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Local stability and parameter dependence of mild solutions for stochastic differential equations

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Abstract

For nonlinear stochastic equations $dx(t) = [Ax(t) + f(t, x(t), \lambda)] dt + g(t, x(t), \lambda) d\omega(t)$ with parameter λ in a Hilbert space, we show the existence and uniqueness of mild solutions. Provided that f satisfies a locally Lipschitz condition and g is a uniformly Lipschitz function, some sufficient conditions for p ($p \geq 2$) moment locally exponential stability as well as almost surely exponential stability of mild solutions are obtained under a sufficiently small initial value ξ. Meanwhile, we also consider parameter dependence of stable mild solutions for the stochastic system if f, g are sufficiently small Lipschitz perturbations in the parameter λ.

Keywords: mild solutions; exponential stability; parameter dependence

1 Introduction

Stochastic differential equation has attracted a great attention from both theoretical and applied disciplines since it has been successfully applied to problems in mechanics, economics, physics and several fields in engineering. For details, see [1–5] and the references therein. In recent years, existence, uniqueness, stability, invariant measures and other quantitative and qualitative properties of solutions to stochastic partial differential equations have been extensively considered by many authors. For example, in 1978, Haussmann [6] studied asymptotic stability of Itô equations in infinite dimensions. Caraballo [7] extended the results from Haussmann to stochastic partial differential equations with delay. In [8], Mao considered exponential stability in the mean square sense for the strong solutions of linear stochastic differential equations. Caraballo and Real [9] considered the stability for the strong solutions of semilinear stochastic evolution equations based on the ideas in [7]. Govindan [10, 11] studied the existence and stability of mild solutions for stochastic partial differential equations by the comparison theorem. Caraballo and Liu [12] discussed the exponential stability for mild solutions of stochastic partial differential equations with delays by employing the well-known Gronwall inequality and stochastic analysis technique under the Lipschitz condition. Liu and Mao [13] established Razuminkhin-type exponential stability for mild solution of stochastic partial functional differential equations. Da Prato *et al.* [14] and Dawson [15] developed this topic by the semigroup approach [16]. In [17], Taniguchi *et al.* discussed the existence, uniqueness and asymptotic behavior of solutions to stochastic partial functional differential equations in a Hilbert space.

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Recently, Burton [18] has successfully extended the fixed point theorem to study the stability for deterministic systems. By employing the contraction mapping principle and stochastic analysis, Luo [19] has obtained some sufficient conditions which ensure exponential stability in p ($p \geq 2$) moment and almost sure exponential stability for mild solutions of stochastic partial differential equations with delays. Following the ideas of Luo, Sakthivel and Luo investigated the existence and asymptotic stability in the pth moment of mild solutions of nonlinear impulsive stochastic differential equations with infinite delay in [20].

Our main objective is to obtain local stability of mild solutions for stochastic parameter systems, with two novelties when compared with the former work in this area:

1. We consider p ($p \geq 2$)th moment locally exponential stability as well as almost surely exponential stability of mild solutions under sufficiently small nonlinear perturbations and sufficiently small initial value.

2. We first discuss parameter dependence of stable mild solutions under sufficiently small Lipschitz perturbation in the parameter space Y.

The rest of this paper is organized as follows. In Section 2, stochastic differential equations with parameter are presented together with some definitions of mild solutions. In Section 3, existence, uniqueness, stability and parameter dependence of mild solutions are derived. An example is given to illustrate our results in Section 4. At last, we present some conclusions of our paper in Section 5.

2 Preliminaries

Let H be a real separable Hilbert space with the inner product $(\cdot, \cdot)$ and the norm $\| \cdot \|$, and let K be another real separable Hilbert space with the inner product $(\cdot, \cdot)_K$ and the norm $\| \cdot \|_K$. $L(K, H)$ denotes the space of bounded operators from K to H. Let $(\Omega, \mathscr{F}, P)$ be a complete probability space equipped with a complete family of right-continuous increasing sub σ-algebras $\{\mathscr{F}_t, t \geq 0\}$ satisfying $\mathscr{F}_t \subset \mathscr{F}$.

Let $\beta_n(t)$, $n = 1, 2, \ldots$, be a sequence of real-valued one-dimensional standard Brownian motions mutually independent over $(\Omega, \mathscr{F}, P)$. Set

$$w(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \beta_n(t) \xi_n, \quad t \geq 0,$$

where $\lambda_n \geq 0$ ($n = 1, 2, \ldots$) are nonnegative real numbers and $\{\xi_n\}$ ($n = 1, 2, \ldots$) is a complete orthonormal basis in K. Let $Q \in L(K, K)$ be an operator defined by $Q\xi_n = \lambda_n \xi_n$ with a finite trace $\mathrm{Tr}(Q) = \sum_{n=1}^{\infty} \lambda_n < \infty$. Then the above K-valued stochastic process $\omega(t)$ is called a Q-Wiener process.

Let $\varphi \in L(K, H)$ and define

$$\|\varphi\|_{L_2^0}^2 = \mathrm{Tr}(\varphi Q \varphi^*) = \left\{ \sum_{n=1}^{\infty} \| \sqrt{\lambda_n} \varphi \xi_n \|^2 \right\}.$$

If $\|\varphi\|_{L_2^0} < \infty$, then φ is called a Q-Hilbert-Schmidt operator, where $L_2^0(K, H)$ denotes the space of all Q-Hilbert-Schmidt operators $\varphi : K \to H$.

