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On a functional equation involving iterates and powers

Janusz Morawiec*

*Correspondence: morawiec@math.us.edu.pl
Institute of Mathematics, University of Silesia, Bankowa 14, Katowice, 40-007, Poland

Abstract

We present a complete list of all continuous solutions $f: (0,+\infty) \to (0,+\infty)$ of the equation $f^2(x) = \gamma[f(x)]^\alpha x^\beta$, where α, β and $\gamma > 0$ are given real numbers.

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1 Introduction

In this note we give a complete list of all continuous solutions $f: (0,+\infty) \to (0,+\infty)$ of the equation

$$f^2(x) = \gamma\big[f(x)\big]^\alpha x^\beta, \tag{1.1}$$

where α, β and $\gamma > 0$ are given real numbers; here and throughout, f^2 denotes the second iterate of f. The motivation for writing this note was two problems concerning continuous solutions $f: (0,+\infty) \to (0,+\infty)$ of some special cases of equation (1.1) (see [1, Problem 11, p.312] and [2, Problem 5, p.22]) as well as conference reports and papers on both problems (see [3, 4] and [5–8]). Let us mention that the problem from booklet [2] is wrongly solved in this booklet. To see that the problem from [1] concerns really equation (1.1) observe that from Remark 1.1 below it follows that in the case where $\beta \neq 0$ equation (1.1) can be rewritten in the form

$$f(x)\big[f^{-1}(x)\big]^{-\beta} = \gamma x^\alpha.$$

Remark 1.1 Assume $\beta \neq 0$. Then every continuous solution $f: (0,+\infty) \to (0,+\infty)$ of equation (1.1) is strictly monotone and maps $(0,+\infty)$ onto $(0,+\infty)$.

Proof Fix $x, y \in (0,+\infty)$ and assume that $f(x) = f(y)$. Then by (1.1) we get

$$\gamma\big[f(x)\big]^\alpha x^\beta = f^2(x) = f^2(y) = \gamma\big[f(y)\big]^\alpha y^\beta = \gamma\big[f(x)\big]^\alpha y^\beta,$$

and since $\beta \neq 0$, we obtain $x = y$. Thus f is injective. This jointly with continuity implies strict monotonicity.

Now suppose that, contrary to our claim, $\lim_{x\to 0} f(x) \in (0,+\infty)$. Then, by the continuity of f and (1.1), we obtain

$$f\Big(\lim_{x\to 0} f(x)\Big) = \lim_{x\to 0} f^2(x) = \lim_{x\to 0} \gamma\big[f(x)\big]^{\alpha} x^{\beta} \in \{0,+\infty\},$$

a contradiction. Thus $\lim_{x\to 0} f(x) \in \{0,+\infty\}$. In the same manner we can prove that $\lim_{x\to +\infty} f(x) \in \{0,+\infty\}$. □

2 Main results

To give a complete list of all continuous solutions $f\colon (0,+\infty)\to(0,+\infty)$ of equation (1.1), we will split our consideration into the following three cases: $\beta = 0$, $\alpha = 0 \neq \beta$ and $\alpha \neq 0 \neq \beta$. It turns out that the description of all continuous solutions of equation (1.1) in the first case is quite easy; whereas in the third case it is much more complicated than in the second one.

2.1 The case $\beta = 0$

If $\beta = 0$, then equation (1.1) reduces to the equation

$$f^2(x) = \gamma\big[f(x)\big]^{\alpha}. \tag{2.1}$$

Equation (2.1) was examined in [9–11]; *cf.* also [12, 13] and the references therein.

We begin with a (rather obvious and simple) characterization of general solutions $f\colon (0,+\infty)\to(0,+\infty)$ of equation (2.1).

Proposition 2.1 *A function* $f\colon (0,+\infty)\to(0,+\infty)$ *satisfies* (2.1) *if and only if*

$$f(x) = \gamma x^{\alpha} \tag{2.2}$$

for all $x \in f((0,+\infty))$.

