and this is the main motivation of this paper. The main objective of this paper is to derive sufficient conditions for approximate controllability and existence of optimal controls for the abstract fractional equation (1.1). The considered system (1.1) is of a more general form, with a coefficient function in front of the nonlinear term. Finally, an example is also given to illustrate the applications of the theory. The previously reported results in [6, 15, 20] are only the special cases of our research.

The rest of this paper is organized as follows. In Section 2, we present some necessary preliminaries and lemmas. In Section 3, we prove the approximate controllability for the system (1.1). In Section 4, we study the existence of optimal controls for the Bolza problem. At last, an example is given to demonstrate the effectiveness of the main results in Section 5.

2 Preliminaries and lemmas

Unless otherwise specified, $\|\cdot\|_{L^p[0,b]}$ represents the $L^p(I,\mathbb{R}^+)$ norm, $1\leq p\leq\infty$, $C(I,\mathbb{X}_\alpha)$ is a Banach space equipped with supnorm given by $\|x\|_\infty=\sup_{t\in I}\|x\|_\alpha$ for $x\in C(I,\mathbb{X}_\alpha)$.

Let $0\in\rho(A)$, here $\rho(A)$ is the resolvent set of A. Define

$$A^{-\alpha}=\frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}T(t)\,dt. \tag{2.1}$$

It follows that each $A^{-\alpha}$ is an injective continuous endomorphism of $\mathbb{X}$. So we can define $A^\alpha=(A^{-\alpha})^{-1}$, which is a closed bijective linear operator in $\mathbb{X}$. It can be shown that A^α has a dense domain and $D(A^\beta)\subset D(A^\alpha)$ for $0\leq\alpha\leq\beta$. Moreover, $A^{\alpha+\beta}x=A^\alpha A^\beta x=A^\beta A^\alpha x$, $x\in D(A^\mu)$ with $\mu:=\max\{\alpha,\beta,\alpha+\beta\}$, where $A^0=I$, I is the identity in $\mathbb{X}$. We have $\mathbb{X}_\beta\hookrightarrow\mathbb{X}_\alpha$ for $0\leq\alpha\leq\beta$ (with $\mathbb{X}_0=\mathbb{X}$), and the embedding is continuous. Moreover, A^α has the following basic properties.

Lemma 2.1 (see [21]) *A^α and $T(t)$ have the following properties*:

(1) $T(t):\mathbb{X}\to\mathbb{X}_\alpha$, *for each $t>0$ and $\alpha\geq0$.*

(2) $A^\alpha T(t)x=T(t)A^\alpha x$, *for each $x\in D(A^\alpha)$ and $t\geq0$.*

(3) *For every $t>0$, $A^\alpha T(t)$ is bounded in $\mathbb{X}$, and there exists $M_\alpha>0$ such that*

$$\|A^\alpha T(t)\|\leq M_\alpha t^{-\alpha}. \tag{2.2}$$

(4) *$A^{-\alpha}$ is a bounded linear operator for $0\leq\alpha\leq1$, and there exists $C_\alpha>0$ such that $\|A^{-\alpha}\|\leq C_\alpha$.*

Definition 2.1 The fractional integral of order q with the lower limit zero for a function f is defined as

$$I^q f(t)=\frac{1}{\Gamma(q)}\int_0^t\frac{f(s)}{(t-s)^{1-q}}\,ds,\quad t>0, q>0, \tag{2.3}$$

provided that the right side is point-wise defined on $[0,+\infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 The Riemann-Liouville derivative of the order q with the lower limit zero for a function $f : [0,\infty] \to \mathbb{R}$ can be written as

$$^{L}D^{q}f(t) = \frac{1}{\Gamma(n-q)}\frac{d^n}{dt^n}\int_0^t \frac{f(s)}{(t-s)^{1-n+q}}\,ds, \quad t>0, n-1<q<n. \tag{2.4}$$

Definition 2.3 The Caputo derivative of the order q for a function $f : [0,\infty] \to \mathbb{R}$ can be written as

$$^{C}D^{q}f(t) = {}^{L}D^{q}\left(f(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0)\right), \quad t>0, n-1<q<n. \tag{2.5}$$

Remark 2.1

(1) If $f(t) \in C^n[0,\infty)$, then

$$^{C}D^{q}f(t) = \frac{1}{\Gamma(n-q)}\int_0^t \frac{f^{(n)}(s)}{(t-s)^{1-n+q}}\,ds = I^{n-q}f^{(n)}(t), \quad t>0, n-1<q<n. \tag{2.6}$$

(2) The Caputo derivative of a constant equals zero.

(3) If f is an abstract function with values in $\mathbb{X}$, then the integrals which appear in Definitions 2.1, 2.2, and 2.3 are taken in Bochner's sense.

Definition 2.4 A solution $x \in C(I, \mathbb{X}_\alpha)$ is said to be a mild solution of the system (1.1), we mean that for any $u(\cdot) \in L^2(I,U)$, the following integral equation holds:

$$\begin{aligned} x(t) &= \mathcal{T}(t)\bigl(x_0 + g(x)\bigr) + \int_0^t (t-s)^{q-1}\mathcal{S}(t-s)a(s)f\bigl(s, x(s), (Hx)(s)\bigr)\,ds \\ &\quad + \int_0^t (t-s)^{q-1}\mathcal{S}(t-s)Bu(s)\,ds, \quad t \in I, \end{aligned} \tag{2.7}$$

where

$$\mathcal{T}(t) = \int_0^\infty \xi_q(\theta)T\bigl(t^q\theta\bigr)\,d\theta, \qquad \mathcal{S}(t) = q\int_0^\infty \theta\xi_q(\theta)T\bigl(t^q\theta\bigr)\,d\theta, \tag{2.8}$$

$$\xi_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}}\varpi_q\bigl(\theta^{-\frac{1}{q}}\bigr) \geq 0, \tag{2.9}$$

$$\varpi_q(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \quad \theta \in (0,\infty), \tag{2.10}$$

ξ_q is a probability density function defined on $(0,\infty)$, that is,

$$\xi_q(\theta) \geq 0, \quad \theta \in (0,\infty) \quad \text{and} \quad \int_0^\infty \xi_q(\theta)\,d\theta = 1. \tag{2.11}$$

Definition 2.5 The system (1.1) is said to be approximately controllable on $[0,b]$ if $\overline{\mathfrak{R}(b,x_0)} = \mathbb{X}_\alpha$, that is, given an arbitrary $\varepsilon > 0$, it is possible to steer from the point x_0 to within a distance $\varepsilon > 0$ for all points in the state space $\mathbb{X}_\alpha$ at time b. Here $\mathfrak{R}(b,x_0) := \{x(b;x_0,u) : u \in L^2([0,b],U_{ad})\}$, $\mathfrak{R}(b,x_0)$ is called the reachable set of the system (1.1) at terminal time b, $x(b;x_0,u)$ is the state value at terminal time b corresponding to the control u and the initial value x_0, $\overline{\mathfrak{R}(b,x_0)}$ represents its closure in $\mathbb{X}_\alpha$.

Denote

$$\Gamma_0^b = \int_0^b (b-s)^{q-1}\mathcal{S}(b-s)BB^*\mathcal{S}^*(b-s)\,ds : \mathbb{X}_\alpha \to \mathbb{X}_\alpha, \tag{2.12}$$

$$R\bigl(\varepsilon, \Gamma_0^b\bigr) = \bigl(\varepsilon I + \Gamma_0^b\bigr)^{-1} : \mathbb{X}_\alpha \to \mathbb{X}_\alpha, \quad \varepsilon > 0, \tag{2.13}$$

where B^* denotes the adjoint of B and $V^*(t)$ is the adjoint of $V(t)$. Obviously, Γ_0^b is a linear bounded operator. We define the following linear fractional control system:

$$\begin{cases} {}^C D^q x(t) = Ax(t) + Bu(t), & t \in I = [0,b], \\ x(0) = x_0 \in \mathbb{X}_\alpha. \end{cases} \tag{2.14}$$

Lemma 2.2 (see [6]) *The linear fractional control system* (2.14) *is approximately controllable on* $[0,b]$ *if and only if* $\varepsilon R(\varepsilon, \Gamma_0^b) \to 0$ *as* $\varepsilon \to 0^+$ *in the strong operator topology.*

Lemma 2.3 (see [20]) *The operators* $\mathcal{T}$ *and* $\mathcal{S}$ *have the following properties*:

(1) *For fixed* $t \geq 0$, $\mathcal{T}(t)$ *and* $\mathcal{S}(t)$ *are linear and bounded operators, that is, for any* $x \in \mathbb{X}$,

$$\bigl\|\mathcal{T}(t)x\bigr\| \leq M\|x\|, \qquad \bigl\|\mathcal{S}(t)x\bigr\| \leq \frac{M}{\Gamma(q)}\|x\|. \tag{2.15}$$

(2) $(\mathcal{T}(t))_{t\geq 0}$ *and* $(\mathcal{S}(t))_{t\geq 0}$ *are strongly continuous.*

(3) *For every* $t > 0$, $\mathcal{T}(t)$ *and* $\mathcal{S}(t)$ *are also compact if* $T(t)$ *is compact.*

(4) *For any* $x \in \mathbb{X}$, $\alpha \in (0,1)$ *and* $\beta \in (0,1)$, *we have*

$$\begin{aligned} &A\mathcal{S}(t)x = A^{1-\beta}\mathcal{S}(t)A^\beta x, \quad t \in I, \\ &\bigl\|A^\alpha \mathcal{S}(t)\bigr\| \leq \frac{M_\alpha q \Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))} t^{-\alpha q}, \quad 0 < t \leq b. \end{aligned} \tag{2.16}$$

