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Transformation technique, fixed point theorem and positive solutions for second-order impulsive differential equations with deviating arguments

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Abstract

This paper investigates the boundary value problems of second-order impulsive differential equations with deviating arguments

$$\begin{cases} x''(t) + \omega(t)f(t, x(\alpha(t))) = 0, & t \in J, t \neq t_k, \\ x(t_k^+) - x(t_k) = c_k x(t_k), & k = 1, 2, \ldots, n, \\ ax(0) - bx'(0) = ax(1) - bx'(1) = \int_0^1 h(s)x(t)\,dt, \end{cases}$$

where $\{c_k\}$ is a real sequence with $c_k > -1$, $k = 1, 2, \ldots, n$, ω may be singular at $t = 0$ and/or $t = 1$. Several new and more general results are obtained for the existence of positive solutions for the above problem by using transformation technique and Krasnosel'skii's fixed point theorem. We discuss our problems under two cases when the deviating arguments are delayed and advanced. The approach to deal with the impulsive term is different from earlier approaches. It is the first paper where the transformation technique and a fixed point theorem for cones are applied to second-order differential equations with impulsive effects and deviating arguments. An example is included to verify the theoretical results.

Keywords: advanced and delayed arguments; impulsive differential equations; transformation technique; fixed point theorem; positive solutions

1 Introduction

Impulsive differential equations, which provide a natural description of observed evolution processes, are regarded as important mathematical tools for better understanding of several real world problems in applied sciences, such as population dynamics, ecology, biological systems, biotechnology, industrial robotic, pharmacokinetics, optimal control, *etc.* Therefore, the study of this class of impulsive differential equations has gained prominence and it is a rapidly growing field. For the general theory of impulsive differential equations, we refer the reader to [1–3], whereas the applications of impulsive differential equations can be found in [4–14]. Nieto and O'Regan [15] pointed out that in a second-order differential equation $u'' = f(t, u, u')$ one usually considers impulses in the position u and the velocity u'. However, in the motion of spacecraft one has to consider instantaneous impulses depending on the position that result in jump discontinuities in velocity,

but with no change in position [16]. The impulses only on velocity occur also in impulsive mechanics [17].

Some classical tools such as bifurcation theory [18, 19], fixed point theorems in cones [20–24], the method of lower and upper solutions [25, 26], the theory of critical point theory and variational methods [7, 15, 27–30] and the technique via appropriate transformation [31–34] have been widely used to study impulsive differential equations. But it is quite difficult to apply these approaches to an impulsive differential equation with deviating arguments; therefore, there was no result in this area for a long time. Only in the recent eight years, there appeared a few articles which dealt with some impulsive differential equations with deviating arguments by using fixed point theorems in cones [35–38]. Motivated by [35–38], in this article we shall use a different approach to discuss the existence of positive solutions for a class of impulsive differential equations with deviating arguments.

Consider the second-order nonlinear impulsive differential equation of the type

$$
\begin{cases}
x''(t) + \omega(t)f(t, x(\alpha(t))) = 0, & t \in J, t \neq t_k, \\
x(t_k^+) - x(t_k) = c_k x(t_k), & k = 1, 2, \ldots, n, \\
ax(0) - bx'(0) = ax(1) - bx'(1) = \int_0^1 h(s)x(t)\, dt,
\end{cases}
\tag{1.1}
$$

where $J = [0,1]$, $f \in C(J \times R^+, R^+)$, $R^+ = [0, +\infty)$, t_k ($k = 1, 2, \ldots, n$, here n is a fixed positive integer) are fixed points with $0 < t_1 < t_2 < \cdots < t_k < \cdots < t_n < 1$, $a, b > 0$, $\{c_k\}$ is a real sequence with $c_k > -1$, $k = 1, 2, \ldots, n$, $x(t_k^+)$ ($k = 1, 2, \ldots, n$) represent the right-hand limit of $x(t)$ at t_k, $h \in C[0,1]$ is nonnegative.

Throughout this paper we assume that $\alpha(t) \not\equiv t$ on $J = [0,1]$. In addition, ω, f, c_k, α and h satisfy

(H$_1$) $\omega \in C((0,1), [0, +\infty))$ with $0 < \int_0^1 \omega(s)\, ds < \infty$ and ω does not vanish on any subinterval of $(0,1)$;

(H$_2$) $f \in C([0,1] \times [0, +\infty), [0, +\infty))$, $\alpha \in C(J, J)$;

(H$_3$) $\{c_k\}$ is a real sequence with $c_k > -1$, $k = 1, 2, \ldots, n$, $c(t) := \prod_{0 < t_k < t}(1 + c_k)$;

(H$_4$) $h \in C[0,1]$ is nonnegative with $\nu \in [0, a)$, where

$$
\nu = \int_0^1 h(t)c(t)\, dt.
\tag{1.2}
$$

Remark 1.1 Throughout this paper, we always assume that a product $c(t) := \prod_{0 < t_k < t}(1 + c_k)$ equals unity if the number of factors is equal to zero, and let

$$
c_M = \max_{t \in J} c(t), \qquad c_m = \min_{t \in J} c(t), \qquad c^{-1}(t) = \prod_{0 < t_k < t}(1 + c_k)^{-1}.
$$

Remark 1.2 Combining (H$_3$) and the definition of $c(t)$, we know that $c(t)$ is a step function and bounded on J, and

$$
c(t) > 0, \quad \forall t \in J, \qquad c(t) = 1, \quad \forall t \in [0, t_1].
$$

Some special cases of (1.1) have been investigated. For example, Zhang and Feng [34] considered problem (1.1) under the case that $\omega(t) \equiv 1$ and $\alpha(t) \equiv t$ on J. By using fixed

point theories in cones, the authors proved the existence of positive solutions for problem (1.1).

At the same time, a class of boundary value problems with delay has been investigated; for example, see [39–44]. It is not difficult to see that the corresponding functions f appearing on the right-hand side depend on $x(t - \tau)$, $\tau > 0$, where initial functions x are given on the initial set, for example, $[-\tau, 0]$. Jankowski [45, 46] pointed out that in such cases $\alpha(t) = t - \tau$, there are some problems with a constant delay τ. If we consider the differential problem on intervals $[0, k]$, where $k \leq \tau$, then it means that we have no delays; we have such a situation in paper [44]. If $k > \tau$, then it is easy to solve the differential equation on $[0, \tau]$, since we have the solution on the initial set $[-\tau, 0]$. Continuing this process, we can find a solution on the whole interval $[0, k]$ by using the method of steps. In the present paper, for example, the deviating argument α can have a form $\alpha(t) = \rho t = t - (1 - \rho)t$ with a fixed number $\rho \in (0, 1)$, so the delay $(1 - \rho)t$ is a function of t. In this case, the initial set reduces to one point $t = 0$, and we cannot apply the step method. To the authors' knowledge, it is the first paper when positive solutions have been investigated for a class of second-order impulsive differential equations with deviating arguments both of advanced and delayed type.