Proof (⇒) If a function $f\colon (0,+\infty)\to(0,+\infty)$ satisfies (1.1), then for every $x = f(z) \in f((0,+\infty))$ we have $f(x) = f^2(z) = \gamma[f(z)]^{\alpha} = \gamma x^{\alpha}$.

(⇐) Fix a function $f\colon (0,+\infty)\to(0,+\infty)$ satisfying (2.2) for all $x \in f((0,+\infty))$. Then, for every $x \in (0,+\infty)$, we have $f(x) \in f((0,+\infty))$. Now, putting $f(x)$ in place of x in (2.2), we obtain (2.1). □

From Proposition 2.1 we obtain the following description of all continuous solutions $f\colon (0,+\infty)\to(0,+\infty)$ of equation (2.1).

Corollary 2.2 *Let* $f\colon (0,+\infty)\to(0,+\infty)$ *be a continuous solution of equation* (2.1). *Then either* f *has form* (2.2) *for all* $x \in (0,+\infty)$ *or there exists a proper subinterval* I (*open or closed or closed on one side*; *possible infinite or degenerated to a single point*) *of the half-line* $(0,+\infty)$ *satisfying*

$$\gamma x^{\alpha} \in I \quad \textit{for all } x \in I \tag{2.3}$$

such that f has form (2.2) *for all* $x \in I$,

$$\lim_{x \to y} f(x) = \gamma y^{\alpha} \quad \textit{for all } y \in \{\inf I, \sup I\} \setminus \{0, +\infty\} \tag{2.4}$$

and

$$f\bigl((0, +\infty) \setminus I\bigr) \subset I. \tag{2.5}$$

Moreover:

(i) *If* $\alpha < -1$, *then* $I = \{\gamma^{\frac{1}{1-\alpha}}\}$;
(ii) *If* $\alpha = -1$, *then* $\operatorname{cl} I = [A, \frac{\gamma}{A}]$ *with arbitrary* $A \in (0, \sqrt{\gamma}]$;
(iii) *If* $\alpha \in (-1, 0)$, *then* $\operatorname{cl} I = [A, B]$ *with arbitrary* $A \in (0, \gamma^{\frac{1}{1-\alpha}}]$ *and* $B \in [\gamma^{\frac{1}{1-\alpha}}, +\infty)$;
(iv) *If* $\alpha \in [0, 1)$, *then either* $\operatorname{cl} I = [0, B]$ *or* $\operatorname{cl} I = [A, B]$ *or* $\operatorname{cl} I = [A, +\infty)$ *with arbitrary* $A \in (0, \gamma^{\frac{1}{1-\alpha}}]$ *and* $B \in [\gamma^{\frac{1}{1-\alpha}}, +\infty)$;
(v) *If* $\alpha = 1 < \gamma$, *then* $\operatorname{cl} I = [A, +\infty)$ *with arbitrary* $A \in (0, +\infty)$;
(vi) *If* $\alpha = 1 = \gamma$, *then no restriction on* I;
(vii) *If* $\alpha = 1 > \gamma$, *then* $\operatorname{cl} I = (0, B]$ *with arbitrary* $B \in (0, +\infty)$;
(viii) *If* $\alpha \in (1, +\infty)$, *then either* $I = \{\gamma^{\frac{1}{1-\alpha}}\}$ *or* $\operatorname{cl} I = [0, B]$ *or* $\operatorname{cl} I = [A, +\infty)$ *with arbitrary* $A \in [\gamma^{\frac{1}{1-\alpha}}, +\infty)$ *and* $B \in (0, \gamma^{\frac{1}{1-\alpha}}]$.

Proof Put $I = f((0, +\infty))$.

If $I = (0, +\infty)$, then (2.2) holds for all $x \in (0, +\infty)$ by Proposition 2.1. Therefore, to the end of the proof, we assume that $I \neq (0, +\infty)$.

Since f is continuous, it follows that I is an interval. By Proposition 2.1 we see that (2.2) holds for all $x \in I$. Thus (2.3) holds. Condition (2.4) follows from the continuity of f and condition (2.5) is a consequence of the definition of I. This completes the proof of the main part of the result.

