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Multiple positive solutions for nonlinear boundary value problem of fractional order differential equation with the Riemann-Liouville derivative

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Abstract

In this paper, by means of the Avery-Peterson fixed point theorem, we establish the existence result of a multiple positive solution of the boundary value problem for a nonlinear differential equation with Riemann-Liouville fractional order derivative. An example illustrating our main result is given. Our results complement previous work in the area of boundary value problems of nonlinear fractional differential equations (Goodrich in Appl. Math. Lett. 23:1050-1055, 2010).

MSC: 26A33; 34B15

Keywords: fractional differential equation; positive solution; fixed point; cone

1 Introduction

In this paper, we consider the positive solution for the following boundary value problem of the differential equation involving the Riemann-Liouville fractional order derivative:

$$D_{0+}^{\alpha} u(t) + f\left(t, u(t), D_{0+}^{\beta} u(t)\right) = 0, \quad t \in (0,1), \tag{1.1}$$

$$u^{(i)}(0) = 0, \quad 0 \le i \le n-2, \tag{1.2}$$

$$\left[D_{0+}^{\delta} u(t)\right]_{t=1} = 0, \quad 1 \le \delta \le n-2, \tag{1.3}$$

where $n-1 < \alpha \le n$, $n > 3$, $n \in \mathbb{N}^+$, $0 < \beta \le 1$ and $f \in C([0,1] \times \mathbb{R}^+ \times \mathbb{R}, \mathbb{R}^+)$.

Due to the development of the theory of fractional calculus and its applications, such as in the fields of physics, Bode's analysis of feedback amplifiers, aerodynamics and polymer rheology, *etc.*, many works on the basic theory of fractional calculus and fractional order differential equations have been published [1–7]. Recently, there have been many papers dealing with the solutions or positive solutions of boundary value problems for nonlinear fractional differential equations (FBVPs); see [8–27] and references along this line.

For example, Bai and Lü [12] considered the following Dirichlet boundary value problem of fractional differential equation:

$$D_{0+}^{\alpha} u(t) + f\left(t, u(t)\right) = 0, \quad t \in (0,1), u(0) = 0 = u(1), 1 < \alpha \le 2. \tag{1.4}$$

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By means of some fixed point theorems on cone, existence and multiplicity results of positive solutions were obtained. Furthermore, Bai and Lü proved that the Green's function of problem (1.4) did not satisfy a Harnack-like inequality, which is a crucial property when seeking the existence of positive solutions by means of cone theory. Jiang and Yuan [18] discussed some new positive properties of the Green's function for problem (1.4) and established the existence results of positive solution. The results were established by using the fixed point theorem on a cone due to Krasnoselskii. Recently, Caballero *et al.* [19] obtained the existence and uniqueness of positive solution for problem (1.4) where the nonlinear term f may be singular at $t = 0$. There are also a few papers dealing with the solutions and positive solutions for nonlocal FBVPs; see [22–28].

In a very recent paper, Goodrich [27] considered a class of fractional boundary value problem of the form

$$D_{0+}^{\alpha} u(t) + f\big(t, u(t)\big) = 0, \quad t \in (0, 1), \tag{1.5}$$

$$u^{(i)}(0) = 0, \quad 0 \le i \le n - 2, \tag{1.6}$$

$$\big[D_{0+}^{\delta} u(t)\big]_{t=1} = 0, \quad 1 \le \delta \le n - 2, \tag{1.7}$$

where $n - 1 < \alpha \le n$, $n \in \mathbb{N}^+$ and $f \in C([0, 1] \times \mathbb{R}^+, \mathbb{R}^+)$ and D_{0+}^{α} is the standard Riemann-Liouville fractional derivative of order α. The author obtained the Green's function of this problem and proved that the Green's function satisfied a Harnack-like inequality. By using a fixed point theorem due to Krasnoselskii, the author established the existence of at least one positive solution of problem (1.5)-(1.7).

Problem (1.1)-(1.3) is of interest because it generalizes many problems in real model and have received much attention in recent years. For example, for $\alpha = 4$, problem (1.1)-(1.3) may be applied to the description of the deformation of beam and has been studied with a variety of boundary conditions and nonlinearities; see, for example, [28, 29] and references therein.