Remark 1.3 There are almost no papers, except [35–37], studying second-order impulsive differential equations with deviating arguments using fixed point theory. However, in [35–37], Jankowski only considered $\omega \in C([0, 1], \infty)$, not ω is singular at $t = 0$ and/or $t = 1$, and dealt with the nonlinear term that is in the form of $f(y(t))$, not $f(t, c(t)y(t))$; see (2.12).

Remark 1.4 Comparing with [35–37], we transform problem (1.1) into a differential system without impulse, *i.e.*, the technique to deal with impulses is completely different from that of [35–37]. According to the authors' knowledge, it is probably the first paper where this technique is applied to second-order impulsive boundary value problems with deviating arguments.

Remark 1.5 The technique to deal with $f(t, c(t)y(t))$ is completely different from that of Zhang *et al.* [31], Zhang *et al.* [32] and Sun *et al.* [33].

Being directly inspired by [31–37], the authors will prove several new and more general results for the existence of positive solutions for problem (1.1) by using fixed point theories.

The organization of this paper is as follows. In Section 2, we present some definitions and lemmas which are needed throughout this paper. In particular, we transform problem (1.1) into a differential system without impulse. In Section 3, we use a fixed point theorem to obtain the existence of positive solutions for problem (1.1) with advanced argument α. Finally, in Section 4, we formulate sufficient conditions under which delayed problem (1.1) has positive solutions. In particular, our results in these sections are new when $\alpha(t) \equiv t$ on $t \in J$.

2 Preliminaries

In this section, we first present some definitions and lemmas which are needed throughout this paper.

Definition 2.1 (see [47]) Let E be a real Banach space over R. A nonempty closed set $P \subset E$ is said to be a cone provided that

(i) $au + bv \in P$ for all $u, v \in P$ and all $a \geq 0$, $b \geq 0$, and

(ii) $u, -u \in P$ implies $u = 0$.

Every cone $P \subset E$ induces an ordering in E given by $x \leq y$ if and only if $y - x \in P$.

Definition 2.2 The map β is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E provided that $\beta : P \to [0, \infty)$ is continuous and

$$\beta\big(tx + (1 - t)y\big) \geq t\beta(x) + (1 - t)\beta(y)$$

for all $x, y \in P$ and $0 \leq t \leq 1$.

Definition 2.3 A function $x(t)$ is said to be a solution of problem (1.1) on J if:

(i) $x(t)$ is absolutely continuous on each interval $(0, t_1]$ and $(t_k, t_{k+1}]$, $k = 1, 2, \ldots, n$;

(ii) for any $k = 1, 2, \ldots, n$, $x(t_k^+)$, $x(t_k^-)$ exist and $x(t_k^-) = x(t_k)$;

(iii) $x(t)$ satisfies (1.1).

We shall reduce problem (1.1) to a system without impulse. To this goal, firstly by means of the transformation

$$x(t) = c(t)y(t), \tag{2.1}$$

we convert problem (1.1) into

$$\begin{cases} -y''(t) = \omega(t)c^{-1}(t)f(t, c(\alpha(t))y(\alpha(t))), & t \in J, \\ ay(0) - by'(0) = \int_0^1 h(s)c(s)y(s)\,ds, \\ ac(1)y(1) - bc(1)y'(1) = \int_0^1 h(s)c(s)y(s)\,ds. \end{cases} \tag{2.2}$$

The following lemmas will be used in the proof of our main results.

Lemma 2.1 *Assume that* (H_1)-(H_4) *hold. Then*

(i) *If* $y(t)$ *is a solution of problem* (2.2) *on* J, *then* $x(t) = c(t)y(t)$ *is a solution of problem* (1.1) *on* J;

(ii) *If* $x(t)$ *is a solution of problem* (1.1) *on* J, *then* $y(t) = c^{-1}(t)x(t)$ *is a solution of problem* (2.2) *on* J.

Proof The proof is similar to that of Lemma 2.1 in [34]. □

Lemma 2.2 *If* (H_1)-(H_4) *hold, then problem* (2.2) *has a solution* y, *and* y *can be expressed in the form*

$$y(t) = \int_0^1 H(t, s)c^{-1}(s)f\big(t, c\big(\alpha(s)\big)y\big(\alpha(s)\big)\big)\,ds, \tag{2.3}$$

where

$$H(t, s) = G(t, s) + \frac{1}{a - \nu}\int_0^1 G(\tau, s)c(\tau)h(\tau)\,d\tau, \tag{2.4}$$

$$G(t,s) = \frac{1}{a(a+2b)} \begin{cases} (b+at)(a(1-s)+b), & 0 \le t \le s \le 1, \\ (b+as)(a(1-t)+b), & 0 \le s \le t \le 1. \end{cases} \tag{2.5}$$

Proof The proof is similar to that of Lemma 2.1 in [48]. □

Lemma 2.3 *Let $\xi \in (0,1)$, G and H be given as in Lemma* 2.2. *Then we have the following results*:

$$G(t,s) \le G(s,s), \qquad H(t,s) \le H(s,s) = \frac{a}{a-v} G(s,s), \quad \forall t,s \in J, \tag{2.6}$$

$$G(t,s) \ge \delta G(s,s), \qquad H(t,s) \ge \delta H(s,s) = \frac{\delta a}{a-v} G(s,s), \quad \forall t \in [\xi,1], s \in J, \tag{2.7}$$

where

$$\delta = \frac{b}{b+a}.$$

Proof Relation (2.6) is simple to prove. Note that

$$\frac{G(t,s)}{G(s,s)} = \frac{b+at}{b+as} \ge \frac{b+a\xi}{b+a} \quad \text{for } s \le t,$$

$$\frac{G(t,s)}{G(s,s)} = \frac{b+a(1-t)}{b+a(1-s)} \ge \frac{b}{b+a} \quad \text{for } t \le s,$$

for $t \in [\xi,1]$, $s \in J$.