To prove the moreover part put $A = \inf I$ and $B = \sup I$. Since $I \neq (0, +\infty)$, it follows that $0 < A$ or $B < +\infty$.

Assume first that $\alpha < 0$.

We will show that $0 < A$ and $B < +\infty$.

If $B = +\infty$, then (2.3) implies $0 = \lim_{x \to B} \gamma x^{\alpha} \in \operatorname{cl} I$, which contradicts $I \neq (0, +\infty)$. Similarly, if $A = 0$, then (2.3) implies $+\infty = \lim_{x \to A^+} \gamma x^{\alpha} \in \operatorname{cl} I$, which contradicts $I \neq (0, +\infty)$.

Applying condition (2.3), we get $0 < A \leq \gamma B^{\alpha}$ and $\gamma A^{\alpha} \leq B < +\infty$. Hence $\gamma A^{\alpha} \leq B \leq \gamma^{-\frac{1}{\alpha}} A^{\frac{1}{\alpha}}$.

(i) If $\alpha < -1$, then $\gamma^{\frac{1}{1-\alpha}} \leq A \leq B \leq \gamma^{\frac{1}{1-\alpha}}$.
(ii) If $\alpha = -1$, then $\frac{\gamma}{A} \leq B \leq \frac{\gamma}{A}$.
(iii) If $\alpha \in (-1, 0)$, then $A \leq \gamma^{\frac{1}{1-\alpha}} \leq B$.

Assume now that $\alpha \geq 0$.

Then (2.3) yields $A \leq \gamma A^{\alpha}$ provided that $A > 0$ and $\gamma B^{\alpha} \leq B$ provided that $B < +\infty$.

(iv) If $\alpha \in [0, 1)$, then $A \leq \gamma^{\frac{1}{1-\alpha}}$ if $A > 0$ and $\gamma^{\frac{1}{1-\alpha}} \leq B$ if $B < +\infty$.
(v) If $\alpha = 1 < \gamma$, then $B = +\infty$ and no restriction on A.
(vi) If $\alpha = 1 = \gamma$, then no restriction on A and B.

(vii) If $\alpha = 1 > \gamma$, then $A = 0$ and no restriction on B.
(viii) If $\alpha > 1$, then $A \geq \gamma^{\frac{1}{1-\alpha}}$ if $A > 0$ and $B \leq \gamma^{\frac{1}{1-\alpha}}$ if $B < +\infty$. □

As a consequence of Corollary 2.2, we have the following direct construction of all continuous solutions $f\colon (0,+\infty) \to (0,+\infty)$ of equation (2.1) (*cf.* [13, Theorem 15.15]).

Corollary 2.3 *Let I be a suitably chosen interval from Corollary* 2.2 *for a given* $\alpha \in \mathbb{R}$, *and let* $f_0\colon I \to I$ *be a function given by* $f_0(x) = \gamma x^{\alpha}$. *Then every extension of* f_0 *to a continuous function* $f\colon (0,+\infty) \to (0,+\infty)$ *satisfying* (2.5) *is a solution of equation* (2.1).

2.2 The case $\alpha = 0 \neq \beta$

If $\alpha = 0 \neq \beta$, then equation (1.1) reduces to the equation

$$f^2(x) = \gamma x^{\beta}. \tag{2.6}$$

Charles Babbage was probably the first who looked for solutions of equation (2.6) in the case where $\beta = \gamma = 1$ (see [14]). For the case where $\gamma = -\beta = 1$, see [15–17]; *cf.* also [18]. Equation (2.6) is a particular case of the equation of iterative roots (see [12, 13, 19–21]). According to known results on the equation of iterative roots, we can formulate a theorem on continuous solutions $f\colon (0,+\infty) \to (0,+\infty)$ of equation (2.6).

Denote by $I(x,y)$ the closed interval $[\min\{x,y\}, \max\{x,y\}]$ with $x, y \in (0,+\infty)$.