However, in these works, the existence results of positive solutions for FBVPs were all established under the assumption that the derivative of the unknown function $u(t)$ was not involved in the nonlinear term explicitly. To the best of our knowledge, few papers can be found in the literature for a positive solution of FBVPs where the derivative of the unknown function $u(t)$ is involved in the nonlinear term. The main purpose of this paper is to fill this gap.

In [30], Avery and Peterson gave a new triple fixed point theorem, which can be regarded as an extension of the Leggett-Williams fixed point theorem. By using this method, many results concerning the existence of at least three positive solutions of boundary value problems of differential equation with integer order were established; see [31–36]. For example, by using the Avery-Peterson fixed point theorem, Yang *et al.* [31] established the existence of at least three positive solutions of the second-order multi-point boundary value problem

$$u'' + f\big(t, u(t), u'(t)\big) = 0, \quad t \in [0, 1],$$

$$u'(0) = \sum_{i=1}^{m-2} \beta_i u'(\xi_i), \qquad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \tag{1.8}$$

where

$$0 < \alpha_i < 1, \qquad 0 < \beta_i < 1, \qquad 0 < \xi_i < 1, \quad i = 1, 2, \ldots, m - 2,$$

$$\sum_{i=1}^{m-2} \alpha_i < 1, \qquad \sum_{i=1}^{m-2} \beta_i < 1.$$

Though the Avery-Peterson fixed point theorem is a classical tool in seeking the multiple positive solutions of BVPs of nonlinear differential equations of integer order, it cannot be used directly to investigate the existence of positive solutions of BVPs of nonlinear differential equations of fractional order. The main reason is that we cannot derive the concavity or convexity of the function $u(t)$ by the sign of its fractional order derivative. In this paper, by obtaining some new inequalities of the unknown function and defining a special cone, we overcome the difficulties brought about by the lack of the concavity or convexity of the unknown function. By an application of the Avery-Peterson fixed point theorem, the existence of at least three positive solutions of problem (1.1)-(1.3) is established. An example is given to illustrate the main results of this paper.

2 Preliminaries

Definition 2.1 The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u(t)$ is given by

$$I_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} u(s) \, ds$$

provided the right side is point-wise defined on $(0, \infty)$.

Definition 2.2 The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $u(t)$ is given by

$$D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n - \alpha)} \left(\frac{d}{dt} \right)^n \int_0^t \frac{u(s)}{(t - s)^{\alpha - n + 1}} \, ds,$$

where $n = [\alpha] + 1$, provided that the right side is point-wise defined on $(0, \infty)$.

For example, for $\lambda > -1$, we have

$$D_{0+}^{\alpha} t^{\lambda} = \frac{\Gamma(1 + \lambda)}{\Gamma(1 + \lambda - \alpha)} t^{\lambda - \alpha}.$$

Lemma 2.1 *Let $\alpha > 0$. The fractional differential equation $D_{0+}^{\alpha} u(t) = 0$ has the solution*

$$u(t) = C_1 t^{\alpha - 1} + C_2 t^{\alpha - 2} + \cdots + C_n t^{\alpha - n}$$

for some $C_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, where n is the smallest integer greater than or equal to α.

Lemma 2.2 *Assume that $u(t)$ with a fractional derivative of order $\alpha > 0$. Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha - 1} + C_2 t^{\alpha - 2} + \cdots + C_n t^{\alpha - n}$$

for some $C_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$.

Lemma 2.3 *Let $\alpha > 0$, $\beta > 0$ and $f \in L_p(0,1)$ ($1 \le p \le \infty$), then the following equation*:

$$I^\alpha I^\beta f(t) = I^{\alpha+\beta} f(t)$$

holds almost everywhere on $[0,1]$.

Definition 2.3 Let E be a real Banach space over $\mathbb{R}$. A nonempty convex closed set $P \subset E$ is said to be a cone provided that:

(1) $au \in P$, for all $u \in P$, $a \ge 0$;

(2) $u, -u \in P$ implies $u = 0$.