We denote by $L^p(\Omega, H)$ ($p \geq 2$) the collection of all strongly-measurable, p-integrable H-valued variables with the norm $\|x(\cdot)\|_{L^p} = (E\|x(\cdot; \omega)\|_H^p)^{\frac{1}{p}}$, where E is defined by $E(h) = \int_{\Omega} h(\omega) \, dP$.

We consider the following stochastic differential equations with parameter in a Hilbert space:

$$dx(t) = \left[Ax(t) + f\big(t, x(t), \lambda\big)\right] dt + g\big(t, x(t), \lambda\big) d\omega(t), \quad t \geq 0,$$
$$x(0) = \xi,$$

(1)

where ξ is an H-valued random variable, A is the infinitesimal generator of a semigroup of bounded linear operators $S(t)$, $t \geq 0$, $f : R^+ \times H \times Y \to H$, $g : R^+ \times H \times Y \to L(K, H)$ are all Borel measurable, Y is a Banach space (which is the parameter space).

Now, we recall the mild solutions for Eq. (1) as follows.

Definition 2.1 A stochastic process $\{x(t, \xi, \lambda) : t \in [0, T], \lambda \in Y\}$, $0 \leq T < \infty$, is called a mild solution of Eq. (1) if

(a) $x(t, \xi, \lambda)$ is adapted to $\mathscr{F}_t$, $t \geq 0$;
(b) $x(t, \xi, \lambda) \in H$ has a càdlàg path on $t \in [0, T]$ almost surely;
(c) for arbitrary $t \in [0, T]$,

$$x(t, \xi, \lambda) = S(t)\xi + \int_0^t S(t-s)f\big(s, x(s), \lambda\big) ds + \int_0^t S(t-s)g\big(s, x(s), \lambda\big) d\omega(s). \quad (2)$$

We assume that the following conditions hold:

(i) A is the infinitesimal generator of a semigroup of bounded linear operators $S(t)$, $t \geq 0$, and there exist $M \geq 1$, $\eta > 0$ such that $\|S(t)\|_H \leq Me^{-\eta t}$ for $t \geq 0$;

(ii) there exist constants $c > 0$, $q > 0$ such that for $t \geq 0$, $x, y \in H$, $\lambda, \mu \in Y$,

$$\left\|f(t, x, \lambda) - f(t, y, \lambda)\right\|_H \leq c\|x - y\|_H \big(\|x\|_H^q + \|y\|_H^q\big),$$

and

$$\left\|f(t, x, \lambda) - f(t, x, \lambda)\right\|_H \leq c\|\lambda - \mu\|_H \|x\|_H^{q+1};$$

(iii) there exists $d > 0$ such that for $t \geq 0$, $x, y \in H$, $\lambda, \mu \in Y$,

$$\left\|g(t, x, \lambda) - g(t, y, \lambda)\right\|_{L_2^0} \leq d\|x - y\|_H,$$

and

$$\left\|g(t, x, \lambda) - g(t, x, \mu)\right\|_{L_2^0} \leq d\|\lambda - \mu\|_Y \|x\|_H;$$

(iv) $q + 1 \leq p$;
(v) $2^p M^p d^p C_p \big(\frac{1}{2\eta}\big)^{\frac{p}{2}} < 1$, where $C_p = (p(2p-1))^p$.

For example, if

$$f(t, x, \lambda) = \sum_{j=1}^m \sum_{i=1}^n \lambda^j f_i(t, x),$$

and

$$g(t, x, \lambda) = \sum_{j=1}^{m} \sum_{i=1}^{n} \lambda^j g_i(t, x)$$

for some functions $f_i, g_i : R^+ \times H \to H$, and there exist constants $c_i > 0$, $d_i > 0$, $i = 1, 2, \ldots, n$, $q > 0$ such that for $t \geq 0$, $x, y \in H$,

$$\left\| f_i(t, x) - f_i(t, y) \right\|_H \leq c_i \|x - y\|_H \left(\|x\|_H^q + \|y\|_H^q \right),$$

and

$$\left\| g_i(t, x) - g_i(t, y) \right\|_{L_2^0} \leq d_i \|x - y\|_H$$

for each $t \geq 0$, $x, y \in H$, then conditions (ii) and (iii) hold for λ, μ in a small neighborhood of zero.

Set $B(\delta) = \{x \in H : E\|x\|_H^p \leq \delta\}$, $\delta > 0$. Then we have the following definitions of locally exponential behavior for mild solutions.

Definition 2.2 Let $p \geq 2$ be an integer, mild solution $x(t, \xi, \lambda)$ of Eq. (1) is said to be pth moment locally exponentially stable if there exist $\gamma > 0$, $L \geq 1$ such that

$$E\left\| x(t, \xi, \lambda) \right\|_H \leq L e^{-\gamma t} E\|\xi\|_H^p$$

for any initial value $\xi \in B(\delta)$, $\lambda \in Y$ and $t \geq 0$.

Definition 2.3 The mild solution $x(t, \xi, \lambda)$ of Eq. (1) is said to be almost surely exponentially stable if for any initial value $\xi \in B(\delta)$ and for any $\lambda \in Y$, there exists $\nu > 0$ such that

$$\limsup_{t \to +\infty} \frac{1}{t} \log \left\| x(t, \xi, \lambda) \right\|_H \leq -\nu, \quad \text{a.s.}$$

Lemma 2.1 [2] *For any $r \geq 1$ and for arbitrary $L_2^0(K, H)$-valued predictable process $\Phi(\cdot)$,*

$$\sup_{s \in [0, t]} E\left\| \int_0^s \Phi(u) \, d\omega(u) \right\|^{2r} \leq C_r \left(\int_0^t \left(E\|\Phi(s)\|_{L_2^0}^{2r} \right)^{\frac{1}{r}} ds \right)^r, \quad t \geq 0,$$

where $C_r = (r(2r - 1))^r$.