Theorem 2.4

(i) *Assume* $\beta < 0$. *Then equation* (2.6) *has no continuous solution* $f\colon (0,+\infty) \to (0,+\infty)$.
(ii) *Assume* $\beta = \gamma = 1$.

(ii_1) *Then the formula* $f(x) = x$ *defines the unique continuous solution* $f\colon (0,+\infty) \to (0,+\infty)$ *of equation* (2.6).
(ii_2) *If* $f\colon (0,+\infty) \to (0,+\infty)$ *is a continuous and decreasing solution of equation* (2.6), *then there exists* $x_0 \in (0,+\infty)$ *such that* $f(x_0) = x_0$ *and* f *maps* $(0,x_0]$ *bijectively onto* $[x_0,+\infty)$. *Conversely, if* $x_0 \in (0,+\infty)$, *then every decreasing bijection* $f_0\colon (0,x_0] \to [x_0,+\infty)$ *such that* $f(x_0) = x_0$ *can be uniquely extended to a continuous and decreasing solution* $f\colon (0,+\infty) \to (0,+\infty)$ *of equation* (2.6).

(iii) *Assume* $\beta = 1 \neq \gamma$.

(iii_1) *Let* $x_0 \in (0,+\infty)$. *If* $f\colon (0,+\infty) \to (0,+\infty)$ *is a continuous and increasing solution of equation* (2.6), *then* f *maps* $I(x_0,f(x_0))$ *bijectively onto* $I(f(x_0),\gamma x_0)$. *Conversely, if* $x_1 \in \operatorname{Int} I(x_0,\gamma x_0)$, *then every increasing bijection* $f_0\colon I(x_0,x_1) \to I(x_1,\gamma x_0)$ *can be uniquely extended to a continuous and increasing solution* $f\colon (0,+\infty) \to (0,+\infty)$ *of equation* (2.6).
(iii_2) *Equation* (2.6) *has no continuous and decreasing solution from* $f\colon (0,+\infty) \to (0,+\infty)$.

(iv) *Assume* $0 < \beta \neq 1$.

(iv_1) *Let* $x_0 \in (0,\gamma^{\frac{1}{1-\beta}})$ *and let* $y_0 \in (\gamma^{\frac{1}{1-\beta}},+\infty)$. *If* $f\colon (0,+\infty) \to (0,+\infty)$ *is a continuous and increasing solution of equation* (2.6), *then* f *maps* $I(x_0,f(x_0)) \cup I(y_0,f(y_0))$ *bijectively onto* $I(f(x_0),\gamma x_0^{\beta}) \cup I(f(y_0),\gamma y_0^{\beta})$. *Conversely, if* $x_1 \in \operatorname{Int} I(x_0,\gamma x_0^{\beta})$

and $y_1 \in \operatorname{Int} I(y_0, \gamma y_0^\beta)$, *then every increasing bijection* $f_0 \colon I(x_0, x_1) \cup I(y_0, y_1) \to I(x_1, \gamma x_0^\beta) \cup I(y_1, \gamma y_0^\beta)$ *can be uniquely extended to a continuous and increasing solution* $f \colon (0, +\infty) \to (0, +\infty)$ *of equation* (2.6).

(iv_2) *If* $f \colon (0, +\infty) \to (0, +\infty)$ *is a continuous and decreasing solution of equation* (2.6), *then* f *maps* $(0, \gamma^{\frac{1}{1-\beta}}]$ *bijectively onto* $[\gamma^{\frac{1}{1-\beta}}, +\infty)$ *and* $\gamma[f(x)]^\beta = f(\gamma x^\beta)$ *for all* $x \in (0, +\infty)$. *Conversely, every decreasing bijection* $f_0 \colon (0, \gamma^{\frac{1}{1-\beta}}] \to [\gamma^{\frac{1}{1-\beta}}, +\infty)$ *such that* $\gamma[f_0(x)]^\beta = f_0(\gamma x^\beta)$ *for all* $x \in (0, \gamma^{\frac{1}{1-\beta}}]$ *can be uniquely extended to a continuous and decreasing solution* $f \colon (0, +\infty) \to (0, +\infty)$ *of equation* (2.6).