Similarly, we can prove that $H(t,s) \ge \delta H(s,s)$ for $t \in [\xi,1]$, $s \in J$. Hence, it follows from $G(t,s) \ge \delta G(s,s)$ that

$$H(t,s) \ge \delta H(s,s) = \frac{\delta a}{a-v} G(s,s), \quad \forall t \in [\xi,1], s \in J.$$

This gives the proof of Lemma 2.3. □

Remark 2.1 Noticing that $a,b > 0$, it follows from (2.4) and (2.5) that

$$0 < \beta_1 \le G(t,s) \le \beta_2, \quad \forall t,s \in J, \tag{2.8}$$

and

$$0 < \beta^* \le H(t,s) \le \beta, \quad \forall t,s \in J, \tag{2.9}$$

where

$$\beta_1 = \frac{b^2}{a(a+2b)}, \qquad \beta_2 = \frac{a+2b}{4a}, \qquad \beta^* = \frac{b^2}{a(a+2b)(a-v)}, \qquad \beta = \frac{a+2b}{4(a-v)}. \tag{2.10}$$

Let $E = C[0,1]$. Then E is a real Banach space with the norm $\| \cdot \|$ defined by

$$\|y\| = \max_{t \in J} |y(t)|, \quad y \in E.$$

Define a cone K in E by

$$K = \left\{ y \in E : y(t) \geq 0, \min_{t \in [\xi, 1]} y(t) \geq \delta \|y\| \right\}. \tag{2.11}$$

Also, define for r a positive number Ω_r by

$$\Omega_r = \left\{ y \in E : \|y\| < r \right\}. $$

Note that $\partial \Omega_r = \{ y \in E : \|y\| = r \}$.

Define $T : K \to K$ by

$$(Ty)(t) = \int_0^1 H(t, s)\omega(s)c^{-1}(s)f\big(s, c(\alpha(s))y(\alpha(s))\big)\, ds. \tag{2.12}$$

Lemma 2.4 *Assume that* (H_1)-(H_4) *hold. Then* $T(K) \subset K$ *and* $T : K \to K$ *is completely continuous.*

Proof For $y \in K$, it follows from (2.6) and (2.12) that

$$
\begin{aligned}
(Ty)(t) &= \int_0^1 H(t, s)\omega(s)c^{-1}(s)f\big(s, c(\alpha(s))y(\alpha(s))\big)\, ds \\
&\leq \int_0^1 H(s, s)\omega(s)c^{-1}(s)f\big(s, c(\alpha(s))y(\alpha(s))\big)\, ds, \quad t \in J.
\end{aligned} \tag{2.13}
$$

It follows from (2.7), (2.12) and (2.13) that

$$
\begin{aligned}
\min_{t \in [\xi, 1]} (Ty)(t) &= \min_{t \in [\xi, 1]} \int_0^1 H(t, s)\omega(s)c^{-1}(s)f\big(s, c(\alpha(s))y(\alpha(s))\big)\, ds \\
&\geq \delta \int_0^1 H(s, s)\omega(s)c^{-1}(s)f\big(s, c(\alpha(s))y(\alpha(s))\big)\, ds \\
&\geq \delta \|Ty\|.
\end{aligned}
$$

Thus, $T(K) \subset K$.

Next, by arguments similar to those of Theorem 1 in [49], one can prove that $T : K \to K$ is completely continuous. So it is omitted, and the lemma is proved. □

Remark 2.2 From (2.12), we know that $y \in E$ is a solution of problem (2.2) if and only if y is a fixed point of the operator T.

From Lemma 2.1 and Remark 2.2, we can obtain the following results.

Lemma 2.5 *Assume that* (H_1)-(H_4) *hold. Then*

(i) *If* $x(t)$ *is a solution of problem* (1.1) *on* J, *then* $y(t) = c^{-1}(t)x(t)$ *is a fixed point of* T;

(ii) *If* $y(t)$ *is a fixed point of* T, *then* $x(t) = c(t)y(t)$ *is a solution of problem* (1.1) *on* J.

In the rest of this section, we state a well-known fixed point theorem which we need later.

Lemma 2.6 (see [47]) *Let P be a cone in a real Banach space E. Assume that Ω_1, Ω_2 are bounded open sets in E with $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$. If*

$$A : P \cap (\bar{\Omega}_2 \backslash \Omega_1) \to P$$

is completely continuous such that either

(i) $\|Ax\| \leq \|x\|$, $\forall x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \geq \|x\|$, $\forall x \in P \cap \partial\Omega_2$, *or*

(ii) $\|Ax\| \geq \|x\|$, $\forall x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \leq \|x\|$, $\forall x \in P \cap \partial\Omega_2$,

then A has at least one fixed point in $P \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

3 Existence of positive solutions for problem (1.1) under $\alpha(t) \geq t$ on J

For convenience, we introduce the following notations:

$$f^0 = \limsup_{y \to 0} \max_{t \in J} \frac{f(t,y)}{y}, \qquad f_0 = \liminf_{y \to 0} \min_{t \in J} \frac{f(t,y)}{y},$$

$$f^\infty = \limsup_{y \to \infty} \max_{t \in J} \frac{f(t,y)}{y}, \qquad f_\infty = \liminf_{y \to \infty} \min_{t \in J} \frac{f(t,y)}{y}.$$

We also define as [50] i_0 = number of zeros in the set $\{f^0, f^\infty\}$ and i_∞ = number of infinities in the set $\{f_0, f_\infty\}$. Sun and Li [51] pointed out that $i_0, i_\infty = 0, 1$ or 2, and there are six possible cases: (i) $i_0 = 0$ and $i_\infty = 0$; (ii) $i_0 = 0$ and $i_\infty = 1$; (iii) $i_0 = 0$ and $i_\infty = 2$; (iv) $i_0 = 1$ and $i_\infty = 0$; (v) $i_0 = 1$ and $i_\infty = 1$; and (vi) $i_0 = 2$ and $i_\infty = 0$. By using Krasnosel'skii's fixed point theorem in a cone, some results are obtained for the existence of at least one or two positive solutions of problem (1.1) for $\alpha(t) \geq t$ on J under the above six possible cases.