Definition 2.4 The map φ is said to be a nonnegative continuous convex functional on cone a P of a real Banach space E provided that $\varphi : P \to [0, +\infty)$ is continuous and

$$\varphi\big(tx + (1-t)y\big) \le t\varphi(x) + (1-t)\varphi(y), \quad x, y \in P, t \in [0,1].$$

Definition 2.5 The map ϕ is said to be a nonnegative continuous concave functional on cone P of a real Banach space E provided that $\phi : P \to [0, +\infty)$ is continuous and

$$\phi\big(tx + (1-t)y\big) \ge t\phi(x) + (1-t)\phi(y), \quad x, y \in P, t \in [0,1].$$

Let γ, θ be nonnegative continuous convex functionals on P, φ be a nonnegative continuous concave functional on P and ψ be a nonnegative continuous functional on P. Then for positive numbers a, b, c, and d, we define the following convex sets:

$$P(\gamma, d) = \big\{x \in P \mid \gamma(x) < d\big\},$$

$$P(\gamma, \varphi, b, d) = \big\{x \in P \mid b \le \alpha(x), \gamma(x) \le d\big\},$$

$$P(\gamma, \theta, \varphi, b, c, d) = \big\{x \in P \mid b \le \varphi(x), \theta(x) \le c, \gamma(x) \le d\big\},$$

and a closed set

$$R(\gamma, \psi, a, d) = \big\{x \in P \mid a \le \psi(x), \gamma(x) \le d\big\}.$$

Lemma 2.4 ([30]) *Let P be a cone in Banach space E. Let γ, θ be nonnegative continuous convex functionals on P, φ be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P satisfying*

$$\psi(\lambda x) \le \lambda \psi(x) \quad \text{for } 0 \le \lambda \le 1, \tag{2.1}$$

such that for some positive numbers l and d,

$$\varphi(x) \le \psi(x), \qquad \|x\| \le l\gamma(x) \tag{2.2}$$

for all $x \in \overline{P(\gamma, d)}$. Suppose $T : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$ is completely continuous and there exist positive numbers a, b, c with $a < b$ such that

(S_1) $\{x \in P(\gamma, \theta, \varphi, b, c, d) \mid \varphi(x) > b\} \neq \emptyset$ and $\alpha(Tx) > b$ for $x \in P(\gamma, \theta, \varphi, b, c, d)$;

(S_2) $\varphi(Tx) > b$ for $x \in P(\gamma, \varphi, b, d)$ with $\theta(Tx) > c$;

(S_3) $0 \notin R(\gamma, \psi, a, d)$ and $\psi(Tx) < a$ for $x \in R(\gamma, \psi, a, d)$ with $\psi(x) = a$.

Then T has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, d)}$ such that

$$
\begin{aligned}
&\gamma(x_i) \leq d, \quad i = 1, 2, 3; \qquad b < \varphi(x_1); \\
&a < \psi(x_2), \qquad \alpha(x_2) < b; \qquad \psi(x_3) < a.
\end{aligned}
\tag{2.3}
$$

3 Main results

Lemma 3.1 ([27]) *Given $y(t) \in C[0,1]$. Then the following FBVPs*:

$$
D_{0+}^{\alpha} u(t) + y(t) = 0, \quad t \in (0,1),
\tag{3.1}
$$

$$
u^{(i)}(0) = 0, \quad 0 \leq i \leq n - 2,
\tag{3.2}
$$

$$
\left[D_{0+}^{\delta} u(t)\right]_{t=1} = 0, \quad 1 \leq \delta \leq n - 2,
\tag{3.3}
$$

are equivalent to the operator equation

$$
u(t) = \int_0^1 G(t,s) y(s) \, ds,
$$

where

$$
G(t,s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-\delta-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-\delta-1}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1. \end{cases}
$$

Lemma 3.2 ([27]) *Let $G(t,s)$ be given as in the statement of Lemma 3.1. Then we find that*:

(1) $G(t,s)$ *is a continuous function on the unit square* $[0,1] \times [0,1]$.

(2) $G(t,s) \geq 0$ *for each* $(t,s) \in [0,1] \times [0,1]$.