(5) *For fixed* $t \geq 0$ *and any* $x \in \mathbb{X}_\alpha$, *we have*

$$\bigl\|\mathcal{T}(t)x\bigr\|_\alpha \leq M\|x\|_\alpha, \qquad \bigl\|\mathcal{S}(t)x\bigr\|_\alpha \leq \frac{M}{\Gamma(q)}\|x\|_\alpha. \tag{2.17}$$

(6) $\mathcal{T}_\alpha(t)$ *and* $\mathcal{S}_\alpha(t)$ *are uniformly continuous, that is, for each fixed* $t > 0$ *and* $\epsilon > 0$, *there exists* $h > 0$ *such that*

$$\begin{aligned} &\bigl\|\mathcal{T}_\alpha(t+\epsilon) - \mathcal{T}_\alpha(t)\bigr\|_\alpha < \varepsilon, \quad \textit{for } t+\epsilon \geq 0 \textit{ and } |\epsilon| < h, \\ &\bigl\|\mathcal{S}_\alpha(t+\epsilon) - \mathcal{S}_\alpha(t)\bigr\|_\alpha < \varepsilon, \quad \textit{for } t+\epsilon \geq 0 \textit{ and } |\epsilon| < h, \end{aligned} \tag{2.18}$$

where

$$\mathcal{T}_\alpha(t) = \int_0^\infty \xi_q(\theta) T_\alpha\bigl(t^q\theta\bigr)\,d\theta, \qquad \mathcal{S}_\alpha(t) = q\int_0^\infty \theta\xi_q(\theta) T_\alpha\bigl(t^q\theta\bigr)\,d\theta. \tag{2.19}$$

Lemma 2.4 (see [22]) *For* $\sigma \in (0,1]$ *and* $0 < c_1 \leq c_2$ *we have* $|c_1^\sigma - c_2^\sigma| \leq (c_2 - c_1)^\sigma$.

Lemma 2.5 (Schauder's fixed point theorem) *If* B *is a closed bounded and convex subset of a Banach space* $\mathbb{X}$ *and* $Q : B \to B$ *is completely continuous, then* Q *has a fixed point in* B.

3 Approximate controllability

In this section, we impose the following assumptions:

(H_1) $f : I \times \mathbb{X}_\alpha \times \mathbb{X}_\alpha \to \mathbb{X}$ is continuous and there exist $m_1, m_2 > 0$ such that

$$\big\|f(t,x_1,x_2) - f(t,y_1,y_2)\big\| \le m_1\|x_1 - y_1\|_\alpha + m_2\|x_2 - y_2\|_\alpha, \tag{3.1}$$

for all $x_i, y_i \in \mathbb{X}_\alpha$, $i = 1,2$, and $t \in I$.

(H_2) $h : \Delta \times \mathbb{X}_\alpha \to \mathbb{X}_\alpha$, there exists a function $m(t,s) \in C(\Delta, \mathbb{R}^+)$ and

$$k^* = \sup_{t\in I} \int_0^b m(t,s)\,ds < \infty$$

such that

$$\big\|h(t,s,x) - h(t,s,y)\big\|_\alpha \le m(t,s)\|x - y\|_\alpha, \tag{3.2}$$

for each $(t,s) \in \Delta$ and $x, y \in \mathbb{X}_\alpha$, where $\Delta = \{(t,s) \in \mathbb{R}^2; 0 \le s, t \le b\}$.

(H_3) $g : C(I, X_\alpha) \to \mathbb{X}_\alpha$ is continuous and there exists a constant $l_g > 0$ such that

$$\big\|g(x) - g(y)\big\|_\alpha \le l_g\|x - y\|_\infty, \tag{3.3}$$

for any $x, y \in C(I, X_\alpha)$.

(H_4) The function $\Omega_\varepsilon : I \to \mathbb{R}^+$ defined by

$$\begin{aligned}\Omega_\varepsilon(t) = {}& Ml_g + \frac{M_\alpha q\Gamma(2-\alpha)}{\varepsilon\Gamma(1+q(1-\alpha))}\|B\|_\alpha \sup_{0\le t\le b}\big\|B^*S^*(b-t)\big\|Mbl_g \\ &+ \frac{M_\alpha q\Gamma(2-\alpha)(m_1+m_2k^*)\|a\|_{L^{p_1}[0,b]}}{\Gamma(1+q(1-\alpha))}\left(\frac{p_1-1}{p_1+p_1(q-1)-1}\right)^{\frac{p_1-1}{p_1}} t^{q-\frac{1}{p_1}-\alpha q} \\ &\times \left(1 + \frac{t}{q-\frac{1}{p_1}-\alpha q+1}\right)\end{aligned} \tag{3.4}$$

satisfies $0 < \Omega_\varepsilon(t) < 1$ for all $t \in I$, where $\max\{\frac{1}{p_1}, \frac{-1+\sqrt{5-4\alpha}}{2(1-\alpha)}\} < q < 1$.

Theorem 3.1 *Assume that conditions* (H_1)-(H_4) *are satisfied. In addition, the functions f and g are bounded and the linear system* (2.14) *is approximately controllable on* $[0,b]$. *Then the fractional system* (1.1) *is approximately controllable on* $[0,b]$.

Proof For arbitrary $x_1 \in \mathbb{X}_\alpha$, define a control function as follows:

$$\begin{aligned}u_{\varepsilon,x}(t) = {}& B^*\mathcal{S}^*(b-t)R\big(\varepsilon,\Gamma_0^b\big)\bigg(x_1 - \mathcal{T}(b)\big(x_0 + g(x)\big) \\ &- \int_0^b (b-s)^{q-1}\mathcal{S}(b-s)a(s)f\big(s,x(s),(Hx)(s)\big)\,ds\bigg)\end{aligned} \tag{3.5}$$

and define the operator Q_ε by

$$(Q_\varepsilon x)(t) = \mathcal{T}(t)\big(x_0 + g(x)\big) + \int_0^t (t-s)^{q-1}\mathcal{S}(t-s)a(s)f\big(s, x(s), (Hx)(s)\big)\,ds \\ + \int_0^t (t-s)^{q-1}\mathcal{S}(t-s)Bu_{\varepsilon,x}(s)\,ds. \tag{3.6}$$

Obviously Q_ε is well defined on $C(I, \mathbb{X}_\alpha)$.

For $x, y \in C(I, \mathbb{X}_\alpha)$, we have

$$\begin{aligned} &\big\|(Q_\varepsilon x)(t) - (Q_\varepsilon y)(t)\big\|_\alpha \\ &\quad\le \big\|\mathcal{T}(t)\big[g(x) - g(y)\big]\big\|_\alpha \\ &\qquad + \int_0^t (t-s)^{q-1}a(s)\big\|\mathcal{S}(t-s)\big[f\big(s, x(s), (Hx)(s)\big) - f\big(s, y(s), (Hy)(s)\big)\big]\big\|_\alpha\,ds \\ &\qquad + \int_0^t (t-s)^{q-1}\big\|\mathcal{S}(t-s)\big[Bu_{\varepsilon,x}(s) - Bu_{\varepsilon,y}(s)\big]\big\|_\alpha\,ds \\ &\quad\le M\big\|g(x) - g(y)\big\|_\alpha \\ &\qquad + \int_0^t (t-s)^{q-1}a(s)\big\|A^\alpha\mathcal{S}(t-s)\big[f\big(s, x(s), (Hx)(s)\big) - f\big(s, y(s), (Hy)(s)\big)\big]\big\|\,ds \\ &\qquad + \int_0^t (t-s)^{q-1}\big\|A^\alpha\mathcal{S}(t-s)\big[Bu_{\varepsilon,x}(s) - Bu_{\varepsilon,y}(s)\big]\big\|\,ds \\ &\quad\le I^1 + I^2 + I^3. \end{aligned} \tag{3.7}$$

By (H_1)-(H_3), Lemma 2.3 and the Hölder inequality, we have $I^1 \le Ml_g\|x-y\|_\infty$ and

$$\begin{aligned} I^2 &\le t^{-\alpha q}\frac{M_\alpha q\Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))}m_1\big\|x(s) - y(s)\big\|_\alpha \int_0^t (t-s)^{q-1}a(s)\,ds \\ &\quad + t^{-\alpha q}\frac{M_\alpha q\Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))}m_2\big\|(Hx)(s) - (Hy)(s)\big\|_\alpha \int_0^t (t-s)^{q-1}a(s)\,ds \\ &\le t^{-\alpha q}\frac{M_\alpha q\Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))}\left[m_1\|x-y\|_\infty + m_2\int_0^b m(s,\tau)\big\|x(\tau) - y(\tau)\big\|_\alpha\,d\tau\right] \\ &\quad \times \left(\int_0^t (t-s)^{(q-1)\frac{p_1}{p_1-1}}\right)^{\frac{p_1-1}{p_1}}\left(\int_0^t \big(a(s)\big)^{p_1}\,ds\right)^{\frac{1}{p_1}} \\ &\le \left[\frac{M_\alpha q\Gamma(2-\alpha)(m_1 + m_2k^*)\|a\|_{L^{p_1}[0,b]}}{\Gamma(1+q(1-\alpha))}\right. \\ &\quad \left.\times\left(\frac{p_1-1}{p_1 + p_1(q-1) - 1}\right)^{\frac{p_1-1}{p_1}} t^{q-\frac{1}{p_1}-\alpha q}\right]\|x-y\|_\infty, \end{aligned} \tag{3.8}$$

$$\begin{aligned} I^3 &\le \frac{M_\alpha q\Gamma(2-\alpha)}{\varepsilon\Gamma(1+q(1-\alpha))}\|B\|_\alpha \sup_{0\le t\le b}\big\|B^*S^*(b-t)\big\|Mb\big\|g(x) - g(y)\big\|_\infty \\ &\quad + \left[\frac{M_\alpha q\Gamma(2-\alpha)(m_1 + m_2k^*)\|a\|_{L^{p_1}[0,b]}}{\Gamma(1+q(1-\alpha))(q - \frac{1}{p1} - \alpha q + 1)}\right. \\ &\quad \left.\times\left(\frac{p_1-1}{p_1 + p_1(q-1) - 1}\right)^{\frac{p_1-1}{p_1}} t^{q-\frac{1}{p_1}-\alpha q+1}\right]\|x-y\|_\infty. \end{aligned} \tag{3.9}$$