3.1 For the case $\alpha(t) \geq t$ on J under $i_0 = 1$ and $i_\infty = 1$

In this subsection, we discuss the existence of a single positive solution for problem (1.1) for $\alpha(t) \geq t$ on J under $i_0 = 1$ and $i_\infty = 1$.

For convenience, we introduce the following notations:

$$\gamma = \int_0^1 \omega(s) \, ds, \qquad \gamma_1 = \int_\xi^1 \omega(s) \, ds.$$

Theorem 3.1 *Assume that* (H_1)-(H_4) *hold. If* $i_0 = 1$ *and* $i_\infty = 1$, *then problem* (1.1) *has at least one positive solution.*

Proof First, we consider the case $f^0 = 0$ and $f_\infty = \infty$. Since $f^0 = 0$, then there exists $r_1 > 0$ such that

$$f(t,y) \leq \frac{c_m}{\beta \gamma c_M} y \quad (\forall t \in J, 0 \leq y \leq r_1).$$

Since $0 \leq t \leq \alpha(t) \leq 1$ on J, it follows from $0 \leq y(t) \leq r_1$ on J that

$$0 \leq y\big(\alpha(t)\big) \leq r_1 \quad \text{for } t \in J.$$

Let $r = \min\{r_1, \frac{1}{c_M}r_1\}$. Then, for $y \in K \cap \partial\Omega_r$, we have $0 \leq y(t) \leq r \leq r_1$ for $t \in J$, and then

$$c(\alpha(t))y(\alpha(t)) \leq c_M\|y\| = c_M r \leq r_1, \quad t \in J.$$

Consequently, for any $t \in J$ and $y \in K \cap \partial\Omega_r$, (2.9) and (2.12) imply

$$\begin{aligned}
(Ty)(t) &= \int_0^1 H(t,s)\omega(s)c^{-1}(s)f\big(s, c(\alpha(s))y(\alpha(s))\big)\,ds \\
&\leq \beta c_m^{-1}\int_0^1 \omega(s)\frac{c_m}{\gamma\beta c_M}c(\alpha(s))y(\alpha(s))\,ds \\
&\leq \beta c_m^{-1}\frac{c_m}{\gamma\beta c_M}c_M\int_0^1 \omega(s)y(\alpha(s))\,ds \\
&\leq \frac{1}{\gamma}\int_0^1 \omega(s)\,ds\|y\| \\
&= \|y\|,
\end{aligned}$$

which implies

$$\|Ty\| \leq \|y\|, \quad \forall y \in K \cap \partial\Omega_r. \tag{3.1}$$

Next turning to $f_\infty = \infty$, there exists $\hat{r}$ satisfying $0 < r_1 < \hat{r}$ such that

$$f(t,y) \geq \frac{c_M}{c_m\delta\beta^*\gamma_1}y, \quad \forall t \in [\xi, 1], y \geq \hat{r}.$$

Since $\xi \leq t \leq \alpha(t) \leq 1$ on J, it follows from $y(t) \geq \hat{r}$ on $[\xi, 1]$ that

$$y(\alpha(t)) \geq \hat{r} \quad \text{for } t \in [\xi, 1].$$

Let $R > \max\{\hat{r}, \frac{\hat{r}}{\delta c_m}\}$. Then, for $y \in K \cap \partial\Omega_R$, we have

$$c(\alpha(t))y(\alpha(t)) \geq c_m y(\alpha(t)) \geq c_m\delta\|y\| \geq \hat{r}, \quad t \in [\xi, 1].$$

Hence, for $y \in K \cap \partial\Omega_R$, it follows from (2.9) and (2.12) that

$$\begin{aligned}
(Ty)(t) &= \int_0^1 H(t,s)\omega(s)c^{-1}(s)f\big(s, c(\alpha(s))y(\alpha(s))\big)\,ds \\
&\geq \beta^*\int_0^1 \omega(s)c^{-1}(s)f\big(s, c(\alpha(s))y(\alpha(s))\big)\,ds \\
&\geq \beta^* c_M^{-1}\int_\xi^1 \omega(s)\frac{c_M}{c_m\delta\beta^*\gamma_1}c(\alpha(s))y(\alpha(s))\,ds \\
&\geq \beta^* c_M^{-1}\frac{c_M}{c_m\delta\beta^*\gamma_1}c_m\int_\xi^1 \omega(s)y(\alpha(s))\,ds \\
&\geq \frac{1}{\delta\gamma_1}\int_\xi^1 \omega(s)\,ds\delta\|y\| \\
&= \|y\|,
\end{aligned}$$

which implies

$$\|Ty\| \geq \|y\|, \quad \forall y \in K \cap \partial\Omega_R. \tag{3.2}$$

Thus by (i) of Lemma 2.6, it follows that T has a fixed point y in $K \cap (\bar{\Omega}_R \backslash \Omega_r)$ with

$$r \leq \|y\| \leq R.$$

Lemma 2.5 implies that problem (1.1) has at least one positive solution x with

$$c_m r \leq \|x\| \leq c_M R.$$

This gives the proof of Theorem 3.1. □

Remark 3.1 For $i_0 = 1$ and $i_\infty = 1$, there is another case $f^\infty = 0$ and $f_0 = \infty$. However, at the moment, we give no information on the existence of a positive solution for problem (1.1) if we change $f^0 = 0$ and $f_\infty = \infty$ into $f^\infty = 0$ and $f_0 = \infty$ in Theorem 3.1.

3.2 For the case $\alpha(t) \geq t$ on J under $i_0 = 0$ and $i_\infty = 0$

In this subsection, we discuss the existence for the positive solutions of problem (1.1) under $i_0 = 0$ and $i_\infty = 0$. For convenience, we introduce the following notations:

$$f_0^\rho = \max\left\{\max_{t \in J} \frac{f(t,y)}{\rho} : y \in [0,\rho]\right\}$$

and

$$l = \frac{c_m}{\beta\gamma}, \qquad L = \frac{c_M}{c_m \beta^* \delta \gamma_1}.$$

Now, we shall state and prove the following main result.

Theorem 3.2 *Suppose that* (H_1)-(H_4) *hold and* $\alpha(t) \geq t$ *on* J. *In addition, let the following two conditions hold*:

(H_5) *There exist* $l > 0$ *and* $\rho_1 > 0$ *such that* $f_0^{\rho_1} \leq l$;
(H_6) *There exist* $\eta > 0$ *and* $\rho_2 > 0$ *such that* $f(t,y) \geq \eta$ *for* $t \in J, y \geq \rho_2$; *furthermore,* $\rho_1 \neq \rho_2$.