(3) $\max_{t \in [0,1]} G(t,s) = G(1,s)$ *for each* $s \in [0,1]$.

(4) *There is a positive constant* $\gamma_0 \in (0,1)$ *such that*

$$
\min_{t \in [\frac{1}{2},1]} G(t,s) \geq \gamma_0 G(1,s), \quad 0 < s < 1,
$$

where

$$
\gamma_0 := \min\left\{ \frac{1}{(2^{\delta}-1)2^{\alpha-\delta-1}}, \frac{1}{2^{\alpha-1}} \right\} > 0.
$$

Lemma 3.3 *Given $y(t) \in C[0,1]$. Assume that $u(t)$ is a solution of problem* (3.1)-(3.3), *then*

$$
\left(D_{0+}^{\beta} u\right)(t) = -\frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} y(s) \, ds + \frac{t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\delta-1} y(s) \, ds.
$$

Lemma 3.4 *Assume that $y(t) > 0$ and $u(t)$ is a solution of problem* (3.1)-(3.3). *Then there exists a positive constant γ_1 such that*

$$
\max_{0 \leq t \leq 1} |u(t)| \leq \gamma_1 \max_{0 \leq t \leq 1} D^{\beta} |u(t)|,
$$

where $\gamma_1 = \frac{\delta \Gamma(\alpha-\beta)}{(\delta-\beta)\Gamma(\alpha)} > 0$.

Proof From Lemmas 3.1-3.3, we have

$$\max_{0 \le t \le 1} |u(t)| = \int_0^1 G(1,s)y(s)\,ds = \int_0^1 \frac{(1-s)^{\alpha-\delta-1} - (1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds,$$

$$\max_{0 \le t \le 1} |D_{0+}^\beta u(t)| \ge |D_{0+}^\beta u(1)| = \int_0^1 \frac{(1-s)^{\alpha-\delta-1} - (1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} y(s)\,ds.$$

Let

$$h(s) = \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} \frac{(1-s)^{\alpha-\delta-1} - (1-s)^{\alpha-1}}{(1-s)^{\alpha-\delta-1} - (1-s)^{\alpha-\beta-1}}, \quad 0 < s < 1.$$

We can check that $h(s)$ is positive and strictly decreasing on $(0,1)$. Furthermore,

$$\lim_{s \to 0^+} h(s) = \frac{\delta\Gamma(\alpha-\beta)}{(\delta-\beta)\Gamma(\alpha)} = \gamma_1.$$

Then

$$h(s) \le \gamma_1, \quad s \in (0,1).$$

Thus,

$$\begin{aligned}
\max_{0 \le t \le 1} |u(t)| &= \int_0^1 \frac{(1-s)^{\alpha-\delta-1} - (1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds \\
&\le \int_0^1 \gamma_1 \frac{(1-s)^{\alpha-\delta-1} - (1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} y(s)\,ds \\
&\le \gamma_1 \max_{0 \le t \le 1} |D_{0+}^\beta u(t)|. \qquad \square
\end{aligned}$$

Let the space $X = \{u \in C[0,1], D_{0+}^\beta u(t) \in C[0,1]\}$ be endowed with the norm

$$\|u\| := \max\Big\{ \max_{0 \le t \le 1} |u(t)|, \max_{0 \le t \le 1} |D_{0+}^\beta u(t)| \Big\}.$$

It is well known that X is a Banach space [37]. Define the cone $K \subset X$ by

$$K = \Big\{ u \in X : u(t) \ge 0, \min_{t \in [\frac{1}{2},1]} u(t) \ge \gamma_0 \max_{0 \le t \le 1} u(t), \max_{0 \le t \le 1} |u(t)| \le \gamma_1 \max_{0 \le t \le 1} D_{0+}^\beta |u(t)| \Big\}.$$

Lemma 3.5 *Let $T : K \to X$ be the operator defined by*

$$Tu(t) := \int_0^1 G(t,s)f\big(s, u(s), D_{0+}^\beta u(s)\big)\,ds.$$

Then $T : K \to K$ is completely continuous.