3 Stability and parameter dependence

In this section, we present and prove our main results. We consider the set $Z(\delta) = \{(t, \xi) : 0 \leq t \leq T, \xi \in B(\delta)\}$.

Theorem 3.1 *Assume that* (i)-(v) *hold. Then, for any $\delta > 0$ sufficiently small and for any $\lambda \in Y$, given $(t, \xi) \in Z(\delta)$, there exists a unique stochastic process $x(t, \xi, \lambda)$ satisfying* (2). *Moreover, the following properties hold*:

1. *There exists $0 < \alpha < p\eta$ such that*

$$2^p M^p d^p C_p \left[\frac{p}{2(p\eta - \alpha)} \right]^{\frac{p}{2}} < 1; \tag{3}$$

2. *$x(t, \xi, \lambda)$ is pth moment locally exponentially stable, that is, there exists $L \geq 1$ such that*

$$E \|x(t, \xi, \lambda)\|_H^p \leq L e^{-\alpha t} E \|\xi\|_H^p \tag{4}$$

   for any $\lambda \in Y$ and $(t, \xi) \in Z(\delta)$;
3. *There exists $K > 1$ such that*

$$E \|x(t, \xi, \lambda) - x(t, \xi, \mu)\|_H^p \leq K \|\lambda - \mu\|_Y^p e^{-\alpha t} \tag{5}$$

   for any $\lambda, \mu \in Y$ and $(t, \xi) \in Z(\delta)$.

Proof By (iv), it follows that there exists $0 < \alpha < \eta$ such that (3) holds. We consider the space χ of all $\mathscr{F}$-adapted processes $x = x(t, \xi, \lambda) : [0, +\infty) \times B(\delta) \times Y \to H$ such that

1. $x(0, \xi, \lambda) = \xi$ for any $\lambda \in Y$;
2. the norm $\|x\|_\chi = \frac{1}{2M} \sup_{t \geq 0} [e^{\alpha t} E \|x(t, \xi, \lambda)\|_H^p]$ satisfies $\|x\|_\chi \leq \delta$.

We can easily see that χ is a complete metric space. Define an operator $J_\lambda : \chi \to \chi$ by

$$
\begin{aligned}
J_\lambda(x)(t, \xi) &= S(t)\xi(0) + \int_0^t S(t-s)f(s, x(s, \xi, \lambda), \lambda) \, ds \\
&\quad + \int_0^t S(t-s)g(s, x(s, \xi, \lambda), \lambda) \, d\omega(s) \\
&= \sum_{i=1}^3 I_i(t)
\end{aligned} \tag{6}
$$

for any $x \in \chi$. We first verify the continuity of J_λ in χ. Let $x \in \chi$, $t_1 \geq 0$, and $|\epsilon|$ be sufficiently small, then

$$E \|J_\lambda(x)(t_1 + \epsilon) - J_\lambda(x)(t_1)\|_H^p \leq 3^{p-1} \sum_{i=1}^3 E \|I_i(t_1 + \epsilon) - I_i(t_1)\|_H^p. \tag{7}$$

Obviously,

$$E \|I_i(t_1 + \epsilon) - I_i(t_1)\|_H^p \to 0, \quad i = 1, 2, \tag{8}$$

as $\epsilon \to 0$. By Lemma 2.1, we have

$$
\begin{aligned}
& E \|I_3(t_1 + \epsilon) - I_3(t_1)\|_H^p \\
& \leq 2^{p-1} E \left\| \int_0^{t_1} (S(t_1 + \epsilon - s) - S(t_1 - s)) g(s, x(s, \xi, \lambda), \lambda) \, d\omega(s) \right\|_H^p \\
& \quad + 2^{p-1} E \left\| \int_{t_1}^{t_1 + \epsilon} S(t_1 + \epsilon - s) g(s, x(s, \xi, \lambda), \lambda) \, d\omega(s) \right\|_H^p
\end{aligned}
$$

$$\leq 2^{p-1} C_p \left(\int_0^{t_1} \left(E \left\| \left(S(t_1 + \epsilon - s) - S(t_1 - s) \right) g \left(s, x(s, \xi, \lambda), \lambda \right) \right\|_{L_2^0}^p \right)^{\frac{2}{p}} ds \right)^{\frac{p}{2}}$$

$$+ 2^{p-1} C_p \left(\int_{t_1}^{t_1 + \epsilon} \left(E \left\| S(t_1 + \epsilon - s) g \left(s, x(s, \xi, \lambda), \lambda \right) \right\|_{L_2^0}^p \right)^{\frac{2}{p}} ds \right)^{\frac{p}{2}} \to 0, \tag{9}$$

as $\epsilon \to 0$, where $C_p = (p(2p-1))^p$. Thus

$$\left\| J_\lambda(x)(t_1 + \epsilon) - J_\lambda(x)(t_1) \right\|_\chi \to 0, \tag{10}$$

as $\epsilon \to 0$, which implies that J_λ is continuous in χ on $[0, T]$. For each $x, y \in \chi$ and $0 \leq t \leq T$, by the Hölder inequality,