Then problem (1.1) *has at least one positive solution*.

Proof Without loss of generality, we may assume that $\rho_1 < \rho_2$. Considering $f_0^{\rho_1} \leq l$, we have $f(t,y) \leq l\rho_1$ for $0 \leq y \leq \rho_1, t \in J$.

Since $0 \leq t \leq \alpha(t) \leq 1$ on J, it follows from $0 \leq y(t) \leq \rho_1$ on J that

$$0 \leq y(\alpha(t)) \leq \rho_1.$$

Let $\rho = \min\{\rho_1, \frac{1}{c_M}\rho_1\}$. Then, for $y \in K \cap \partial\Omega_\rho$, we have $0 \leq y(t) \leq \rho \leq \rho_1$ for $t \in J$, and then

$$c(\alpha(t))y(\alpha(t)) \leq c_M \|y\| = c_M \rho \leq \rho_1, \quad t \in J.$$

Consequently, for any $t \in J$ and $y \in K \cap \partial\Omega_\rho$, (2.9) and (2.12) imply

$$
\begin{aligned}
(Ty)(t) &= \int_0^1 H(t,s)\omega(s)c^{-1}(s)f\big(s,c(\alpha(s))y(\alpha(s))\big)\,ds \\
&\le \beta c_m^{-1}l\rho_1 \int_0^1 \omega(s)\,ds \\
&= \beta c_m^{-1}l\rho_1\gamma \\
&= \rho_1,
\end{aligned}
$$

which implies

$$
\|Ty\| \le \|y\|, \quad \forall y \in K \cap \partial\Omega_\rho. \tag{3.3}
$$

On the other hand, from (H$_6$), when ρ_2 is fixed, there exists $\eta > 0$ such that

$$
f(t,y) \ge \eta \ge \max\left\{\rho_2, \frac{\rho_2}{\delta c_m}\right\} \times \frac{c_M}{\beta^*\gamma_1}
$$

for $t \in J$ and $y \ge \rho_2$. Since $0 \le t \le \alpha(t) \le 1$ on J, it follows from $y(t) \ge \rho_2$ on J that

$$
y(\alpha(t)) \ge \rho_2.
$$

Let $\bar\rho = \max\{\rho_2, \frac{\rho_2}{\delta c_m}\}$. Then, for $y \in K \cap \partial\Omega_{\bar\rho}$, we have

$$
c(t)y(t) \ge c_m y(t) \ge c_m \delta \|y\| \ge \rho_2, \quad t \in [\xi, 1].
$$

Hence, for $y \in K \cap \partial\Omega_{\bar\rho}$, it follows from (2.9) and (2.12) that

$$
\begin{aligned}
(Ty)(t) &= \int_0^1 H(t,s)\omega(s)c^{-1}(s)f\big(s,c(\alpha(s))y(\alpha(s))\big)\,ds \\
&\ge \beta^* \int_0^1 \omega(s)c^{-1}(s)f\big(s,c(\alpha(s))y(\alpha(s))\big)\,ds \\
&\ge \beta^* c_M^{-1} \int_\xi^1 \omega(s)\eta\,ds \\
&\ge \beta^* c_M^{-1}\eta \int_\xi^1 \omega(s)\,ds \\
&= \beta^* c_M^{-1}\eta\gamma_1 \\
&\ge \beta^* c_M^{-1}\gamma_1\bar\rho\,\frac{c_M}{\beta^*\gamma_1} \\
&= \bar\rho,
\end{aligned}
$$

which implies

$$
\|Ty\| \ge \|y\|, \quad \forall y \in K \cap \partial\Omega_{\bar\rho}. \tag{3.4}
$$

Thus by (i) of Lemma 2.6, it follows that T has a fixed point y in $K \cap (\bar{\Omega}_{\rho_2} \setminus \Omega_{\rho_1})$ with

$$\rho \leq \|y\| \leq \bar{\rho}.$$

Thus, it follows from Lemma 2.5 that problem (1.1) has at least one positive solution x with

$$c_m \rho \leq \|x\| \leq c_M \bar{\rho}.$$

This finishes the proof of Theorem 3.2. □

We remark that condition (H_5) in Theorem 3.2 can be replaced by the following condition:

(H_5)′ $f^0 \leq l$,

which is a special case of (H_5).

Corollary 3.1 *Suppose that* (H_1)-(H_4), (H_5)′, (H_6) *hold and* $\alpha(t) \geq t$ *on* J. *Then problem* (1.1) *has at least one positive solution.*

Proof We show that (H_5)′ implies (H_5). Suppose that (H_5)′ holds. Then there exists a positive number $\rho_1 \neq \rho_2$ such that

$$\frac{f(t,y)}{y} \leq l, \quad t \in J, 0 < y \leq \rho_1.$$

Hence, we obtain

$$f(t,y) \leq ly \leq l\rho_1, \quad t \in J, 0 < y \leq \rho_1.$$

Therefore, (H_5) holds. Hence, by Theorem 3.2, problem (1.1) has at least one positive solution. □

Theorem 3.3 *Suppose that* (H_1)-(H_5) *hold and* $\alpha(t) \geq t$ *on* J. *In addition, let the following condition hold*:

(H_7) $f_\infty \geq L$.

Then problem (1.1) *has at least one positive solution.*

Proof The proof is similar to those of (3.2) and (3.3), respectively. □

Corollary 3.2 *Suppose that* (H_1)-(H_4), (H_5)′, (H_7) *hold and* $\alpha(t) \geq t$ *on* J. *Then problem* (1.1) *has at least one positive solution.*

3.3 For the case $\alpha(t) \geq t$ on J under $i_0 = 1$ and $i_\infty = 0$ or $i_0 = 0$ and $i_\infty = 1$

In this subsection, we discuss the existence for the positive solutions of problem (1.1) for the case $\alpha(t) \geq t$ on J under $i_0 = 1$ and $i_\infty = 0$ or $i_0 = 0$ and $i_\infty = 1$.