Then we can deduce that

$$\begin{aligned}
&\|Q_\varepsilon x - Q_\varepsilon y\|_\infty \\
&\quad\le \Bigg[Ml_g + \frac{M_\alpha q\Gamma(2-\alpha)}{\varepsilon\Gamma(1+q(1-\alpha))}\|B\|_\alpha \sup_{0\le t\le b}\big\|B^*S^*(b-t)\big\|Mbl_g \\
&\qquad + \frac{M_\alpha q\Gamma(2-\alpha)(m_1+m_2k^*)\|a\|_{L^{p_1}[0,b]}}{\Gamma(1+q(1-\alpha))} \\
&\qquad \times\left(\frac{p_1-1}{p_1+p_1(q-1)-1}\right)^{\frac{p_1-1}{p_1}} t^{q-\frac{1}{p_1}-\alpha q}\left(1+\frac{t}{q-\frac{1}{p_1}-\alpha q+1}\right)\Bigg]\|x-y\|_\infty \\
&\quad\le \Omega_\varepsilon(t)\|x-y\|_\infty.
\end{aligned} \tag{3.10}$$

From (H_4) and the contraction mapping principle, we conclude that the operator Q_ε has a fixed point in $C(I,\mathbb{X}_\alpha)$. Since f and g are bounded, for definiteness and without loss of generality, let x_ε be a fixed point of Q_ε in $B_{r(\varepsilon)}$, where $B_{r(\varepsilon)} = \{x \in C([0,b],\mathbb{X}_\alpha) \mid \|x\|_\alpha \le r(\varepsilon)\}$. From the boundedness of x_ε, there is a subsequence denoted by $\{x_\varepsilon\}$ which converges weakly to x as $\varepsilon \to 0^+$, and $\|x_\varepsilon\|_\alpha \to \|x\|_\alpha$ as $\varepsilon \to 0^+$. Then $\lim_{\varepsilon\to 0^+}\|x_\varepsilon - x\|_\alpha = 0$. Any fixed point x_ε is a mild solution of (1.1) under the control

$$\begin{aligned}
u_{\varepsilon,x}(t) &= B^*\mathcal{S}^*(b-t)R\big(\varepsilon,\Gamma_0^b\big)\bigg(x_1 - \mathcal{T}(b)\big(x_0+g(x_\varepsilon)\big) \\
&\quad - \int_0^b (b-s)^{q-1}\mathcal{S}(b-s)a(s)f\big(s,x_\varepsilon(s),(Hx_\varepsilon)(s)\big)\,ds\bigg).
\end{aligned} \tag{3.11}$$

Then

$$\begin{aligned}
x_\varepsilon(t) &= \mathcal{T}(t)\big(x_0+g(x_\varepsilon)\big) + \int_0^t (t-s)^{q-1}\mathcal{S}(t-s)a(s)f\big(s,x_\varepsilon(s),(Hx_\varepsilon)(s)\big)\,ds \\
&\quad + \int_0^t (t-s)^{q-1}\mathcal{S}(t-s)BB^*\mathcal{S}^*(b-t)R\big(\varepsilon,\Gamma_0^b\big)p(x_\varepsilon)\,ds,
\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}
p(x_\varepsilon) &= x_1 - \mathcal{T}(b)\big(x_0+g(x_\varepsilon)\big) \\
&\quad - \int_0^b (b-s)^{q-1}\mathcal{S}(b-s)a(s)f\big(s,x_\varepsilon(s),(Hx_\varepsilon)(s)\big)\,ds.
\end{aligned} \tag{3.13}$$

Therefore we have

$$x_\varepsilon(b) = x_1 - \varepsilon R\big(\varepsilon,\Gamma_0^b\big)p(x_\varepsilon). \tag{3.14}$$

Define

$$w = x_1 - \mathcal{T}(b)\big(x_0+g(x)\big) - \int_0^b (b-s)^{q-1}\mathcal{S}(b-s)a(s)f\big(s,x(s),(Hx)(s)\big)\,ds, \tag{3.15}$$

it follows that

$$\begin{aligned}\|p(x_\varepsilon)-w\|_\alpha \le M\|g(x_\varepsilon)-g(x)\|_\alpha + \bigg\| \int_0^b (b-s)^{q-1}\mathcal{S}(b-s)a(s)\big(f\big(s,x_\varepsilon(s),(Hx_\varepsilon)(s)\big) \\ -f\big(s,x(s),(Hx)(s)\big)\big)\,ds\bigg\|_\alpha. \end{aligned} \tag{3.16}$$

By assumptions (H$_1$)-(H$_3$), it is easy to get $\|p(x_\varepsilon)-w\|_\alpha \to 0$ as $\varepsilon \to 0^+$. Then

$$\begin{aligned}\|x_\varepsilon(b)-x_1\|_\alpha &\le \|\varepsilon R(\varepsilon,\Gamma_0^b)(w)\|_\alpha + \|\varepsilon R(\varepsilon,\Gamma_0^b)\|\,\|p(x_\varepsilon)-w\|_\alpha \\ &\le \|\varepsilon R(\varepsilon,\Gamma_0^b)(w)\|_\alpha + \|p(x_\varepsilon)-w\|_\alpha \to 0. \end{aligned} \tag{3.17}$$

This proves the approximate controllability of (1.1). □

In order to obtain approximate controllability results by the Schauder fixed point theorem, we pose the following conditions:

(H$_5$) $(T(t))_{t>0}$ is a compact analytic semigroup in $\mathbb{X}$.

(H$_6$) There exist constants $\alpha \le \beta \le 1$ such that $f:[0,b]\times\mathbb{X}_\alpha\times\mathbb{X}_\alpha \to \mathbb{X}_\beta$ and f satisfies:

  (1) For each $(x,y)\in\mathbb{X}_\alpha\times\mathbb{X}_\alpha$, the function $f(\cdot,x,y)$ is measurable.

  (2) For each $t\in[0,b]$, the function $f(t,\cdot,\cdot):\mathbb{X}_\alpha\times\mathbb{X}_\alpha\to\mathbb{X}_\beta$ is continuous.

  (3) For any $r>0$, there exist functions $\varphi_r \in L^\infty([0,b],\mathbb{R}^+)$ such that

$$\sup\big\{\|f(t,x,y)\|_\beta : \|x\|_\alpha \le r, \|y\|_\alpha \le k^*br\big\} \le \varphi_r(t), \quad t\in[0,b] \tag{3.18}$$

  and there exists a constant $\gamma_1>0$ such that

$$\liminf_{r\to+\infty}\frac{1}{r}\int_0^t \frac{a(s)\varphi_r(s)}{(t-s)^{1-q}}\,ds \le \gamma_1 < +\infty, \tag{3.19}$$

  where k^* has been specified in assumption (H$_2$).

(H$_7$) $g: C(I,\mathbb{X}_\alpha)\to\mathbb{X}_\alpha$ is completely continuous. For any $r>0$, there exist constants ψ_r such that

$$\big\{\|g(x)\|_\alpha : \|x\|_\infty \le r\big\} \le \psi_r \tag{3.20}$$

and there exists a constant $\gamma_2>0$ such that

$$\lim_{r\to+\infty}\frac{\psi_r}{r} \le \gamma_2 < +\infty. \tag{3.21}$$

(H$_8$) The following inequality holds:

$$\gamma_1\frac{MC_{\beta-\alpha}}{\Gamma(q)}\left(1+\frac{1}{\varepsilon}\frac{b^q}{q}\frac{M}{\Gamma(q)}C_B\right) + \gamma_2 M\left(1+\frac{1}{\varepsilon}\frac{b^q}{q}\frac{M}{\Gamma(q)}C_B\right) < 1, \tag{3.22}$$

where C_B will be specified in the following theorem.

Theorem 3.2 *Assume that conditions* (H_3), (H_5)-(H_8) *are satisfied. In addition, the linear system* (2.14) *is approximately controllable on* $[0,b]$. *Then the fractional system* (1.1) *is approximately controllable on* $[0,b]$.

Proof For $r(\varepsilon) > 0$, we set $B_{r(\varepsilon)} = \{x \in C([0,b],\mathbb{X}_\alpha) \mid \|x\|_\alpha \leq r(\varepsilon)\}$. For arbitrary $x_1 \in \mathbb{X}_\alpha$, define the control function as follows:

$$\begin{aligned} u_{\varepsilon,x}(t) = B^*\mathcal{S}^*(b-t)R\big(\varepsilon,\Gamma_0^b\big)\bigg(& x_1 - \mathcal{T}(b)\big(x_0+g(x)\big) \\ & - \int_0^b (b-s)^{q-1}\mathcal{S}(b-s)a(s)f\big(s,x(s),(Hx)(s)\big)\,ds\bigg) \end{aligned} \tag{3.23}$$

and define the operator Q_ε by

$$\begin{aligned} (Q_\varepsilon x)(t) = {} & \mathcal{T}(t)\big(x_0+g(x)\big) + \int_0^t (t-s)^{q-1}\mathcal{S}(t-s)a(s)f\big(s,x(s),(Hx)(s)\big)\,ds \\ & + \int_0^t (t-s)^{q-1}\mathcal{S}(t-s)Bu_{\varepsilon,x}(s)\,ds. \end{aligned} \tag{3.24}$$

We divide the proof into five steps.

Step 1: Q_ε maps bounded sets into bounded sets, that is, for arbitrary $\varepsilon > 0$, there is a positive constant $r(\varepsilon)$ such that $Q_\varepsilon(B_{r(\varepsilon)}) \subset B_{r(\varepsilon)}$.