$$E \left\| x^* y \right\|_H \leq \left(E \|x\|_H^r \right)^{\frac{1}{r}} \left(E \|y\|_H^s \right)^{\frac{1}{s}}, \quad x, y \in H, \frac{1}{r} + \frac{1}{s} = 1, \tag{11}$$

where $*$ represents the transpose, and the Lyapunov inequality

$$\left(E \|x\|_H^m \right)^{\frac{1}{m}} \leq \left(E \|x\|_H^n \right)^{\frac{1}{n}}, \quad x \in H, 0 < m \leq n < +\infty \tag{12}$$

and (ii), we have

$$E \left\| f \left(s, x(s, \xi, \lambda), \lambda \right) - f \left(s, y(s, \xi, \lambda), \lambda \right) \right\|_H$$

$$\leq c E \left[\left\| x(s, \xi, \lambda) - y(s, \xi, \lambda) \right\|_H \left(\left\| x(s, \xi, \lambda) \right\|_H^q + \left\| y(s, \xi, \lambda) \right\|_H^q \right) \right]$$

$$\leq c \left(E \left\| x(s, \xi, \lambda) - y(s, \xi, \lambda) \right\|_H^{q+1} \right)^{\frac{1}{q+1}} \left[\left(E \left\| x(s, \xi, \lambda) \right\|_H^{q+1} \right)^{\frac{q}{q+1}} + \left(E \left\| y(s, \xi, \lambda) \right\|_H^{q+1} \right)^{\frac{q}{q+1}} \right]$$

$$\leq c \left(E \left\| x(s, \xi, \lambda) - y(s, \xi, \lambda) \right\|_H^p \right)^{\frac{1}{p}} \left[\left(E \left\| x(s, \xi, \lambda) \right\|_H^p \right)^{\frac{q}{p}} + \left(E \left\| y(s, \xi, \lambda) \right\|_H^p \right)^{\frac{q}{p}} \right]$$

$$\leq 2c (2M)^{\frac{q+1}{p}} e^{-\frac{(q+1)\alpha s}{p}} \delta^{\frac{q}{p}} \left(\left\| x(s, \xi, \lambda) - y(s, \xi, \lambda) \right\|_\chi \right)^{\frac{1}{p}}. \tag{13}$$

By (iii), we obtain

$$E \left\| g \left(s, x(s, \xi, \lambda), \lambda \right) - g \left(s, y(s, \xi, \lambda), \lambda \right) \right\|_{L_2^0}^p \leq d^p E \left\| x(s, \xi, \lambda) - y(s, \xi, \lambda) \right\|_H^p$$

$$\leq 2M d^p e^{-\alpha s} \left\| x(s, \xi, \lambda) - y(s, \xi, \lambda) \right\|_\chi. \tag{14}$$

Then, by (13), (14) and Lemma 2.1, we have

$$E \left\| J_\lambda(x)(t, \xi) - J_\lambda(y)(t, \xi) \right\|_H^p$$

$$\leq 2^{p-1} \left[E \left\| \int_0^t S(t - s) \left(f \left(s, x(s, \xi, \lambda), \lambda \right) - f \left(s, y(s, \xi, \lambda), \lambda \right) \right) ds \right\|_H^p \right.$$

$$\left. + E \left\| \int_0^t S(t - s) \left(g \left(s, x(s, \xi, \lambda), \lambda \right) - g \left(s, x(s, \xi, \lambda), \lambda \right) \right) d\omega(s) \right\|_H^p \right]$$

$$\leq 2^{p-1} \left[\left(\int_0^t E \left\| S(t - s) \left(f \left(s, x(s, \xi, \lambda), \lambda \right) - f \left(s, y(s, \xi, \lambda), \lambda \right) \right) \right\|_H ds \right)^p \right.$$

$$\left. + C_p \left(\int_0^t \left(E \left\| S(t - s) \left(g \left(s, x(s, \xi, \lambda), \lambda \right) - g \left(s, x(s, \xi, \lambda), \lambda \right) \right) \right\|_{L_2^0}^p \right)^{\frac{2}{p}} ds \right)^{\frac{p}{2}} \right]$$

$$\leq 2^{p-1}\left[E\left(\int_0^t Me^{-\eta(t-s)}\left\|f\left(s,x(s,\xi,\lambda),\lambda\right)-f\left(s,y(s,\xi,\lambda),\lambda\right)\right\|_H ds\right)^p\right.$$

$$\left.+ C_p\left(\int_0^t \left(M^p e^{-p\eta(t-s)}E\left\|g\left(s,x(s,\xi,\lambda),\lambda\right)-g\left(s,x(s,\xi,\lambda),\lambda\right)\right\|_{L_2^0}^p\right)^{\frac{2}{p}} ds\right)^{\frac{p}{2}}\right]$$

$$\leq \left[2^{2p+q}M^{p+q+1}c^p\delta^q e^{-\eta pt}\left(\int_0^t e^{\eta s-\frac{(q+1)\alpha s}{p}} ds\right)^p\right.$$

$$\left.+ 2^p M^{p+1}d^p C_p e^{-p\eta t}\left(\int_0^t e^{\frac{2p\eta s-2\alpha s}{p}} ds\right)^{\frac{p}{2}}\right]\|x-y\|_\chi$$

$$\leq \left\{2^{2p+q}M^{p+q+1}c^p\delta^q\left|\frac{p}{p\eta-\alpha(q+1)}\right|^p + 2^p M^{p+1}d^p C_p\left[\frac{p}{2(p\eta-\alpha)}\right]^{\frac{p}{2}}\right\}$$