Theorem 3.4 *Suppose that* (H_1)-(H_4) *hold,* $\alpha(t) \geq t$ *on J and* $f^0 \in [0, l)$ *and* $f_\infty \in (L, \infty)$. *Then problem* (1.1) *has at least one positive solution.*

Proof The proof is similar to that of Theorem 3.2. $\qquad\qquad\square$

Theorem 3.5 *Suppose that* (H_1)-(H_4) *hold,* $\alpha(t) \geq t$ *on J and* $f_0 \in (L, \infty)$ *and* $f^\infty \in [0, l)$. *Then problem* (1.1) *has at least one positive solution.*

Proof Consider $f_0 \in (L, \infty)$, then there exists $\rho_1 > 0$ such that $f(t, y) > Ly$ for $0 \leq y \leq \rho_1$, $t \in J$.

Since $0 \leq t \leq \alpha(t) \leq 1$ on J, it follows from $0 \leq y(t) \leq \rho_1$ on J that

$$0 \leq y(\alpha(t)) \leq \rho_1.$$

Let $\rho = \min\{\rho_1, \frac{1}{c_M}\rho_1\}$. Then, for $y \in K \cap \partial\Omega_\rho$, we have $0 \leq y(t) \leq \rho \leq \rho_1$ for $t \in J$, and then

$$c(\alpha(t))y(\alpha(t)) \leq c_M\|y\| = c_M\rho \leq \rho_1, \quad t \in J.$$

Consequently, for $y \in K \cap \partial\Omega_\rho$, it follows from (2.9) and (2.12) that

$$
\begin{aligned}
(Ty)(t) &= \int_0^1 H(t,s)\omega(s)c^{-1}(s)f(s, c(\alpha(s))y(\alpha(s)))\, ds \\
&\geq \beta^* \int_0^1 \omega(s)c^{-1}(s)f(s, c(\alpha(s))y(\alpha(s)))\, ds \\
&\geq \beta^* c_M^{-1} \int_\xi^1 \omega(s)Lc(\alpha(s))y(\alpha(s))\, ds \\
&\geq \beta^* c_M^{-1} Lc_m\delta\|y\| \int_\xi^1 \omega(s)\, ds \\
&= \beta^* c_M^{-1} Lc_m\delta\|y\|\gamma_1 \\
&\geq \|y\|,
\end{aligned}
$$

which implies

$$\|Ty\| \geq \|y\|, \quad \forall y \in K \cap \partial\Omega_\rho. \tag{3.5}$$

Next, turn to $f^\infty \in [0, l)$. In fact, we can show that $f^\infty \in [0, l)$ implies (H_5).

Let $\tau \in (f^\infty, l)$. Then there exists $r > \tau$ such that $\max_{t \in J} f(t, y) \leq \tau y$ for $y \in [r, \infty)$. Let

$$\beta = \max\left\{\max_{t \in J} f(t, y) : 0 \leq y \leq r\right\} \quad \text{and} \quad \rho_1^* > \max\left\{\frac{\beta}{l - \tau}, \rho, c_M\rho\right\}.$$

Then we have

$$\max_{0 \leq t \leq 1} f(t, y) \leq \tau y + \beta \leq \tau\rho_1^* + \beta < l\rho_1^*, \quad \forall y \in [0, \rho_1^*].$$

This implies that $f_0^{\rho_1^*} \leq l$. Hence, $f^\infty \in [0, l)$ implies that (H_5).

Similarly to the proof of (3.3), we have

$$\|Ty\| \le \|y\|, \quad \forall y \in K \cap \partial\Omega_{\rho^*}, \tag{3.6}$$

where $\rho^* = \min\{\rho_1^*, \frac{1}{c_M}\rho_1^*\}$.

Thus by (ii) of Lemma 2.6, it follows that T has a fixed point y in $K \cap (\bar{\Omega}_{\rho^*} \setminus \Omega_\rho)$ with

$$\rho \le \|y\| \le \rho^*.$$

This finishes the proof of Theorem 3.5. □

From Theorems 3.4 and 3.5, we have the following result.

Corollary 3.3 *Suppose that $f^0 = 0$ and condition (H_6) in Theorem 3.2 hold. Then problem* (1.1) *has at least one positive solution.*

Theorem 3.6 *Suppose that (H_1)-(H_4), $\alpha(t) \ge t$ on J, $f^0 \in (0, l)$ and $f_\infty = \infty$. Then problem* (1.1) *has at least one positive solution.*

Proof The proof is similar to that of Theorem 3.2. □

Theorem 3.7 *Suppose (H_1)-(H_4), $\alpha(t) \ge t$ on J, $f_0 = \infty$ and $f^\infty \in (0, l)$. Then problem* (1.1) *has at least one positive solution.*

Proof The proof is similar to that of Theorem 3.2. □

From Theorems 3.6 and 3.7, the following corollaries are easily obtained.

Corollary 3.4 *Suppose that $f^0 = \infty$ and condition (H_5) in Theorem 3.2 hold. Then problem* (1.1) *has at least one positive solution.*

Corollary 3.5 *Suppose that $f_\infty = \infty$ and condition (H_5) in Theorem 3.2 hold. Then problem* (1.1) *has at least one positive solution.*

3.4 For the case $\alpha(t) \ge t$ on J under $i_0 = 0$ and $i_\infty = 2$ or $i_0 = 2$ and $i_\infty = 0$

In this subsection we study the existence of multiple positive solutions for problem (1.1) for the case $\alpha(t) \ge t$ on J under $i_0 = 0$ and $i_\infty = 2$ or $i_0 = 2$ and $i_\infty = 0$.

Combining the proofs of Theorems 3.1 and 3.2, the following theorem is easily proved.

Theorem 3.8 *Suppose that (H_1)-(H_4), $\alpha(t) \ge t$ on J, $i_0 = 0$ and $i_\infty = 2$ and condition (H_5) of Theorem 3.2 hold. Then problem* (1.1) *has at least two positive solutions.*

Corollary 3.6 *Suppose that (H_1)-(H_4), $\alpha(t) \ge t$ on J, $i_0 = 0$ and $i_\infty = 2$ and condition $(H_5)'$ of Corollary 3.1 hold. Then problem* (1.1) *has at least two positive solutions.*

Remark 3.2 Noticing Remark 3.1, at the moment we give no information on the existence of a positive solution for problem (1.1) under $i_0 = 2$ and $i_\infty = 0$.

4 Positive solutions of problem (1.1) for the case of $\alpha(t) \leq t$ on J

Now we deal with problem (1.1) for the case of $\alpha(t) \leq t$ on J. Similarly as in Lemmas 2.3 and 2.4, we can prove the following results.