Let $x \in B_{r(\varepsilon)}$, from (2.12), (2.13), and (3.23), we have

$$\begin{aligned} \big\|Bu_{\varepsilon,x}(t)\big\|_\alpha \leq {} & \frac{1}{\varepsilon}\big\|A^\alpha BB^*\mathcal{S}^*(b-t)\big\|\bigg[\big\|A^\alpha x_1\big\| + \big\|\mathcal{T}(b)A^\alpha x_0\big\| + \big\|\mathcal{T}(b)A^\alpha g(x)\big\| \\ & + \bigg\|\int_0^b (b-s)^{q-1}A^{\alpha-\beta}\mathcal{S}(b-s)A^\beta a(s)f\big(s,x(s),(Hx)(s)\big)\,ds\bigg\|\bigg] \\ \leq {} & \frac{1}{\varepsilon}\big\|A^\alpha B\big\|\sup_{0\leq t\leq b}\big\|B^*\mathcal{S}^*(b-t)\big\| \\ & \times\bigg[\|x_1\|_\alpha + M\|x_0\|_\alpha + M\psi_r + \frac{MC_{\beta-\alpha}}{\Gamma(q)}\int_0^b (b-s)^{q-1}a(s)\varphi_r(s)\,ds\bigg] \\ \leq {} & \frac{1}{\varepsilon}C_u, \end{aligned} \tag{3.25}$$

where

$$C_u = C_B\bigg[\|x_1\|_\alpha + M\|x_0\|_\alpha + M\psi_r + \frac{MC_{\beta-\alpha}}{\Gamma(q)}\int_0^b (b-s)^{q-1}a(s)\varphi_r(s)\,ds\bigg], \tag{3.26}$$

$$C_B = \|B\|_\alpha \sup_{0\leq t\leq b}\big\|B^*\mathcal{S}^*(b-t)\big\|. \tag{3.27}$$

Then we get

$$\int_0^t (t-s)^{q-1}\big\|A^\alpha Bu_{\varepsilon,x}(s)\big\|\,ds \leq \frac{1}{\varepsilon}\frac{t^q}{q}C_u. \tag{3.28}$$

If operator Q_ε is not bounded, for each $r > 0$, there would exist $x \in B_{r(\varepsilon)}$ and $t_r \in [0,b]$ such that

$$\begin{aligned}
r &< \|(Q_\varepsilon x)(t)\|_\alpha \\
&\le M\|x_0\|_\alpha + M\|g(x)\|_\alpha + \left\|\int_0^{t_r}(t_r-s)^{q-1}\mathcal{S}(t_r-s)a(s)f(s,x(s),(Hx)(s))\,ds\right\|_\alpha \\
&\quad + \left\|\int_0^{t_r}(t_r-s)^{q-1}\mathcal{S}(t_r-s)Bu_{\varepsilon,x}(s)\,ds\right\|_\alpha \\
&\le M\|x_0\|_\alpha + M\|g(x)\|_\alpha + \int_0^{t_r}(t_r-s)^{q-1}\|A^{\alpha-\beta}\mathcal{S}(t_r-s)a(s)A^\beta f(s,x(s),(Hx)(s))\|\,ds \\
&\quad + \int_0^{t_r}(t_r-s)^{q-1}\|\mathcal{S}(t_r-s)A^\alpha Bu_{\varepsilon,x}(s)\|\,ds \\
&\le M\|x_0\|_\alpha + M\psi_r + \frac{MC_{\beta-\alpha}}{\Gamma(q)}\int_0^{t_r}(t_r-s)^{q-1}a(s)\varphi_r(s)\,ds \\
&\quad + \frac{1}{\varepsilon}\frac{b^q}{q}\frac{M}{\Gamma(q)}\|B\|_\alpha \sup_{0\le t\le b}\|B^*\mathcal{S}^*(b-t)\| \\
&\quad \times\left(\|x_1\|_\alpha + M\|x_0\|_\alpha + M\psi_r + \frac{MC_{\beta-\alpha}}{\Gamma(q)}\int_0^b(b-s)^{q-1}a(s)\varphi_r(s)\,ds\right).
\end{aligned} \tag{3.29}$$

Dividing both sides by r and taking the lower as $r\to\infty$, we have

$$\gamma_1\frac{MC_{\beta-\alpha}}{\Gamma(q)}\left(1+\frac{1}{\varepsilon}\frac{b^q}{q}\frac{M}{\Gamma(q)}C_B\right) + \gamma_2 M\left(1+\frac{1}{\varepsilon}\frac{b^q}{q}\frac{M}{\Gamma(q)}C_B\right) > 1, \tag{3.30}$$

which is a contradiction to (H_8). Then Q_ε maps bounded sets into bounded sets.

Step 2. Q_ε is continuous.

Let $\{x_n\}\subset B_{r(\varepsilon)}$ and $x_n\to x\in B_{r(\varepsilon)}$ as $n\to\infty$. From assumptions (H_6)-(H_7), for each $s\in[0,b]$, we have

$$a(s)\|f(s,x_n,(Hx_n)(s)) - f(s,x,(Hx)(s))\|_\beta \le 2a(s)\varphi_r(s), \tag{3.31}$$

$$\|Bu_{\varepsilon,x_n}(s) - Bu_{\varepsilon,x}(s)\|_\alpha \le \frac{2}{\varepsilon}C_u. \tag{3.32}$$

By the Lebesgue dominated convergence theorem, for each $s\in[0,b]$, we get

$$\begin{aligned}
&\|(Q_\varepsilon x_n)(t) - (Q_\varepsilon x)(t)\|_\alpha \\
&\quad \le Ml_g\|x_n-x\|_\infty + \int_0^t(t-s)^{q-1}\|A^{\alpha-\beta}\mathcal{S}(t-s)A^\beta a(s) \\
&\qquad \times[f(s,x_n,(Hx_n)(s)) - f(s,x,(Hx)(s))]\|\,ds \\
&\qquad + \int_0^t(t-s)^{q-1}\|\mathcal{S}(t-s)A^\alpha B[u_{\varepsilon,x_n}(s)-u_{\varepsilon,x}(s)]\|\,ds \\
&\quad \le Ml_g\|x_n-x\|_\infty + \frac{MC_{\beta-\alpha}}{\Gamma(q)}\int_0^t(t-s)^{q-1}a(s) \\
&\qquad \times\|f(s,x_n(s),(Hx_n)(s)) - f(s,x(s),(Hx)(s))\|_\beta\,ds \\
&\qquad + \frac{M}{\Gamma(q)}\int_0^t(t-s)^{q-1}\|B[u_{\varepsilon,x_n}(s)-u_{\varepsilon,x}(s)]\|_\alpha\,ds \to 0,
\end{aligned} \tag{3.33}$$

which implies that $Q_\varepsilon : B_{r(\varepsilon)}\to B_{r(\varepsilon)}$ is continuous.

Step 3. For each $\varepsilon > 0$, the set $V(t) = \{(Q_\varepsilon x)(t) : x \in B_{r(\varepsilon)}\}$ is relatively compact in $\mathbb{X}_\alpha$.

The case $t = 0$ is trivial, $V(0) = \{(Q_\varepsilon x)(0) : x(\cdot) \in B_{r(\varepsilon)}\} = \{x_0 + g(x)\}$ is compact in $\mathbb{X}_\alpha$ (see (H_7)). So let $t \in (0, b]$ be a fixed real number, and let h be given a real number satisfied $0 < h < t$. For any $\delta > 0$, define $V_h(t) = \{(Q_\varepsilon^{h,\delta} x)(t) : x \in B_{r(\varepsilon)}\}$,

$$\begin{aligned}
\big(Q_\varepsilon^{h,\delta} x\big)(t) &= \int_\delta^\infty \xi_q(\theta) T\big(t^q \theta\big)\, d\theta \big(x_0 + g(x)\big) \\
&\quad + q \int_0^{t-h} \int_\delta^\infty \theta (t-s)^{q-1} \xi_q(\theta) T\big((t-s)^q \theta\big) a(s) f\big(s, x(s), (Hx)(s)\big)\, d\theta\, ds \\
&\quad + q \int_0^{t-h} \int_\delta^\infty \theta (t-s)^{q-1} \xi_q(\theta) T\big((t-s)^q \theta\big) B u_{\varepsilon,x}(s)\, d\theta\, ds \\
&= T\big(h^q \delta\big) \int_\delta^\infty \xi_q(\theta) T\big(t^q \theta - h^q \delta\big)\, d\theta \big(x_0 + g(x)\big) \\
&\quad + T\big(h^q \delta\big) q \int_0^{t-h} \int_\delta^\infty \theta (t-s)^{q-1} \xi_q(\theta) T\big((t-s)^q \theta - h^q \delta\big) a(s) \\
&\quad \times f\big(s, x(s), (Hx)(s)\big)\, d\theta\, ds \\
&\quad + T\big(h^q \delta\big) q \int_0^{t-h} \int_\delta^\infty \theta (t-s)^{q-1} \xi_q(\theta) T\big((t-s)^q \theta - h^q \delta\big) B u_{\varepsilon,x}(s)\, d\theta\, ds \\
&= T\big(h^q \delta\big) y(t,h).
\end{aligned} \tag{3.34}$$