Proof The operator $T : K \to X$ is continuous in view of the nonnegativity and continuity of the functions $G(t,s)$ and $f(t, u(t), D_{0+}^\beta u(t))$. Let $\Omega \subset K$ be bounded. Then there exists a positive constant $R_1 > 0$ such that $\|u\| \le R_1$, $u \in \Omega$. Denote

$$R = \sup_{0 \le t \le 1, u \in \Omega} |f\big(t, u(t), D_{0+}^\beta u(t)\big)| + 1.$$

Then for $u \in \Omega$, we have

$$|Tu| \leq \int_0^1 G(t,s)\big|f\big(s,u(s),D_{0+}^\beta u(s)\big)\big|\,ds$$

$$\leq R \int_0^1 G(1,s)\,ds = \frac{\delta R}{\Gamma(\alpha+1)(\alpha-\delta)},$$

$$\big|D_{0+}^\beta(Tu)(t)\big| = \left|-\frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}f\big(s,u(s),D_{0+}^\beta u(s)\big)\,ds\right.$$

$$\left.+\frac{t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\int_0^1 (1-s)^{\alpha-\delta-1}f\big(s,u(s),D_{0+}^\beta u(s)\big)\,ds\right|$$

$$\leq \frac{R}{\Gamma(\alpha-\beta)}\left(\frac{1}{\alpha-\beta}+\frac{1}{\alpha-\delta}\right).$$

Hence $T(\Omega)$ is bounded. For $u \in \Omega$, $t_1, t_2 \in [0,1]$, one has

$$\big|Tu(t_2) - Tu(t_1)\big|$$

$$\leq \frac{|t_2^{\alpha-1}-t_1^{\alpha-1}|}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\delta-1}f\big(s,u(s),D_{0+}^\beta u(s)\big)\,ds$$

$$+\left|\frac{1}{\Gamma(\alpha)}\left(\int_0^{t_2}(t_2-s)^{\alpha-1}f\big(s,u,D_{0+}^\beta u\big)\,ds - \int_0^{t_1}(t_1-s)^{\alpha-1}f\big(s,u,D_{0+}^\beta u\big)\,ds\right)\right|$$

$$\leq \left[\frac{|t_2^{\alpha-1}-t_1^{\alpha-1}|}{\Gamma(\alpha)(\alpha-\delta)}+\frac{|t_2^\alpha-t_1^\alpha|}{\Gamma(\alpha+1)}\right]\times R,$$

$$\big|D_{0+}^\beta\big(Tu(t_2)\big)-D_{0+}^\beta\big(Tu(t_1)\big)\big|$$

$$\leq \left|\frac{1}{\Gamma(\alpha-\beta)}\left(\int_0^{t_1}(t_1-s)^{\alpha-\beta-1}f\big(s,u,D_{0+}^\beta u\big)\,ds - \int_0^{t_2}(t_2-s)^{\alpha-\beta-1}f_1\big(s,u,D_{0+}^\beta u\big)\,ds\right)\right|$$

$$+\left|\frac{t_2^{\alpha-\beta-1}-t_1^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\int_0^1 (1-s)^{\alpha-\delta-1}f\big(s,u(s),D_{0+}^\beta u(s)\big)\,ds\right|$$

$$\leq \frac{R}{\Gamma(\alpha-\beta)}\frac{|t_2^{\alpha-\beta}-t_1^{\alpha-\beta}|}{\alpha-\beta}+\frac{R}{\Gamma(\alpha-\beta)}\frac{|t_2^{\alpha-\beta-1}-t_1^{\alpha-\beta-1}|}{\alpha-\delta}.$$

Thus,

$$\big\|Tu(t_2)-Tu(t_1)\big\|\to 0 \quad \text{for } t_1 \to t_2, u \in \Omega.$$