Lemma 4.1 *Let $\xi \in (0,1)$, G and H be given as in Lemma* 2.2. *Then we have the following results*:

$$G(t,s) \geq \delta G(s,s), \qquad H(t,s) \geq \delta H(s,s) = \frac{\delta a}{a-v}G(s,s), \quad \forall t \in [0,\xi], s \in J, \tag{4.1}$$

where

$$\delta = \frac{b}{b+a}.$$

Proof Note that

$$\frac{G(t,s)}{G(s,s)} = \frac{b+at}{b+as} \geq \frac{b}{b+a} \quad \text{for } s \leq t,$$

$$\frac{G(t,s)}{G(s,s)} = \frac{b+a(1-t)}{b+a(1-s)} \geq \frac{b+a(1-\xi)}{b+a} \quad \text{for } t \leq s,$$

for $t \in [0,\xi]$, $s \in J$.

Similarly, we can prove that $H(t,s) \geq \delta H(s,s)$ for $t \in [0,\xi]$, $s \in J$. Hence, it follows from $G(t,s) \geq \delta G(s,s)$ that

$$H(t,s) \geq \delta H(s,s) = \frac{\delta a}{a-v}G(s,s), \quad \forall t \in [0,\xi], s \in J.$$

This gives the proof of Lemma 4.1. □

Let E be as defined in Section 2. We define a cone K^* in E by

$$K^* = \left\{ u \in E : u \geq 0, \min_{t \in [0,\xi]} u(t) \geq \delta \|u\| \right\}.$$

It is easy to see that K^* is a closed convex cone of E.

Define $T^* : K^* \to E$ by

$$\left(T^*y\right)(t) = \int_0^1 H(t,s)\omega(s)c^{-1}(s)f\left(s, c(\alpha(s))y(\alpha(s))\right) ds. \tag{4.2}$$

It is clear that $y(t)$ is a positive solution of problem (1.1) if and only of y is a fixed point of T^*.

By analogous methods, we have the following results. Here we only give the proof of Lemma 4.1.

Lemma 4.2 *Assume that* (H_1)-(H_4) *hold. Then* $T^*(K^*) \subset K^*$ *and* $T^* : K^* \to K^*$ *is completely continuous.*

Lemma 4.3 *Assume that* (H_1)-(H_4) *hold. Then*

(i) *If $x(t)$ is a solution of problem* (1.1) *on J, then $y(t) = c^{-1}(t)x(t)$ is a fixed point of T^*;*

(ii) *If $y(t)$ is a fixed point of T^*, then $x(t) = c(t)y(t)$ is a solution of problem* (1.1) *on J.*

Similar to the proof of that in Section 3, we have the following results.

4.1 For the case $\alpha(t) \le t$ on J under $i_0 = 1$ and $i_\infty = 1$

For convenience, we introduce the following notation:

$$\gamma_1^* = \int_0^\xi \omega(s)\,ds.$$

Theorem 4.1 *Assume that* (H_1)-(H_4) *hold. If* $i_0 = 1$ *and* $i_\infty = 1$, *then problem* (1.1) *has at least one positive solution.*

Proof First, we consider the case $f^0 = 0$ and $f_\infty = \infty$. Since $f^0 = 0$, then there exists $r_1 > 0$ such that

$$f(t,y) \le \frac{c_m}{\beta\gamma c_M}y \quad (\forall t \in J, 0 \le y \le r_1).$$

Since $0 \le \alpha(t) \le t \le 1$ on J, it follows from $0 \le y(t) \le r_1$ on J that

$$0 \le y(\alpha(t)) \le r_1 \quad \text{for } t \in J.$$

Let $r = \min\{r_1, \frac{1}{c_M}r_1\}$. Then, for $y \in K^* \cap \partial\Omega_r$, we have $0 \le y(t) \le r \le r_1$ for $t \in J$, and then

$$c(\alpha(t))y(\alpha(t)) \le c_M\|y\| = c_M r \le r_1, \quad t \in J.$$

Consequently, for any $t \in J$ and $y \in K^* \cap \partial\Omega_r$, (2.9) and (2.12) imply

$$\begin{aligned}
(T^*y)(t) &= \int_0^1 H(t,s)\omega(s)c^{-1}(s)f(s, c(\alpha(s))y(\alpha(s)))\,ds \\
&\le \beta c_m^{-1} \int_0^1 \omega(s)\frac{c_m}{\gamma\beta c_M}c(\alpha(s))y(\alpha(s))\,ds \\
&\le \beta c_m^{-1}\frac{c_m}{\gamma\beta c_M}c_M \int_0^1 \omega(s)y(\alpha(s))\,ds \\
&\le \frac{1}{\gamma}\int_0^1 \omega(s)\,ds\|y\| \\
&= \|y\|,
\end{aligned}$$

which implies

$$\|T^*y\| \le \|y\|, \quad \forall y \in K^* \cap \partial\Omega_r. \tag{4.3}$$

Next we consider $f_\infty = \infty$, there exists $\hat{r}$ satisfying $0 < r_1 < \hat{r}$ such that

$$f(t,y) \ge \frac{c_M}{c_m\delta\alpha\gamma_1}y, \quad \forall t \in [0,\xi], y \ge \hat{r}.$$

Since $\alpha(t) \le t \le \xi$ on J, it follows from $y(t) \ge \hat{r}$ on $[0,\xi]$ that

$$y(\alpha(t)) \ge \hat{r} \quad \text{for } t \in [0,\xi].$$

Let $R > \max\{\hat{r}, \frac{\hat{r}}{\delta c_m}\}$. Then, for $y \in K^* \cap \partial\Omega_R$, we have

$$c(\alpha(t))y(\alpha(t)) \geq c_m y(\alpha(t)) \geq c_m \delta \|y\| \geq \hat{r}, \quad t \in [0, \xi].$$

Hence, for $y \in K^* \cap \partial\Omega_R$, it follows from (2.9) and (2.12) that

$$
\begin{aligned}
(T^*y)(t) &= \int_0^1 H(t,s)\omega(s)c^{-1}(s)f\left(s, c(\alpha(s))y(\alpha(s))\right) ds \\
&\geq \alpha c_M^{-1} \int_0^\xi \omega(s)\frac{c_M}{c_m \delta \alpha \gamma_1} c(\alpha(s))y(\alpha(s)) ds \\
&\geq \alpha c_M^{-1} \frac{c_M}{c_m \delta \alpha \gamma_1} c_m \int_0^\xi \omega(s)y(\alpha(s)) ds \\
&\geq \frac{1}{\delta \gamma_1^*} \int_0^\xi \omega(s) ds \delta \|y\| \\
&= \|y\|,
\end{aligned}
$$

which implies

$$\left\|T^*y\right\| \geq \|y\|, \quad \forall y \in K^* \cap \partial\Omega_R. \tag{4.4}$$