$$\times e^{-\alpha t}\|x-y\|_\chi. \tag{15}$$

Therefore

$$\left\|J_\lambda(x)-J_\lambda(y)\right\|_\chi \leq \theta\|x-y\|_\chi,$$

where $\theta = 2^{2p+q-1}M^{p+q}c^p\delta^q\left|\frac{p}{p\eta-\alpha(q+1)}\right|^p + 2^{p-1}M^p d^p C_p\left[\frac{p}{2(p\eta-\alpha)}\right]^{\frac{p}{2}}$. Taking δ sufficiently small so that $\theta < \frac{1}{2}$, the operator J_λ becomes a contraction. In addition, we can obtain

$$\left\|J_\lambda(x)\right\|_\chi \leq \left\|S(t)\xi\right\|_\chi + \theta\|x\|_\chi \leq \frac{1}{2}\delta + \frac{1}{2}\delta = \delta. \tag{16}$$

This shows that $J_\lambda(\chi) \subset \chi$. Thus, there exists a unique function $x \in \chi$ such that $J_\lambda(x) = x$. By the above inequality, we have $\|x\|_\chi \leq \frac{\|\xi\|_H^p}{2(1-\theta)}$, which means that (4) holds. Writing $y_\lambda = x(\cdot,\xi,\lambda)$, we have

$$y_\lambda - y_\mu = J_\lambda y_\lambda - J_\mu y_\mu = J_\lambda y_\lambda - J_\lambda y_\mu + J_\lambda y_\mu - J_\mu y_\mu \tag{17}$$

for any $\lambda,\mu \in Y$. Thus

$$\|y_\lambda - y_\mu\|_\chi \leq \theta\|y_\lambda - y_\mu\|_\chi + \|J_\lambda y_\mu - J_\mu y_\mu\|_\chi. \tag{18}$$

It follows that

$$\|y_\lambda - y_\mu\|_\chi \leq (1-\theta)^{-1}\|J_\lambda y_\mu - J_\mu y_\mu\|_\chi. \tag{19}$$

Using the Lyapunov inequality

$$\left(E\|x\|_H^m\right)^{\frac{1}{m}} \leq \left(E\|x\|_H^n\right)^{\frac{1}{n}}, \quad x \in H, 0 < m \leq n < +\infty, \tag{20}$$

from (ii), (iii) and (4), we get

$$a(s) = E\left\|f\left(s,x(s,\xi,\mu),\lambda\right)-f\left(s,x(s,\xi,\mu),\mu\right)\right\|_H$$

$$\leq c\|\lambda-\mu\|_Y E\left\|x(s,\xi,\mu)\right\|_H^{q+1}$$

$$\leq c\|\lambda - \mu\|_Y \left(E\|x(s,\xi,\mu)\|_H^p\right)^{\frac{q+1}{p}}$$

$$\leq c(L\delta)^{\frac{q+1}{p}} e^{-\frac{(q+1)s}{p}} \|\lambda - \mu\|_Y \tag{21}$$

and

$$b(s) = E\|g(s,x(s,\xi,\mu),\lambda) - g(s,x(s,\xi,\mu),\mu)\|_{L_2^0}^p$$

$$\leq d^p \|\lambda - \mu\|_Y^p E\|x(s,\xi,\mu)\|_H^p$$

$$\leq d^p L\delta e^{-\alpha s} \|\lambda - \mu\|_Y^p. \tag{22}$$

Thus

$$\|(J_\lambda y_\mu)(t) - (J_\mu y_\mu)(t)\|_\chi$$

$$= \frac{1}{2M} \sup_{t \geq 0} \left[e^{\alpha t} E\|(J_\lambda y_\mu)(t) - (J_\mu y_\mu)(t)\|_H^p \right]$$

$$\leq \frac{2^{p-2}}{M} \sup_{t \geq 0} \left\{ e^{\alpha t} \left[E\left\| \int_0^t S(t-s)\left(f(s,x(s,\xi,\mu),\lambda) - f(s,x(s,\xi,\mu),\mu)\right) ds \right\|_H^p \right. \right.$$

$$\left. \left. + E\left\| \int_0^t S(t-s)g(s,x(s,\xi,\mu),\lambda) - g(s,x(s,\xi,\mu),\mu) \, d\omega(s) \right\|_H^p \right] \right\}$$

$$\leq \frac{2^{p-2}}{M} \sup_{t \geq 0} \left\{ e^{\alpha t} \left[\left(\int_0^t E\|S(t-s)\left(f(s,x(s,\xi,\mu),\lambda) - f(s,x(s,\xi,\mu),\mu)\right)\|_H \, ds \right)^p \right. \right.$$

$$\left. \left. + C_p \left(\int_0^t \left(E\|S(t-s)\left(g(s,x(s,\xi,\mu),\lambda) - g(s,x(s,\xi,\mu),\mu)\right)\|_{L_2^0}^p \right)^{\frac{2}{p}} ds \right)^{\frac{p}{2}} \right] \right\}$$

$$\leq 2^{p-2}M^{p-1} \sup_{t \geq 0} \left\{ e^{\alpha t} \left[\left(\int_0^t e^{-\eta(t-s)} a(s) \, ds \right)^p + C_p \left(\int_0^t \left(e^{-p\eta(t-s)} b(s) \right)^{\frac{2}{p}} ds \right)^{\frac{p}{2}} \right] \right\}$$