By means of the Arzela-Ascoli theorem, we claim that T is completely continuous. Finally, we see that

$$\min_{1/2\leq t\leq 1}\big|Tu(t)\big| \geq \gamma_0 \int_0^1 G(1,s)f\big(s,u(s),D_{0+}^\beta u(s)\big)\,ds$$

$$= \gamma_0 \max_{0\leq t\leq 1}\int_0^1 G(t,s)f\big(s,u(s),D_{0+}^\beta u(s)\big)\,ds$$

$$= \gamma_0 \max_{0\leq t\leq 1}(Tu)(t),$$

$$\max_{0\leq t\leq 1}\big|Tu(t)\big| = \int_0^1 \frac{(1-s)^{\alpha-\delta-1}-(1-s)^{\alpha-1}}{\Gamma(\alpha)}f\big(s,u(s),D_{0+}^\beta u(s)\big)\,ds$$

$$\leq \int_0^1 \gamma_1 \frac{(1-s)^{\alpha-\delta-1} - (1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} f\big(s, u(s), D_{0+}^\beta u(s)\big)\, ds$$

$$\leq \gamma_1 \max_{0 \leq t \leq 1} \big|D_{0+}^\beta (Tu)(t)\big|.$$

Thus, we show that $T: K \to K$ is a completely continuous operator. □

Let the nonnegative continuous concave functional φ, the nonnegative continuous convex functionals γ, θ and the nonnegative continuous functional ψ be defined on the cone by

$$\gamma(u) = \max_{0 \leq t \leq 1} \big|D_{0+}^\beta u(t)\big|, \qquad \theta(u) = \psi(u) = \max_{0 \leq t \leq 1} \big|u(t)\big|, \qquad \varphi(u) = \min_{\frac{1}{2} \leq t \leq 1} \big|u(t)\big|.$$

By Lemmas 3.3 and 3.4, the functionals defined above satisfy

$$\gamma_0 \theta(u) \leq \varphi(u) \leq \theta(u) = \psi(u), \qquad \|u\| \leq \gamma_2 \gamma(u), \quad u \in P,$$

where $\gamma_2 = \max\{\gamma_1, 1\}$. Therefore condition (2.2) of Lemma 2.4 is satisfied.

Assume that there exist constants $0 < a, b, d$ with $a < b < d$, $c = \frac{b}{\gamma_0}$ and

$$\Gamma(\alpha+1)(2\alpha - \delta - \beta)b < \gamma_0 \delta \Gamma(\alpha-\beta)(\alpha-\beta)d$$

such that

(A$_1$) $f(t, u, v) \leq \dfrac{\Gamma(\alpha-\beta)(\alpha-\beta)(\alpha-\delta)}{2\alpha-\delta-\beta}d, \quad (t, u, v) \in [0,1] \times [0, \gamma_2 d] \times [-d, d];$

(A$_2$) $f(t, u, v) > \dfrac{\Gamma(\alpha+1)(\alpha-\delta)}{\gamma_0 \delta}b, \quad (t, u, v) \in [0,1] \times \left[b, \dfrac{b}{\gamma_0}\right] \times [-d, d];$

(A$_3$) $f(t, u, v) < \dfrac{\Gamma(\alpha+1)(\alpha-\delta)}{\delta}a, \quad (t, u, v) \in [0,1] \times [0, a] \times [-d, d].$

Theorem 3.1 *Under assumptions* (A$_1$)-(A$_3$), *problem* (1.1)-(1.3) *has at least three positive solutions* u_1, u_2, u_3 *satisfying*

$$\max_{0 \leq t \leq 1} \big|D_{0+}^\beta u(t)\big| \leq d, \quad i = 1, 2, 3;$$

$$b < \min_{1/2 \leq t \leq 1} \big|u_1(t)\big|, \qquad a < \max_{0 \leq t \leq 1} \big|u_2(t)\big|;$$

$$\min_{1/2 \leq t \leq 1} \big|u_2(t)\big| < b, \qquad \max_{0 \leq t \leq 1} \big|u_3(t)\big| \leq a.$$

Proof Problem (1.1)-(1.3) has a solution $u = u(t)$ if and only if u solves the operator equation

$$u(t) = \int_0^1 G(t, s) f\big(s, u(s), D_{0+}^\beta u(s)\big)\, ds = (Tu)(t).$$