Thus by (i) of Lemma 2.6, it follows that T^* has a fixed point y in $K^* \cap (\bar{\Omega}_R \setminus \Omega_r)$ with

$$r \leq y(t) \leq R.$$

This finishes the proof of Theorem 4.1. □

Remark 4.1 For $i_0 = 1$ and $i_\infty = 1$, there is another case $f^\infty = 0$ and $f_0 = \infty$. However, at the moment, we give no information on the existence of a positive solution for problem (1.1) if we change $f^0 = 0$ and $f_\infty = \infty$ into $f^\infty = 0$ and $f_0 = \infty$ in Theorem 4.1.

4.2 For the case $\alpha(t) \leq t$ on J under $i_0 = 0$ and $i_\infty = 0$

In this subsection, we discuss the existence for the positive solutions of problem (1.1) under $i_0 = 0$ and $i_\infty = 0$. For convenience, we introduce the following notation:

$$L^* = \frac{c_M}{c_m \alpha \delta \gamma_1^*}.$$

Now, we shall state and prove the following main result.

Theorem 4.2 *Suppose that* (H_1)-(H_5) *hold and* $\alpha(t) \leq t$ *on* J. *In addition, let the following condition hold*:

(H_6^*) *There exist* $\eta^* > 0$ *and* $\rho_2 > 0$ *such that* $f(t,y) \geq \eta^*$ *for* $t \in J$, $y \geq \rho_2$; *furthermore,*
$\rho_1 \neq \rho_2$.

Then problem (1.1) *has at least one positive solution.*

Corollary 4.1 *Suppose that* (H_1)-(H_4), $(H_5)'$, (H_6^*) *hold and* $\alpha(t) \leq t$ *on* J. *Then problem* (1.1) *has at least one positive solution.*

Theorem 4.3 *Suppose that* (H_1)-(H_5) *hold and* $\alpha(t) \leq t$ *on* J. *In addition, let the following condition hold*:

(H_7^*) $f_\infty \geq L^*$.

Then problem (1.1) *has at least one positive solution.*

Corollary 4.2 *Suppose that* (H_1)-(H_4), $(H_5)'$, (H_7^*) *hold and* $\alpha(t) \geq t$ *on* J. *Then problem* (1.1) *has at least one positive solution.*

4.3 For the case $\alpha(t) \leq t$ on J under $i_0 = 1$ and $i_\infty = 0$ or $i_0 = 0$ and $i_\infty = 1$

Theorem 4.4 *Suppose that* (H_1)-(H_4) *hold,* $\alpha(t) \leq t$ *on* J *and* $f^0 \in [0, l)$ *and* $f_\infty \in (L^*, \infty)$. *Then problem* (1.1) *has at least one positive solution.*

Theorem 4.5 *Suppose that* (H_1)-(H_4) *hold,* $\alpha(t) \leq t$ *on* J *and* $f_0 \in (L^*, \infty)$ *and* $f^\infty \in [0, l)$. *Then problem* (1.1) *has at least one positive solution.*

4.4 For the case $\alpha(t) \leq t$ on J under $i_0 = 0$ and $i_\infty = 2$ or $i_0 = 2$ and $i_\infty = 0$

Combining the proofs of Theorems 4.1 and 4.2, the following theorem is easily proved.

Theorem 4.6 *Suppose that* (H_1)-(H_4), $\alpha(t) \leq t$ *on* J, $i_0 = 0$ *and* $i_\infty = 2$ *and condition* (H_5) *of Theorem* 4.2 *hold. Then problem* (1.1) *has at least two positive solutions.*

Corollary 4.3 *Suppose that* (H_1)-(H_4), $\alpha(t) \leq t$ *on* J, $i_0 = 0$ *and* $i_\infty = 2$ *and condition* $(H_5)'$ *of Corollary* 4.1 *hold. Then problem* (1.1) *has at least two positive solutions.*

5 An example

To illustrate how our main results can be used in practice, we present an example.

Example 5.1 Consider the following boundary value problem:

$$
\begin{cases}
x''(t) + \omega(t)f(t, x(\alpha(t))) = 0, & t \in J, t \neq \frac{1}{2}, \\
x(\frac{1}{2}^+) - x(\frac{1}{2}) = \frac{1}{2}x(\frac{1}{2}), & k = 1, \\
2x(0) - x'(0) = 2x(1) - x'(1) = \int_0^1 x(t)\, dt,
\end{cases}
\tag{5.1}
$$

where $\alpha \in C(J, J)$, $\alpha(t) \geq t$ on J and

$$
\omega(t) = \frac{1}{\sqrt{t}}, \qquad f(t, x) = \sqrt[n]{1 + t^n} x^n,
$$

here $n \geq 2$ is a positive integral number.

This means that problem (5.1) involves the advanced argument α. For example, we can take $\alpha(t) = \sqrt[3]{t}$. It is clear that ω is singular at $t = 0$ and f is both nonnegative and continuous.

Problem (5.1) can be regarded as a problem of the form (1.1), where $a = 2$, $b = 1$, $h(t) \equiv 1$, and

$$c(t) = \begin{cases} 1, & 0 \le t \le \frac{1}{2}, \\ \frac{3}{2}, & \frac{1}{2} < t \le 1. \end{cases}$$

Hence

$$\delta = \frac{1}{3}, \qquad \nu = \int_0^1 h(t)c(t)\,dt = \frac{5}{4}, \qquad c_M = \frac{3}{2}, \qquad c_m = 1,$$

and

$$\beta^* = \frac{b^2}{a(a+2b)(a-\nu)} = \frac{1}{6}, \qquad \beta = \frac{a+2b}{4(a-\nu)} = \frac{4}{3}.$$

It follows from the definition of ω, f, c and h that (H_1)-(H_4) hold, and

$$f^0 = 0, \qquad f_\infty = \infty.$$

Hence, by Theorem 3.1, the conclusion follows, and the proof is complete.

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