$$\leq 2^{p-2}M^{p-1} \sup_{t \geq 0} \left\{ e^{\alpha t} \left[c^p(L\delta)^{q+1} \left(\int_0^t e^{\frac{-p\eta t + p\eta s - (q+1)\alpha s}{p}} ds \right)^p \right. \right.$$

$$\left. \left. + L\delta d^p C_p \left(\int_0^t \left(e^{-p\eta(t-s)-\alpha s} \right)^{\frac{2}{p}} ds \right)^{\frac{p}{2}} \right] \right\} \|\lambda - \mu\|_Y^p$$

$$\leq \theta_1 \|\lambda - \mu\|_Y^p, \tag{23}$$

where $\theta_1 = 2^{p-2}M^{p-1} c^p (L\delta)^{q+1} \left| \frac{p}{p\eta - \alpha(q+1)} \right|^p + 2^{p-2}M^{p-1} d^p C_p L\delta \left[\frac{p}{2(p\eta - \alpha)} \right]^{\frac{p}{2}}$. Therefore

$$\|y_\lambda - y_\mu\|_\chi \leq \frac{\theta_1}{1-\theta} \|\lambda - \mu\|_Y^p, \tag{24}$$

which yields (5). This completes the proof of the theorem. □

Remark 3.1 From the proof of Theorem 3.1, if we take δ sufficiently small so that $\theta < \frac{1}{2}$, the mild solutions (1) with δ-domain initial value are p moment locally exponentially stable. In this case, the property of p moment stability is local.

Theorem 3.2 *Assume that* (i)-(v) *hold. Then, for any* $\delta > 0$ *sufficiently small, mild solution* $x(t, \xi, \lambda)$ *of* (1) *is almost surely exponentially stable, that is, there exists* $0 < \alpha < p\eta$ *such that*

$$\limsup_{t \to +\infty} \frac{1}{t} \log \|x(t, \xi, \lambda)\|_H \le -\frac{\alpha}{2p}, \quad a.s., \tag{25}$$

for any $\lambda \in Y$ *and* $\xi \in B(\delta)$. *Furthermore,*

$$\limsup_{t \to +\infty} \frac{1}{t} \log \frac{\|x(t, \xi, \lambda) - x(t, \xi, \mu)\|_H}{\|\lambda - \mu\|_Y} \le -\frac{\alpha}{2p}, \quad a.s., \tag{26}$$

for any $\lambda, \mu \in Y$, $\lambda \ne \mu$ *and* $\xi \in B(\delta)$.

Proof For any fixed positive real number $\varepsilon_N > 0$ ($N = 2, 3, \ldots$),

$$P\left(\sup_{N \le t \le N+1} \|x(t, \xi, \lambda)\|_H > \varepsilon_N \right)$$

$$\le \left(\frac{3}{\varepsilon_N} \right)^p E\left(\sup_{N \le t \le N+1} \|S(t-N)x(N, \xi, \lambda)\|_H^p \right)$$

$$+ \left(\frac{3}{\varepsilon_N} \right)^p E\left(\sup_{N \le t \le N+1} \left\| \int_N^t S(t-s)f(s, x(s, \xi, \lambda), \lambda) \, ds \right\|^p \right)$$

$$+ \left(\frac{3}{\varepsilon_N} \right)^p E\left(\sup_{N \le t \le N+1} \left\| \int_N^t S(t-s)g(s, x(s, \xi, \lambda), \lambda) \, d\omega(s) \right\|_H^p \right)$$

$$= \sum_{i=1}^3 I_i. \tag{27}$$

By (i) and (4), we have

$$I_1 \le \left(\frac{3}{\varepsilon_N} \right)^p M^p E \|x(N, \xi, \lambda)\|_H^p \le \left(\frac{3}{\varepsilon_N} \right)^p M^p L\delta e^{-\alpha N}. \tag{28}$$

It follows from (ii) and (4) that

$$E\|f(s, x(s, \xi, \lambda), \lambda\|_H \le cE\|x(s, \xi, \lambda)\|^{q+1}$$

$$\le c\left(E\|x(s, \xi, \lambda)\|^p\right)^{\frac{q+1}{p}} \le c(L\delta)^{\frac{q+1}{p}} e^{-\frac{(q+1)\alpha s}{p}}. \tag{29}$$

This implies that

$$I_2 \le c^p M^p (L\delta)^{q+1} \left(\frac{3}{\varepsilon_N} \right)^p \sup_{N \le t \le N+1} \left\| \int_N^t e^{-\eta(t-s)} e^{-\frac{(q+1)\alpha s}{p}} \, ds \right\|^p$$

$$\le c^p M^p (L\delta)^{q+1} \left(\frac{3}{\varepsilon_N} \right)^p \left| \frac{p}{p\eta - (q+1)\alpha} \right|^p e^{-\alpha N}. \tag{30}$$

From (iii) and (4), we have the following estimation:

$$E\|g(s, x(s, \xi, \lambda), \lambda\|_{L_2^0}^p \le d^p E\|x(s, \xi, \lambda)\|_H^p \le d^p L\delta e^{-\alpha s}, \tag{31}$$

and

$$I_3 \leq M^p C_p \left(\frac{3}{\varepsilon_N}\right)^p \sup_{N \leq t \leq N+1} \left(\int_N^t \left(e^{-p\eta(t-s)} E \|g(s, x(s, \xi, \lambda), \lambda)\|_{L_2^0}^p\right)^{\frac{2}{p}} ds\right)^{\frac{p}{2}}$$

$$\leq M^p d^p C_p L\delta \left(\frac{3}{\varepsilon_N}\right)^p \sup_{N \leq t \leq N+1} \left(\int_N^t \left(e^{-p\eta(t-s)} e^{-\alpha s}\right)^{\frac{2}{p}} ds\right)^{\frac{p}{2}}$$

$$\leq M^p d^p C_p L\delta \left(\frac{3}{\varepsilon_N}\right)^p \left[\frac{p}{2(p\eta - \alpha)}\right]^{\frac{p}{2}} e^{-\alpha N}. \tag{32}$$