For $u \in \overline{P(\gamma, d)}$, we have $\gamma(u) = \max_{0 \leq t \leq 1} |D_{0+}^\beta u(t)| < d$. From assumption (A$_1$), we obtain

$$f\big(t, u(t), D_{0+}^\beta u(t)\big) \leq \frac{\Gamma(\alpha-\beta)(\alpha-\beta)(\alpha-\delta)}{2\alpha-\delta-\beta}d.$$

Thus,

$$
\begin{aligned}
\gamma(Tu) &= \max_{0 \le t \le 1} \left| -\frac{1}{\Gamma(\alpha - \beta)} \int_0^t (t-s)^{\alpha - \beta - 1} f\big(s, u(s), D_{0+}^\beta u(s)\big)\, ds \right. \\
&\qquad \left. + \frac{t^{\alpha - \beta - 1}}{\Gamma(\alpha - \beta)} \int_0^1 (1-s)^{\alpha - \delta - 1} f\big(s, u(s), D_{0+}^\beta u(s)\big)\, ds \right| \\
&\le \left[\frac{1}{\Gamma(\alpha - \beta)} \frac{1}{\alpha - \beta} + \frac{1}{\Gamma(\alpha - \beta)} \frac{1}{\alpha - \delta} \right] \times \frac{\Gamma(\alpha - \beta)(\alpha - \beta)(\alpha - \delta)}{2\alpha - \delta - \beta} d \\
&= d.
\end{aligned}
$$

Hence, $T : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$.

The facts that the constant function $u(t) = \frac{b}{\gamma_0} \in P(\gamma, \theta, \varphi, b, c, d)$ and $\varphi(\frac{b}{\gamma_0}) > b$ imply that $\{u \in P(\gamma, \theta, \varphi, b, c, d) \mid \varphi(u) > b)\} \ne \emptyset$. For $u \in P(\gamma, \theta, \varphi, b, c, d)$, we have $b \le u(t) \le \frac{b}{\gamma_0}$ and $|D_{0+}^\beta u(t)| < d$ for $\frac{1}{2} \le t \le 1$. From assumption (A$_2$), we see

$$
f(t, u, v) > \frac{\Gamma(\alpha + 1)(\alpha - \delta)}{\gamma_0 \delta} b.
$$

Thus,

$$
\begin{aligned}
\varphi(Tu) &\ge \gamma_0 \max_{0 \le t \le 1} \left| \int_0^1 G(t, s) f\big(s, u(s), D_{0+}^\beta u(s)\big)\, ds \right| \\
&\ge \int_0^1 \gamma_0 G(1, s)\, ds \times \frac{\Gamma(\alpha + 1)(\alpha - \delta)}{\gamma_0 \delta} b = b,
\end{aligned}
$$

which means $\varphi(Tu) > b$, $\forall u \in P(\gamma, \theta, \varphi, b, \frac{b}{\gamma_0}, d)$. These ensure that condition (S$_1$) of Lemma 2.4 is satisfied. Secondly, for all $u \in P(\gamma, \varphi, b, d)$ with $\theta(Tu) > \frac{b}{\gamma_0}$,

$$
\varphi(Tu) \ge \gamma_0 \theta(Tu) > \gamma_0 \times c = \gamma_0 \times \frac{b}{\gamma_0} = b.
$$

Thus, condition (S$_2$) of Lemma 2.4 holds. Finally we show that (S$_3$) also holds. We see that $\psi(0) = 0 < a$ and $0 \notin R(\gamma, \psi, a, d)$. Suppose that $u \in R(\gamma, \psi, a, d)$ with $\psi(x) = a$. Then by assumption (A$_3$),

$$
\begin{aligned}
\psi(Tu) &= \max_{0 \le t \le 1} \left| \int_0^1 G(t, s) f\big(s, u(s), D_{0+}^\beta u(s)\big)\, ds \right| \\
&\le \int_0^1 G(1, s)\, ds \times \frac{\Gamma(\alpha + 1)(\alpha - \delta)}{\delta} a = a.
\end{aligned}
$$