Since $T(h^q \delta)$ is compact in $\mathbb{X}_\alpha$ and $y(t,h)$ is bounded on $B_{r(\varepsilon)}$, then the set $V_h(t)$ is a relatively compact set in $\mathbb{X}_\alpha$. On the other hand,

$$\begin{aligned}
&\big\| (Q_\varepsilon x)(t) - \big(Q_\varepsilon^{h,\delta} x\big)(t) \big\|_\alpha \\
&\quad \le \left\| \int_0^\delta \xi_q(\theta) T\big(t^q \theta\big)\, d\theta \big(x_0 + g(x)\big) \right\|_\alpha \\
&\qquad + \left\| q \int_0^t \int_0^\delta \theta (t-s)^{q-1} \xi_q(\theta) T\big((t-s)^q \theta\big) \big[a(s) f\big(s, x(s), (Hx)(s)\big) + B u_{\varepsilon,x}(s)\big]\, d\theta\, ds \right\|_\alpha \\
&\qquad + \left\| q \int_{t-h}^t \int_\delta^\infty \theta (t-s)^{q-1} \xi_q(\theta) T\big((t-s)^q \theta\big) \right. \\
&\qquad \left. \times \big[a(s) f\big(s, x(s), (Hx)(s)\big) + B u_{\varepsilon,x}(s)\big]\, d\theta\, ds \right\|_\alpha \\
&\quad \le M \|x_0\|_\alpha \int_0^\delta \xi_q(\theta)\, d\theta + \psi_r \int_0^\delta \xi_q(\theta)\, d\theta + qM \left(C_{\beta-\alpha} \|\varphi_r\|_{L^\infty[0,b]} + \frac{1}{\varepsilon} C_u \right) \\
&\qquad \times \int_0^t (t-s)^{q-1} a(s) \int_0^\delta \theta \xi_q(\theta)\, d\theta\, ds \\
&\qquad + qM \left(C_{\beta-\alpha} \|\varphi_r\|_{L^\infty[0,b]} + \frac{1}{\varepsilon} C_u \right) \int_{t-h}^t (t-s)^{q-1} a(s) \int_0^\infty \theta \xi_q(\theta)\, d\theta\, ds.
\end{aligned} \tag{3.35}$$

This implies that there are relatively compact sets arbitrarily close to the set $V(t)$ for each $t \in (0, b]$. Then $V(t)$, $t \in (0, b]$ is relatively compact in $\mathbb{X}_\alpha$. Since it is compact at $t = 0$, we have the relatively compactness of $V(t)$ in $\mathbb{X}_\alpha$ for all $t \in [0, b]$.

Step 4. $V := \{Q_\varepsilon x \in C([0,b], \mathbb{X}_\alpha) \mid x \in B_{r(\varepsilon)}\}$ is an equicontinuous family of functions on $[0, b]$.

For $0 < t_1 < t_2 < b$,

$$\begin{aligned}
&\bigl\|Q_\varepsilon x(t_2) - Q_\varepsilon x(t_1)\bigr\|_\alpha \\
&\quad\le \bigl\|\mathcal{T}(t_2)x_0 - \mathcal{T}(t_1)x_0\bigr\|_\alpha + \bigl\|\mathcal{T}(t_2)g(x) - \mathcal{T}(t_1)g(x)\bigr\|_\alpha \\
&\qquad + \left\|\int_{t_1}^{t_2}(t_2-s)^{q-1}\mathcal{S}(t_2-s)a(s)f\bigl(s,x(s),(Hx)(s)\bigr)\right\|_\alpha \\
&\qquad + \left\|\int_0^{t_1}\bigl[(t_2-s)^{q-1}-(t_1-s)^{q-1}\bigr]\mathcal{S}(t_2-s)a(s)f\bigl(s,x(s),(Hx)(s)\bigr)\right\|_\alpha \\
&\qquad + \left\|\int_0^{t_1}(t_1-s)^{q-1}\bigl[\mathcal{S}(t_2-s)-\mathcal{S}(t_1-s)\bigr]a(s)f\bigl(s,x(s),(Hx)(s)\bigr)\right\|_\alpha \\
&\qquad + \left\|\int_{t_1}^{t_2}(t_2-s)^{q-1}\mathcal{S}(t_2-s)Bu_{\varepsilon,x}(s)\right\|_\alpha \\
&\qquad + \left\|\int_0^{t_1}\bigl[(t_2-s)^{q-1}-(t_1-s)^{q-1}\bigr]\mathcal{S}(t_2-s)Bu_{\varepsilon,x}(s)\right\|_\alpha \\
&\qquad + \left\|\int_0^{t_1}(t_1-s)^{q-1}\bigl[\mathcal{S}(t_2-s)-\mathcal{S}(t_1-s)\bigr]Bu_{\varepsilon,x}(s)\right\|_\alpha \\
&\quad =: I_1+I_2+I_3+I_4+I_5+I_6+I_7.
\end{aligned} \tag{3.36}$$

Form the Hölder inequality, Lemmas 2.1, 2.3, and assumption (H_6), we obtain

$$\begin{aligned}
I_2 &= \left\|\int_{t_1}^{t_2}(t_2-s)^{q-1}\mathcal{S}(t_2-s)a(s)f\bigl(s,x(s),(Hx)(s)\bigr)\,ds\right\|_\alpha \\
&\le \left\|\int_{t_1}^{t_2}(t_2-s)^{q-1}A^{\alpha-\beta}\mathcal{S}(t_2-s)A^\beta a(s)f\bigl(s,x(s),(Hx)(s)\bigr)\,ds\right\| \\
&\le \frac{MC_{\beta-\alpha}\|\varphi_r\|_{L^\infty[0,b]}}{\Gamma(q)}\left\|\int_{t_1}^{t_2}(t_2-s)^{q-1}a(s)\,ds\right\| \\
&\le \frac{MC_{\beta-\alpha}\|\varphi_r\|_{L^\infty[0,b]}}{\Gamma(q)}\left(\int_{t_1}^{t_2}(t_2-s)^{\frac{p_1(q-1)}{p_1-1}}\,ds\right)^{\frac{p_1-1}{p_1}} \times \|a\|_{L^{p_1}[t_1,t_2]} \\
&\le \frac{MC_{\beta-\alpha}\|\varphi_r\|_{L^\infty[0,b]}\|a\|_{L^{1/q_1}[t_1,t_2]}}{\Gamma(q)}\left(\frac{p_1-1}{p_1q-1}\right)^{\frac{p_1-1}{p_1}}(t_2-t_1)^{q-\frac{1}{p_1}}.
\end{aligned} \tag{3.37}$$

From Lemma 2.4, we have

$$\begin{aligned}
I_3 &= \left\|\int_0^{t_1}\bigl[(t_2-s)^{q-1}-(t_1-s)^{q-1}\bigr]\mathcal{S}(t_2-s)a(s)f\bigl(s,x(s),(Hx)(s)\bigr)\right\|_\alpha \\
&\le \frac{MC_{\beta-\alpha}\|\varphi_r\|_{L^\infty[0,b]}}{\Gamma(q)}\left\|\int_0^{t_1}\bigl[(t_2-s)^{q-1}-(t_1-s)^{q-1}\bigr]a(s)\,ds\right\| \\
&\le \frac{MC_{\beta-\alpha}\|\varphi_r\|_{L^\infty[0,b]}}{\Gamma(q)}(t_2-t_1)^{1-q}\int_0^{t_1}\frac{a(s)}{(t_1-s)^{1-q}(t_2-s)^{1-q}}\,ds.
\end{aligned} \tag{3.38}$$

By (3.25), it is easy to see that

$$I_5 = \left\|\int_{t_1}^{t_2}(t_2-s)^{q-1}\mathcal{S}(t_2-s)Bu_{\varepsilon,x}(s)\,ds\right\|_\alpha \le \frac{M(t_2-t_1)^q}{q\Gamma(q)}\frac{1}{\varepsilon}C_u. \tag{3.39}$$

Similar to (3.39), we obtain

$$\begin{aligned} I_6 &= \left\| \int_0^{t_1} \left[(t_2-s)^{q-1}-(t_1-s)^{q-1}\right]\mathcal{S}(t_2-s)Bu_{\varepsilon,x}(s)\,ds \right\|_\alpha \\ &\le \frac{M}{\Gamma(q)}(t_2-t_1)^{1-q}\int_0^{t_1}\frac{1}{(t_1-s)^{1-q}(t_2-s)^{1-q}}\,ds\frac{1}{\varepsilon}C_u. \end{aligned} \tag{3.40}$$

For $t_1 = 0$, $0 < t_2 \le b$, it can easily be seen that $I_4 = I_7 = 0$. For $t_1 > 0$, when $\eta > 0$ is small enough, we have

$$\begin{aligned} I_4 &\le \int_0^{t_1-\eta}(t_1-s)^{q-1}a(s)\left\|\mathcal{S}(t_2-s)-S(t_1-s)\right\|\cdot\left\|f\left(s,x(s),(Hx)(s)\right)\right\|_\alpha ds \\ &\quad + \int_{t_1-\eta}^{t_1}(t_1-s)^{q-1}a(s)\left\|\mathcal{S}(t_2-s)-S(t_1-s)\right\|\cdot\left\|f\left(s,x(s),(Hx)(s)\right)\right\|_\alpha ds \\ &\le \sup_{s\in[0,t_1-\eta]}\left\|\mathcal{S}(t_2-s)-\mathcal{S}(t_1-s)\right\| \\ &\quad \times\left[-\frac{p_1-1}{p_1q-1}\left(\eta^{\frac{p_1q-1}{p_1-1}}-t_1^{\frac{p_1q-1}{p_1-1}}\right)\|a\|_{L^{p_1}[t_1,t_2]}\right]C_{\beta-\alpha}\|\varphi_r\|_{L^\infty[0,b]} \\ &\quad + \frac{2MC_{\beta-\alpha}\|\varphi_r\|_{L^\infty[0,b]}}{\Gamma(q)}\frac{p_1-1}{p_1q-1}\eta^{\frac{p_1q-1}{p_1-1}}\|a\|_{L^{p_1}[t_1-\eta,t_1]} \end{aligned} \tag{3.41}$$

and

$$\begin{aligned} I_7 &\le \int_0^{t_1-\eta}(t_1-s)^{q-1}\left\|\mathcal{S}(t_2-s)-\mathcal{S}(t_1-s)\right\|\left\|Bu_{\varepsilon,x}(s)\right\|ds \\ &\quad + \int_{t_1-\eta}^{t_1}(t_1-s)^{q-1}\left\|\mathcal{S}(t_2-s)-\mathcal{S}(t_1-s)\right\|\left\|Bu_{\varepsilon,x}(s)\right\|ds \\ &\le \sup_{s\in[0,t_1-\eta]}\left\|\mathcal{S}(t_2-s)-\mathcal{S}(t_1-s)\right\|\frac{1}{q\varepsilon}C_u\left(t_1^q-\eta^q\right) \\ &\quad + \frac{2MC_\alpha}{\Gamma(q)}\frac{1}{q\varepsilon}C_u\eta^q. \end{aligned} \tag{3.42}$$

Since we have assumption (H_5), $\mathcal{S}(t)$, $t > 0$ in t is continuous in the uniformly operator topology, it can easily be seen that I_4 and I_7 tend to zero independently of $x \in B_{r(\varepsilon)}$ as $t_2 \to t_1$, $\eta \to 0$. It is clear that $I_i \to 0$, $i = 1,2,\ldots,7$, as $t_2 \to t_1$. Then $V(t)$ is equicontinuous and bounded. By the Ascoli-Arzela theorem, $V(t)$ is relatively compact in $C(I,\mathbb{X}_\alpha)$. Hence Q_ε is a completely continuous operator. From the Schauder fixed point theorem, Q_ε has a fixed point, that is, the fractional control system (1.1) has a mild solution on $[0,b]$.