Substituting (28), (30) and (32) into (27), we have

$$P\left\{\sup_{N \leq t \leq N+1} \|x(t, \xi, \lambda)\|_H > \varepsilon_N\right\} \leq \kappa \left(\frac{3}{\varepsilon_N}\right)^p e^{-\alpha N}, \tag{33}$$

where $\kappa = M^p L\delta + c^p M^p (L\delta)^{q+1} |\frac{p}{p\eta - (q+1)\alpha}|^p + M^p d^p C_p L\delta [\frac{p}{2(p\eta - \alpha)}]^{\frac{p}{2}}$. As ε_N is an arbitrarily given real number, let $\varepsilon_N = e^{-\frac{\alpha N}{2p}}$ such that

$$P\left\{\sup_{N \leq t \leq N+1} \|x(t, \xi, \lambda)\|_H > e^{-\frac{\alpha N}{2p}}\right\} \leq \kappa 3^p e^{-\frac{\alpha N}{2}}. \tag{34}$$

Consequently, from the Borel-Cantelli lemma, there exists $\overline{T}(\omega) > 0$ such that for any $t \geq \overline{T}(\omega)$,

$$\|x(t, \xi, \lambda)\|_H^p \leq e^{-\frac{\alpha t}{2p}}, \quad \text{a.s.} \tag{35}$$

It remains to verify that (26) holds. The argument is similar to the above proof of (27). For any $\varepsilon_N > 0$ ($N = 2, 3, \ldots$),

$$P\left(\sup_{N \leq t \leq N+1} \frac{\|x(t, \xi, \lambda) - x(t, \xi, \mu)\|_H}{\|\lambda - \mu\|_Y} > \varepsilon_N\right)$$

$$\leq \left(\frac{3}{\varepsilon_N}\right)^p E\left(\sup_{N \leq t \leq N+1} \frac{\|S(t-N)(x(N, \xi, \lambda) - x(N, \xi, \mu))\|_H^p}{\|\lambda - \mu\|_Y^p}\right)$$

$$+ \left(\frac{3}{\varepsilon_N}\right)^p E\left(\sup_{N \leq t \leq N+1} \frac{\|\int_N^t S(t-s)(f(s, x(s, \xi, \lambda), \lambda) - f(s, x(s, \xi, \mu), \mu)) ds\|^p}{\|\lambda - \mu\|_Y^p}\right)$$

$$+ \left(\frac{3}{\varepsilon_N}\right)^p E\left(\sup_{N \leq t \leq N+1} \frac{\|\int_N^t S(t-s)(g(s, x(s, \xi, \lambda), \lambda) - g(s, x(s, \xi, \mu), \mu)) d\omega(s)\|_H^p}{\|\lambda - \mu\|_Y^p}\right)$$

$$= \sum_{i=1}^{3} I_i'. \tag{36}$$

From (i) and (5), we obtain

$$I_1' \leq \left(\frac{3}{\varepsilon_N}\right)^p M^p K e^{-\alpha N}. \tag{37}$$

By (4), (5) and (ii), applying inequalities (12) and (13), we get

$$
\begin{aligned}
E\big\|f\big(s,&x(s,\xi,\lambda),\lambda\big)-f\big(s,x(s,\xi,\mu),\mu\big)\big\|_H \\
&\leq E\big\|f\big(s,x(s,\xi,\lambda),\lambda\big)-f\big(s,x(s,\xi,\mu),\lambda\big)\big\|_H \\
&\quad+E\big\|f\big(s,x(s,\xi,\mu),\lambda\big)-f\big(s,x(s,\xi,\mu),\mu\big)\big\|_H \\
&\leq cE\big[\big\|x(s,\xi,\lambda)-x(s,\xi,\mu)\big\|_H\big(\big\|x(s,\xi,\lambda)\big\|^q+\big\|x(s,\xi,\mu)\big\|^q\big)\big] \\
&\quad+cE\|\lambda-\mu\|_Y\big\|x(s,\xi,\mu)\big\|_H^{q+1} \\
&\leq c\big(E\big\|x(s,\xi,\lambda)-x(s,\xi,\mu)\big\|_H^p\big)^{\frac{1}{p}}\big[\big(E\big\|x(s,\xi,\lambda)\big\|^p\big)^{\frac{q}{p}}+\big(E\big\|x(s,\xi,\mu)\big\|^p\big)^{\frac{q}{p}}\big] \\
&\quad+c\|\lambda-\mu\|_Y E\big(\big\|x(s,\xi,\mu)\big\|_H^p\big)^{\frac{q+1}{p}} \\
&\leq c\big[2K^{\frac{1}{p}}(L\delta)^{\frac{q}{p}}+(L\delta)^{\frac{q+1}{p}}\big]e^{-\frac{(q+1)\alpha s}{p}}\|\lambda-\mu\|_Y.
\end{aligned}
\tag{38}
$$