Thus, all conditions of Lemma 2.4 are satisfied. Hence problem (1.1)-(1.2) has at least three positive solutions u_1, u_2, u_3 satisfying

$$
\max_{0 \le t \le 1} \left| D_{0+}^\beta u(t) \right| \le d, \quad i = 1, 2, 3;
$$

$$
b < \min_{1/2 \le t \le 1} |u_1(t)|, \qquad a < \max_{0 \le t \le 1} |u_2(t)|;
$$

$$
\min_{1/2 \le t \le 1} |u_2(t)| < b, \qquad \max_{0 \le t \le 1} |u_3(t)| \le a. \qquad\qquad \square
$$

4 Example

Consider the nonlinear FBVPs

$$D_{0+}^{\alpha} u(t) + f\left(t, u(t), D_{0+}^{\beta} u(t)\right) = 0, \quad t \in (0,1), \tag{4.1}$$

$$u(0) = u'(0) = u''(0) = 0, \qquad \left[D_{0+}^{\delta} u(t)\right]_{t=1} = 0, \tag{4.2}$$

where $\alpha = 3.6$, $\delta = 1.5$, $\beta = 0.7$, $n = 4$, and

$$f(t, u, v) = \begin{cases} \frac{1}{\pi^4} e^t + 5u^4 + \frac{1}{100} \sin(\frac{v}{10,000}), & 0 \le u \le 7, \\ \frac{1}{\pi^4} e^t + 12{,}005 + \frac{1}{100} \sin(\frac{v}{10,000}), & u > 7. \end{cases}$$

Choose $a = 1$, $b = 3$, $d = 10{,}000$. By a simple computation, we obtain

$$\gamma_0 = \left(\frac{1}{2}\right)^{2.6}, \qquad \gamma_1 = \frac{1.5\Gamma(2.9)}{0.8\Gamma(3.6)} \approx 0.9218, \qquad \gamma_2 = 1,$$

$$\frac{\Gamma(\alpha+1)(\alpha-\delta)}{\delta} a \approx 18.7338,$$

$$\frac{\Gamma(\alpha+1)(\alpha-\delta)}{\gamma_0 \delta} b \approx 340.7416,$$

$$\frac{\Gamma(\alpha-\beta)(\alpha-\beta)(\alpha-\delta)}{2\alpha-\delta-\beta} d \approx 22{,}257.$$

We can check that the nonlinear term $f(t, u, v)$ satisfies

(1) $f(t, u, v) < \dfrac{\Gamma(\alpha-\beta)(\alpha-\beta)(\alpha-\delta)}{2\alpha-\delta-\beta} d,$

   $(t, u, v) \in [0,1] \times [0, 10{,}000] \times [-10{,}000, 10{,}000];$

(2) $f(t, u, v) > \dfrac{\Gamma(\alpha+1)(\alpha-\delta)}{\gamma_0 \delta} b,$

   $(t, u, v) \in \left[\dfrac{1}{2}, 1\right] \times \left[3, 3 \times 2^{2.6}\right] \times [-10{,}000, 10{,}000];$

(3) $f(t, u, v) < \dfrac{\Gamma(\alpha+1)(\alpha-\delta)}{\delta} a, \quad (t, u, v) \in [0,1] \times [0,1] \times [-10{,}000, 10{,}000].$

Then all assumptions of Theorem 3.1 are satisfied. Thus problem (4.1)-(4.2) has at least three positive solutions $u_1(t)$, $u_2(t)$, $u_3(t)$ satisfying

$$\max_{0 \le t \le 1} \left|D_{0+}^{\beta} u(t)\right| \le 10{,}000, \quad i = 1, 2, 3;$$

$$3 < \min_{1/2 \le t \le 1} \left|u_1(t)\right|, \qquad 1 < \max_{0 \le t \le 1} \left|u_2(t)\right|;$$

$$\min_{1/2 \le t \le 1} \left|u_2(t)\right| < 3, \qquad \max_{0 \le t \le 1} \left|u_3(t)\right| \le 1.$$

Remark We see that the fractional derivative of function $u(t)$ of order β is involved in the nonlinear term of problem (4.1)-(4.2) explicitly. The early results for positive solutions of FBVPs, to the best of our knowledge, are not applicable to this problem. Our results complement some previous works in the area of FBVPs, such as those due to Goodrich [27].

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