Step 5. Similar to the proof in Theorem 3.1, it is easy to show that the semilinear fractional system (1.1) is approximately controllable on $[0,b]$.

Since the nonlinear term f is bounded, for any $x_\varepsilon \in B_{r(\varepsilon)}$, there exists a constant $N > 0$ such that

$$\int_0^b \left\|f\left(s,x_\varepsilon(s),(Hx_\varepsilon)(s)\right)\right\|_\alpha^2 ds \le N. \tag{3.43}$$

Consequently, the sequence $\{f(s,x_\varepsilon,(Hx_\varepsilon)(s))\}$ is bounded in $L^2([0,b],\mathbb{X}_\alpha)$, then there is a subsequence denoted by $\{f(s,x_\varepsilon,(Hx_\varepsilon)(s))\}$, which converges weakly to $f(s,x,(Hx)(s))$ in $L^2([0,b],\mathbb{X}_\alpha)$.

It follows that

$$\begin{aligned}\|p(x_\varepsilon)-w\|_\alpha \le M\|g(x_\varepsilon)-g(x)\|_\alpha + \bigg\|\int_0^b (b-s)^{q-1}\mathcal{S}(b-s)a(s)\big(f\big(s,x_\varepsilon(s),(Hx_\varepsilon)(s)\big)\\ -f\big(s,x(s),(Hx)(s)\big)\big)\,ds\bigg\|_\alpha.\end{aligned}\tag{3.44}$$

Now, by the compactness of the operator $l(\cdot)\to\int_0^{\cdot}(\cdot-s)^{q-1}\mathcal{S}(\cdot-s)l(s)\,ds : L^2(I,\mathbb{X}_\alpha)\to C(I,\mathbb{X}_\alpha)$ and (H$_7$), it is easy to get $\|p(x_\varepsilon)-w\|_\alpha\to 0$ as $\varepsilon\to 0^+$. Then

$$\begin{aligned}\|x_\varepsilon(b)-x_1\|_\alpha &= \big\|\varepsilon R\big(\varepsilon,\Gamma_0^b\big)(w)\big\|_\alpha + \big\|\varepsilon R\big(\varepsilon,\Gamma_0^b\big)\big\|\,\|p(x_\varepsilon)-w\|_\alpha\\ &\le \big\|\varepsilon R\big(\varepsilon,\Gamma_0^b\big)(w)\big\|_\alpha + \|p(x_\varepsilon)-w\|_\alpha \to 0.\end{aligned}\tag{3.45}$$

This proves the approximate controllability of (1.1). The proof is completed. □

4 Optimal controls

In this section, we assume that $\mathbb{Y}$ is another separable reflexive Banach space from which the controls u take the values. We define the admissible control set $U_{ad}=\{u\in L^{p_2}(E)\mid u(t)\in w(t)\text{ a.e.}\}$, $1<p_2\le p_1<\infty$, where the multifunction $w:I\to w_f(\mathbb{Y})$ is measurable, $w_f(\mathbb{Y})$ represents a class of nonempty closed and convex subsets of $\mathbb{Y}$, and $w(\cdot)\subset E$, E is a bounded set of $\mathbb{Y}$.

We consider the following controlled system:

$$\begin{cases}{}^CD^qx(t)=-Ax(t)+a(t)f(t,x(t),(Hx)(t))+C(t)u(t), & t\in I, u\in U_{ad},\\ x(0)=g(x)+x_0\in\mathbb{X}_\alpha,\end{cases}\tag{4.1}$$

where $a\in L^{p_1}(I,\mathbb{R}^+)$, $C\in L^\infty(I,L(\mathbb{Y},\mathbb{X}_\alpha))$, it is easy to see that $Cu\in L^{p_2}(I,\mathbb{X}_\alpha)$ for all $u\in U_{ad}$.

Let x^u denote a mild solution of the system (4.1) corresponding to $u\in U_{ad}$. Here we consider the Bolza problem (P), which means that we shall find an optimal pair $(x^0,u^0)\in C(I,\mathbb{X}_\alpha)\times U_{ad}$ such that

$$J\big(x^0,u^0\big)\le J\big(x^u,u\big),\quad \text{for all } u\in U_{ad},\tag{4.2}$$

where

$$J\big(x^u,u\big)=\phi\big(x^u(b)\big)+\int_0^b l\big(t,x^u(t),u(t)\big)\,dt.\tag{4.3}$$

We list here some suitable hypotheses on the operator C, ϕ, and l as follows:

(HL) (1) The functional $l:I\times\mathbb{X}_\alpha\times\mathbb{Y}\to\mathbb{R}\cup\{\infty\}$ is Borel measurable.
(2) $l(t,\cdot,\cdot)$ is sequentially lower semicontinuous on $\mathbb{X}_\alpha\times\mathbb{Y}$ for almost all $t\in I$.
(3) $l(t,x,\cdot)$ is convex on $\mathbb{Y}$ for each $x\in\mathbb{X}_\alpha$ and almost all $t\in I$.

(4) There exist $d \geq 0$, $e > 0$, and $\lambda \in L^1(I, \mathbb{R})$ such that
$l(t,x,u) \geq \lambda(t) + d\|x\|_\alpha + e\|u\|_{\mathbb{Y}}^{p_2}$.

(5) The functional $\phi : \mathbb{X} \to \mathbb{R}$ is continuous and nonnegative.

(6) The following inequality holds:

$$\begin{aligned}
&\gamma_1 \frac{MC_{\beta-\alpha}}{\Gamma(q)} \left(1 + \frac{1}{\varepsilon}\frac{b^q}{q}\frac{M}{\Gamma(q)}\|C\|_\infty \sup_{0\leq t\leq b}\big\|B^*\mathcal{S}^*(b-t)\big\|\right) \\
&\quad + \gamma_2 M\left(1 + \frac{1}{\varepsilon}\frac{b^q}{q}\frac{M}{\Gamma(q)}\|C\|_\infty \sup_{0\leq t\leq b}\big\|B^*\mathcal{S}^*(b-t)\big\|\right) < 1.
\end{aligned} \tag{4.4}$$

Theorem 4.1 *Assume that assumptions* (H_3), (H_5)-(H_7), *and* (HL) *are satisfied. Then the Bolza problem* (*P*) *admits at least one optimal pair on* $C(I, \mathbb{X}_\alpha) \times U_{ad}$ *provided that*

$$Ml_g < 1. \tag{4.5}$$

Proof Firstly, we show that the system (4.1) has a mild solution corresponding to u given by the following integral equation:

$$\begin{aligned}
x^u(t) &= \mathcal{T}(t)\big(x_0 + g(x^u)\big) + \int_0^t (t-s)^{q-1}\mathcal{S}(t-s)a(s)f\big(s, x^u(s), \big(Hx^u\big)(s)\big)\,ds \\
&\quad + \int_0^t (t-s)^{q-1}\mathcal{S}(t-s)C(s)u(s)\,ds.
\end{aligned} \tag{4.6}$$

From Lemmas 2.1, 2.3, and (3.11), we have

$$\begin{aligned}
&\left\|\int_0^t (t-s)^{q-1}\mathcal{S}(t-s)C(s)u(s)\,ds\right\|_\alpha \\
&\quad \leq \frac{1}{\varepsilon}\frac{b^q}{q}\frac{M}{\Gamma(q)}\|C\|_\infty^L \big\|B^*\mathcal{S}^*(b-t)\big\| \\
&\qquad \times \left(\|x_1\|_\alpha + M\|x_0\|_\alpha + M\psi_r + \frac{MC_{\beta-\alpha}}{\Gamma(q)}\int_0^b (b-s)^{q-1}a(s)\varphi_r(s)\,ds\right),
\end{aligned} \tag{4.7}$$

where $\|\cdot\|_\infty^L$ is the norm of Banach space $L^\infty(I, L(\mathbb{Y}, \mathbb{X}))$. Meanwhile, assumptions (H_5)-(H_7) and (HL) are satisfied. Similar to the proof of Theorem 3.2, we can verify that the system (4.1) has a mild solution x^u corresponding to u easily.

Secondly, we discuss the existence of optimal controls. If $\inf\{J(x^u, u) \mid u \in U_{ad}\} = +\infty$, there is nothing to prove. We assume that $\inf\{J(x^u, u) \mid u \in U_{ad}\} = J_0 < +\infty$. Using condition (HL), we know

$$\begin{aligned}
J\big(x^u, u\big) = J_0 &\geq \phi\big(x^u(b)\big) + \int_0^b \lambda(t)\,dt + d\int_0^b \big\|x^u(t)\big\|_\alpha\,dt + e\int_0^b \big\|u(t)\big\|_{\mathbb{Y}}^{p_2}\,dt \\
&\geq -J' > -\infty,
\end{aligned} \tag{4.8}$$

here $J_0 \geq -J'$, $J' > 0$ is a constant.