This implies that

$$
\begin{aligned}
I_2' &\leq c^p M^p\Big(\frac{3}{\varepsilon_N}\Big)^p\big[2K^{\frac{1}{p}}(L\delta)^{\frac{q}{p}}+(L\delta)^{\frac{q+1}{p}}\big]^p\sup_{N\leq t\leq N+1}\Big\|\int_N^t e^{-\eta(t-s)}e^{-\frac{(q+1)\alpha s}{p}}\,ds\Big\|^p \\
&\leq c^p M^p\Big(\frac{3}{\varepsilon_N}\Big)^p\big[2K^{\frac{1}{p}}(L\delta)^{\frac{q}{p}}+(L\delta)^{\frac{q+1}{p}}\big]^p\Big|\frac{p}{p\eta-(q+1)\alpha}\Big|^p e^{-\alpha N}.
\end{aligned}
\tag{39}
$$

It follows from (4), (5) and (iii) that

$$
\begin{aligned}
E\big\|g\big(s,&x(s,\xi,\lambda),\lambda\big)-g\big(s,x(s,\xi,\mu),\mu\big)\big\|_{L_2^0}^P \\
&\leq 2^{P-1}\big[E\big\|g\big(s,x(s,\xi,\lambda),\lambda\big)-g\big(s,x(s,\xi,\mu),\lambda\big)\big\|_{L_2^0}^P \\
&\quad+E\big\|g\big(s,x(s,\xi,\mu),\lambda\big)-g\big(s,x(s,\xi,\mu),\mu\big)\big\|_{L_2^0}^P\big] \\
&\leq 2^{p-1}d^p\big[E\big\|x(s,\xi,\lambda)-x(s,\xi,\mu)\big\|_H^P+\|\lambda-\mu\|_Y^p E\big\|x(s,\xi,\mu)\big\|_H^P\big] \\
&\leq 2^{p-1}d^p\delta(L+K)e^{-\alpha s}\|\lambda-\mu\|_Y^p.
\end{aligned}
\tag{40}
$$

So

$$
\begin{aligned}
I_3' &\leq 2^{p-1}d^p M^p\delta C_p(L+K)\Big(\frac{3}{\varepsilon_N}\Big)^p\sup_{N\leq t\leq N+1}\Big(\int_N^t\big(e^{-p\eta(t-s)}e^{-\alpha s}\big)^{\frac{2}{p}}\,ds\Big)^{\frac{p}{2}} \\
&\leq 2^{p-1}d^p M^p\delta C_p\Big(\frac{3}{\varepsilon_N}\Big)^p(L+K)\Big[\frac{p}{2(p\eta-\alpha)}\Big]^{\frac{p}{2}}e^{-\alpha N}.
\end{aligned}
\tag{41}
$$

Substituting (37), (39) and (41) into (36), we have

$$
P\Big(\sup_{N\leq t\leq N+1}\frac{\|x(t,\xi,\lambda)-x(t,\xi,\mu)\|_H}{\|\lambda-\mu\|_Y}>\varepsilon_N\Big)\leq \overline{\kappa}\Big(\frac{3}{\varepsilon_N}\Big)^p e^{-\alpha N},
\tag{42}
$$

where $\overline{\kappa}=M^p K\delta+c^p M^p(\frac{3}{\varepsilon_N})^p[2K^{\frac{1}{p}}(L\delta)^{\frac{q}{p}}+(L\delta)^{\frac{q+1}{p}}]^p|\frac{p}{p\eta-(q+1)\alpha}|^p+2^{p-1}d^p M^p\delta C_p(L+K)\times[\frac{p}{2(p\eta-\alpha)}]^{\frac{p}{2}}$. We can proceed to the remaining proof in a similar manner. $\qquad\square$

4 Example

In this section, an example is provided to illustrate our results.

Example 4.1 Consider the following stochastic partial equation:

$$
dZ(t,x) = \left[\frac{\partial^2}{\partial x^2} Z(t,x) + \frac{\lambda^2 Z^2(t,x)}{1 + |Z(t,x)|} \right] dt + \frac{\lambda Z(t,x)}{20(1 + |Z(t,x)|)} \, d\omega(t), \quad t \geq 0,
$$

$$
Z(t,0) = Z(t,\pi) = 0, \quad t \geq 0,
$$

$$
Z(0,x) = \xi(x),
$$

(43)

where $\omega(t)$ is a standard one-dimensional Brownian motion, $\lambda \in Y = [\frac{1}{4}, \frac{1}{3}]$ is a parameter. Let $A = \frac{\partial^2}{\partial x^2}$ with the domain

$$
\mathcal{D}(A) = \left\{ u \in H : \frac{\partial}{\partial u}, \frac{\partial^2}{\partial u^2} \in H, u(0) = u(\pi) = 0 \right\},
$$

(44)

then $\|S(t)\|_H \leq e^{-t}, \forall t \geq 0$, which implies that $M = 1$. It is easy to see that (ii)-(iii) hold and $c = 1$, $q = 1$, $d = \frac{1}{20}$. Taking $p = 4$, we may show that $2^p d^p C_p (\frac{1}{2\pi^2})^{\frac{p}{2}} < 1$. Then the mild solution of Eq. (43) has locally stable behavior and parameter dependence for $\delta > 0$ sufficiently small.

5 Conclusion

This paper is devoted to locally stable behavior and parameter dependence for stochastic differential equations. Our results are derived under sufficiently small Lipschitz perturbation in the parameter system. Meanwhile, a lot of stochastic inequality techniques are used to estimate the bound of mild solution. The future research topics would be extending locally stable behavior and parameter dependence to the stochastic delayed parameter system.

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