By the definition of an infimum, there exists a minimizing sequence of the feasible pair $\{x^n, u^n\} \subset A_{ad}$, where $A_{ad} := \{(x, u) \mid x$ is a mild solution of the system (4.1) corresponding to $u \in U_{ad}\}$, such that $J(x^n, u^n) \to J_0$ as $n \to +\infty$. Since $\{u_n\} \subseteq U_{ad}$, $\{u_n\}$ is bounded in $L^{p_2}(I, \mathbb{Y})$, there exists a subsequence, relabeled as $\{u^n\}$, and $u^0 \in L^{p_2}(I, \mathbb{Y})$ satisfies

$$u^n \to u^0 \quad \text{(weakly)} \tag{4.9}$$

in $L^{p_2}(I, \mathbb{Y})$. Since U_{ad} is closed and convex, by the Marzur lemma, we have $u^0 \in U_{ad}$.

Let x^n be a mild solution of the system (4.1) corresponding to u^n ($n = 0, 1, 2, \ldots$), then x^n satisfies the following integral equation:

$$\begin{aligned} x^n(t) = {}& \mathcal{T}(t)\bigl(x_0 + g\bigl(x^n\bigr)\bigr) + \int_0^t (t-s)^{q-1}\mathcal{S}(t-s)a(s)f\bigl(s, x^n(s), \bigl(Hx^n\bigr)(s)\bigr)\,ds \\ & + \int_0^t (t-s)^{q-1}\mathcal{S}(t-s)C(s)u^n(s)\,ds. \end{aligned} \tag{4.10}$$

Noting that $f(\cdot, x^n(s), (Hx^n)(s))$ is a bounded continuous operator from I into $\mathbb{X}_\beta$, we have

$$f\bigl(\cdot, x^n(s), \bigl(Hx^n\bigr)(s)\bigr) \in L^{p_2}(I, \mathbb{X}_\beta). \tag{4.11}$$

Furthermore, $\{f(\cdot, x^n(s), (Hx^n)(s))\}$ is bounded in $L^{p_2}(I, \mathbb{X}_\beta)$, so there exists a subsequence, relabeled $\{f(s, x^n(s), (Hx^n)(s))\}$ such that

$$f\bigl(s, x^n(s), \bigl(Hx^n\bigr)(s)\bigr) \to \widehat{f}\bigl(s, x(s), (Hx)(s)\bigr) \quad \text{(weakly)}, \tag{4.12}$$

where $\widehat{f}(\cdot, x(s), (Hx)(s)) \in L^{p_2}(I, \mathbb{X}_\beta)$.

We denote the operators Q_1 and Q_2 by

$$(Q_1x)(t) = \int_0^t (t-s)^{q-1}\mathcal{S}(t-s)a(s)f\bigl(s, x, (Hx)(s)\bigr)\,ds, \tag{4.13}$$

$$(Q_2x)(t) = \int_0^t (t-s)^{q-1}\mathcal{S}(t-s)C(s)u(s)\,ds. \tag{4.14}$$

Since $\{f(s, x(s), (Hx)(s))\} \subseteq L^{p_2}(I, \mathbb{X}_\beta)$ is bounded, similar to the proof of Theorem 3.2, it is easy to see that $\|(Q_1x)(t)\|_\alpha$ is bounded. It is easy to verify that $(Q_1x)(t)$ is compact and equicontinuous in $\mathbb{X}_\alpha$. Due to the Ascoli-Arzela theorem, $\{(Q_1x)(t)\}$ is relatively compact in $C(I, \mathbb{X}_\alpha)$. Obviously, Q_1 is linear and continuous, then Q_1 is a strongly continuous operator, and we obtain

$$\begin{aligned} & \int_0^\cdot (\cdot - s)^{q-1}\mathcal{S}(\cdot - s)a(s)f\bigl(s, x^n(s), \bigl(Hx^n\bigr)(s)\bigr)\,ds \\ & \quad \to \int_0^\cdot (\cdot - s)^{q-1}\mathcal{S}(\cdot - s)a(s)\widehat{f}\bigl(s, x(s), (Hx)(s)\bigr)\,ds \quad \text{(strongly)}. \end{aligned} \tag{4.15}$$

Similarly, we can verify Q_2 is a strongly continuous operator and

$$\int_0^\cdot (\cdot - s)^{q-1}\mathcal{S}(\cdot - s)C(s)u^n(s)\,ds \to \int_0^\cdot (\cdot - s)^{q-1}\mathcal{S}(\cdot - s)C(s)u^0(s)\,ds \quad \text{(strongly)}. \tag{4.16}$$

Now, we turn to considering the following controlled system:

$$\begin{cases} {}^{C}D^{q}x(t) = -Ax(t) + a(t)\widehat{f}(t, x(t), (Hx)(t)) + C(t)u^{0}(t), \quad t \in I, u^{0} \in U_{ad}, \\ x(0) = g(x) + x_{0} \in \mathbb{X}_{\alpha}. \end{cases} \tag{4.17}$$

By Theorem 3.2, the above system has a mild solution $\widehat{x}$ corresponding to u^{0}, and

$$\begin{aligned} \widehat{x}(t) &= \mathcal{T}(t)\bigl[x_{0} + g(\widehat{x})\bigr] + \int_{0}^{t} (t-s)^{q-1}\mathcal{S}(t-s)a(s)\widehat{f}\bigl(s, x(s), (Hx)(s)\bigr)\,ds \\ &\quad + \int_{0}^{t} (t-s)^{q-1}\mathcal{S}(t-s)C(s)u^{0}(s)\,ds. \end{aligned} \tag{4.18}$$

From Lemma 2.3 and (H_3), we obtain

$$I_{1}' := \bigl\|\mathcal{T}(t)\bigl[g\bigl(x^{m}\bigr) - g(\widehat{x})\bigr]\bigr\|_{\alpha} \leq Ml_{g}\bigl\|x^{m} - \widehat{x}\bigr\|_{\infty} \tag{4.19}$$

and

$$\begin{aligned} I_{2}' &:= \biggl\| \int_{0}^{t} (t-s)^{q-1}a(s)\mathcal{T}(t-s)\bigl(f\bigl(s, x^{m}(s), \bigl(Hx^{m}\bigr)(s)\bigr) - \widehat{f}\bigl(s, x(s), (Hx)(s)\bigr)\bigr) \biggr\|_{\alpha} \\ &\leq \biggl(\int_{0}^{t} (t-s)^{(q-1)\frac{p_2}{p_2-1}}\,ds\biggr)^{\frac{p_2-1}{p_2}} \\ &\quad \times \biggl(\int_{0}^{t} \bigl(a(s)\bigr)^{p_2} \bigl\|\mathcal{S}(t-s)\bigl(f\bigl(s, x^{m}(s), \bigl(Hx^{m}\bigr)(s)\bigr) - \widehat{f}\bigl(s, x(s), (Hx)(s)\bigr)\bigr)\bigr\|_{\alpha}^{p_2}\,ds\biggr)^{\frac{1}{p_2}} \\ &\leq \biggl(\frac{p_2 - 1}{p_2 + p_2(q-1) - 1}\biggr)^{\frac{p_2-1}{p_2}} b^{\frac{p_2+p_2(q-1)-1}{p_2}} \biggl(\int_{0}^{t} \bigl(a(s)\bigr)^{p_2} \bigl\|\mathcal{S}(t-s)\bigl(f\bigl(s, x^{m}(s), \bigl(Hx^{m}\bigr)(s)\bigr) \\ &\quad - \widehat{f}\bigl(s, x(s), (H)(s)\bigr)\bigr)\bigr\|_{\alpha}^{p_2}\,ds\biggr)^{\frac{1}{p_2}}. \end{aligned} \tag{4.20}$$

Similarly, we have

$$\begin{aligned} I_{3}' &:= \biggl\| \int_{0}^{t} (t-s)^{q-1}a(s)\mathcal{T}(t-s)C(s)\bigl(u^{m}(s) - u^{0}(s)\bigr)\,ds \biggr\|_{\alpha} \\ &\leq \biggl(\frac{p_2 - 1}{p_2 + p_2(q-1) - 1}\biggr)^{\frac{p_2-1}{p_2}} b^{\frac{p_2+p_2(q-1)-1}{p_2}} \|C\|_{\infty}^{L} \\ &\quad \times \biggl(\int_{0}^{t} \bigl(a(s)\bigr)^{p_2} \bigl\|\mathcal{S}(t-s)\bigl(u^{m}(s) - u^{0}(s)\bigr)\bigr\|_{\alpha}^{p_2}\,ds\biggr)^{\frac{1}{p_2}}. \end{aligned} \tag{4.21}$$

By (4.15), (4.16), and the Lebesgue dominated convergence theorem, we can deduce that $I_{2}', I_{3}' \to 0$ as $m \to 0$.

For each $t \in I$, $x^{n}(\cdot), \widehat{x}(\cdot) \in \mathbb{X}_{\alpha}$, we have

$$\bigl\|x^{n}(t) - \widehat{x}(t)\bigr\|_{\alpha} \leq Ml_{g}\bigl\|x^{n}(t) - \widehat{x}(t)\bigr\|_{\alpha} + I_{2}' + I_{3}'. \tag{4.22}$$

Noting (4.5), we get

$$\|x^n - \widehat{x}\|_\infty \le \frac{I_2' + I_3'}{1 - Ml_g}. \tag{4.23}$$

Then

$$x^n \to \widehat{x} \quad \text{in } C(I, \mathbb{X}_\alpha) \text{ as } n \to \infty \text{ (strongly)}. \tag{4.24}$$

Furthermore, we can infer that

$$f\big(s, x^n(s), \big(Hx^n\big)(s)\big) \to f\big(s, \widehat{x}(s), (H\widehat{x})(s)\big) \quad \text{in } C(I, \mathbb{X}_\alpha) \text{ as } n \to \infty \text{ (strongly)}. \tag{4.25}$$

Using the uniqueness of the limit, we have

$$\widehat{f}\big(s, x(s), (Hx)(s)\big) = f\big(s, \widehat{x}(s), (H\widehat{x})(s)\big). \tag{4.26}$$

Therefore

$$\begin{aligned}\widehat{x}(t) = {}& \mathcal{T}(t)\big[x_0 + g(\widehat{x})\big] + \int_0^t (t-s)^{q-1}\mathcal{S}(t-s)a(s)f\big(s, \widehat{x}(s), (H\widehat{x})(s)\big)\,ds \\ & + \int_0^t (t-s)^{q-1}\mathcal{S}(t-s)C(s)u^0(s)\,ds,\end{aligned} \tag{4.27}$$

which is just a mild solution of the system (4.1) corresponding to u^0.

Since $C(I, \mathbb{X}_\alpha) \hookrightarrow L^1(I, \mathbb{X}_\alpha)$, using assumption (HL) and the Balder theorem, we have

$$\begin{aligned}J_0 &= \lim_{n\to\infty} \phi\big(x^n(b)\big) + \lim_{n\to\infty} \int_0^b l\big(t, x^n(t), u^n(t)\big)\,dt \\ &\ge \phi\big(\widehat{x}(b)\big) + \int_0^b l\big(t, \widehat{x}(t), u^0(t)\big)\,dt = J\big(\widehat{x}, u^0\big) \ge J_0.\end{aligned} \tag{4.28}$$

This implies that J attains its minimum at $(\widehat{x}, u^0) \in C(I, \mathbb{X}_\alpha) \times U_{ad}$. □

5 Applications

Example 5.1 Consider optimal controls for the following fractional controlled system:

$$\begin{cases} \frac{\partial^{\frac{2}{3}}}{\partial t^{\frac{2}{3}}} x(t,y) = \frac{\partial^2}{\partial y^2} x(t,y) + \frac{a(t)e^{-t}}{e^t + e^{-t}} \cos\big[\frac{6}{13} x(t,y) + \frac{1}{2}\int_0^t \sin(\frac{s}{15}) x(s,y)\,ds\big] + \frac{7a(t)e^{-t}}{e^t+e^{-t}} \\ \quad + \int_0^1 k_0(t,s)u(s,y)\,ds, \quad y \in [0,1], t \in [0,1], u \in U_{ad}, \\ x(t,0) = x(t,1) = 0, \quad t > 0, \\ x(0,y) = \sum_{i=0}^{\sigma} \int_0^1 k_1(t,s)x(s_i,y)\,dy + \sum_{i=0}^{\sigma}\int_0^1 k_2(t,s)\frac{\partial}{\partial y}x(s_i,y)\,dy, \end{cases} \tag{5.1}$$

with the cost function

$$J(x,u) = \int_0^1\int_0^1 \big|x(t,y)\big|^2\,dy\,dt + \int_0^1\int_0^1 \big|u(t,y)\big|^2\,dy\,dt + \int_0^1 \big|x(b,y)\big|^2\,dy, \tag{5.2}$$

where $\sigma \in \mathbb{N}$, $0 < s_0 < s_1 < \cdots < s_\sigma < 1$, $k_0 \in C([0,1] \times [0,1], \mathbb{R}^+)$, and $k_1, k_2 \in L^2([0,1] \times [0,1], \mathbb{R}^+)$.

Let $\mathbb{X} = \mathbb{Y} = (L^2([0,1]), \|\cdot\|_2)$. The operator $A : D(A) \to \mathbb{X}$ is defined by $D(A) = \{x \in \mathbb{X} \mid x', x'' \in \mathbb{X}, x(0) = x(1) = 0\}$ with $Ax = -x''$, then A generates a compact, analytic semigroup $T(\cdot)$ of uniformly bounded linear operator. Clearly, assumption (H_5) is satisfied. Moreover, the eigenvalues of A are $n^2\pi^2$ and the corresponding normalized eigenvectors are $e_n(u) = \sqrt{2}\sin(n\pi u)$, $n = 1, 2, \ldots$.

Define the control function $u : Tx([0,1]) \to \mathbb{R}$ such that $u \in L^2(Tx([0,1]))$. It means that $t \to u(t,\cdot)$ going from $[0,1]$ into $\mathbb{Y}$ is measurable. Set $U(t) = \{u \in \mathbb{Y} \mid \|u\|_{\mathbb{Y}} \le \vartheta\}$ where $\vartheta \in L^2(I, \mathbb{R}^+)$. We restrict the admissible controls U_{ad} to be all the $u \in L^2(Tx([0,1]))$ such that $\|u(t,\cdot)\|_2 \le \vartheta(t)$.

Let $\mathbb{X}_{\frac{1}{2}} = (D(A^{\frac{1}{2}}), \|\cdot\|_{\frac{1}{2}})$, where $\|\cdot\|_{\frac{1}{2}} = \|A^{\frac{1}{2}}\|_2$ and the operator $A^{\frac{1}{2}}$ is given by

$$A^{\frac{1}{2}} = \sum_{n=1}^{\infty} \langle z, e_n\rangle e_n, \tag{5.3}$$

for each $z \in D(A^{\frac{1}{2}}) = \{f \in \mathbb{X} \mid \sum_{n=1}^{\infty} \langle z, e_n\rangle e_n \in \mathbb{X}\}$ and $\|A^{-\frac{1}{2}}\| = 1$.

Suppose that $C(I, \mathbb{X}_{\frac{1}{2}})$ is a Banach space equipped with the supnorm $\|\cdot\|_\infty$, $x(t)(y) = x(t,y)$, $C(t)u(t)(y) = (\int_0^1 k_0(t,s)u(s)\,ds)(y)$. Define $f : [0,1] \times \mathbb{X}_{\frac{1}{2}} \times \mathbb{X}_{\frac{1}{2}} \to \mathbb{X}_{\frac{1}{2}}$ by

$$\begin{aligned} &f\bigl(t, x(t), (Hx)(t)\bigr)(y) \\ &\quad = \frac{e^{-t}}{e^t + e^{-t}} \cos\left[\frac{6}{13}x(t) + \frac{1}{2}\int_0^t \sin\left(\frac{s}{15}\right)x(s)\,ds\right](y) + \frac{7e^{-t}}{e^t + e^{-t}} \end{aligned} \tag{5.4}$$

and $g : C(I, \mathbb{X}_{\frac{1}{2}}) \to \mathbb{X}_{\frac{1}{2}}$ by

$$g(x)(y) = \left(\sum_{i=0}^{\sigma} (Kx)(t_i)\right)(y) \quad \text{for } x \in C(I, \mathbb{X}_{\frac{1}{2}}), \tag{5.5}$$

where $K : \mathbb{X}_{\frac{1}{2}} \to \mathbb{X}_{\frac{1}{2}}$ is defined by

$$(Kx)(s) = \int_0^1 k_1(t,s)x(s)\,ds + \int_0^1 k_2(t,s)x'(s)\,ds, \quad \text{for all } x \in \mathbb{X}_{\frac{1}{2}}. \tag{5.6}$$

Obviously, we have

$$\bigl(K(x-y)\bigr)(s) = \int_0^1 k_1(t,s)(x-y)(s)\,ds + \int_0^1 k_2(t,s)(x'-y')(s)\,ds, \quad \text{for all } x \in \mathbb{X}_{\frac{1}{2}}. \tag{5.7}$$

The system (5.1) can be transformed into

$$\begin{cases} {}^C D^q x(t) = -Ax(t) + a(t)f(t, x(t), (Hx)(t)) + C(t)u(t), & t \in I, u \in U_{ad} \\ x(0) = g(x) + x_0 \in \mathbb{X}_\alpha, \end{cases} \tag{5.8}$$

with the cost function

$$J(x,u) = \|x(b)\|_{\frac{1}{2}} + \int_0^1 \left(\|x(t)\|_{\frac{1}{2}}^2 + \|u(t)\|_{\mathbb{Y}}^2\right) dt, \tag{5.9}$$

and we can verify (HL)(1)-(5) are satisfied. It is also not difficult to see that

$$\bigl\|f\bigl(t,x(t),(Hx)(t)\bigr)\bigr\|_{\frac{1}{2}} \le \frac{8e^{-t}}{e^t+e^{-t}} = \varphi(t), \quad \varphi(t)\in L^\infty\bigl(I,\mathbb{R}^+\bigr), \tag{5.10}$$

then there exists $\psi_r(t)\equiv\psi$ and $\gamma_1 = 0$ such that (3.19) holds, and conditions (H_6) is satisfied. Meanwhile, it comes from the example in [15] that g is a completely continuous operator from $C(I,\mathbb{X}_{\frac{1}{2}})\to\mathbb{X}_{\frac{1}{2}}$ and there exist constants c_{12} and c_{22} such that

$$\bigl\|g(x)\bigr\|_{\frac{1}{2}} \le \sigma(c_{12}+c_{22})\|x\|_\infty, \qquad \bigl\|g(x)-g(z)\bigr\|_{\frac{1}{2}} \le \sigma(c_{12}+c_{22})\|x-z\|_\infty. \tag{5.11}$$

Let $\psi_r\equiv\sigma(c_{12}+c_{22}) = l_g$, $\gamma_2 = 0$, it is easy to verify that (H_3) and (H_7) hold. Since $\gamma_1 = \gamma_2 = 0$, condition (H_8) and condition (HL)(6) are satisfied automatically. By Theorem 4.1, we can conclude that the system (5.1) has at least one optimal pair while the condition $2M\sigma(c_{12}+c_{22}) < 1$ holds.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors have contributed to this work on an equal basis. All authors read and approved the final manuscript.

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