Proof All the assertions can be derived from [13, Chapter XV]) as it has been noticed earlier. However, most of the assertions have evident proofs, so we present them for the convenience of the reader.

(i) Suppose that, contrary to our claim, equation (2.6) has a continuous solution $f \colon (0, +\infty) \to (0, +\infty)$. Then the second iterate f^2 of f is strictly increasing. Now by (2.6) we conclude that $\beta > 0$, a contradiction.

(ii_1) It is clear that the identity function on $(0, +\infty)$ satisfies (2.6). Suppose the contrary, and let $f \colon (0, +\infty) \to (0, +\infty)$ be another increasing solution of equation (2.6). Then there exists $x \in (0, +\infty)$ such that $f(x) \neq x$. By the monotonicity of f, we conclude that $f^2(x) \neq x$, which contradicts (2.6).

(ii_2) The first assertion is clear. To prove the second one, fix a decreasing bijection $f_0 \colon (0, x_0] \to [x_0, +\infty)$ such that $f(x_0) = x_0$ and extend it to a function $f \colon (0, +\infty) \to (0, +\infty)$ putting $f(x) = f_0^{-1}(x)$ for all $x \in (x_0, +\infty)$. It is easy to see that f is a decreasing bijection satisfying (2.6).

(iii_1) The first assertion is evident. The second one can be deduced from [13, Lemma 15.6].

(iii_2) Suppose, to derive a contradiction, that $f \colon (0, +\infty) \to (0, +\infty)$ is a continuous and decreasing solution of equation (2.6). Then there exists $x \in (0, +\infty)$ such that $f(x) = x$. Hence $f^2(x) = x \neq \gamma x$, a contradiction.

(iv_1) The first assertion is easy to verify. The second one can be inferred from [13, Theorem 15.7].

(iv_2) Let f be a continuous and decreasing solution of equation (2.6). Then there exists $x_0 \in (0, +\infty)$ such that $f(x_0) = x_0$. Hence by (2.6) we get $x_0 = f^2(x_0) = \gamma x_0^\beta$, and thus $x_0 = \gamma^{\frac{1}{1-\beta}}$. Consequently, f maps $(0, \gamma^{\frac{1}{1-\beta}}]$ bijectively onto $[\gamma^{\frac{1}{1-\beta}}, +\infty)$. Moreover, (2.6) yields $f(\gamma x^\beta) = f^3(x) = \gamma[f(x)]^\beta$ for all $x \in (0, +\infty)$. To prove the second part of the assertion, fix a decreasing bijection $f_0 \colon (0, \gamma^{\frac{1}{1-\beta}}] \to [\gamma^{\frac{1}{1-\beta}}, +\infty)$ such that $\gamma[f_0(x)]^\beta = f_0(\gamma x^\beta)$ for all $x \in (0, \gamma^{\frac{1}{1-\beta}}]$ and extend it to a function $f \colon (0, +\infty) \to (0, +\infty)$ putting $f(x) = f_0^{-1}(\gamma x^\beta)$ for all $x \in (\gamma^{\frac{1}{1-\beta}}, +\infty)$. It is easy to calculate that f is a decreasing bijection satisfying (2.6). □

2.3 The case $\alpha \neq 0 \neq \beta$

In this case an explicit description of all continuous solutions $f \colon (0, +\infty) \to (0, +\infty)$ of equation (1.1) is much more involved than in both the previous cases. We begin with an observation which allows us to rewrite equation (1.1) in an equivalent form.

Lemma 2.5 *If* $f\colon (0,+\infty) \to (0,+\infty)$ *is a solution of equation* (1.1), *then the function* $g\colon \mathbb{R} \to \mathbb{R}$ *given by* $g(x) = \log f(e^x)$ *satisfies*

$$g^2(x) = \log\gamma + \alpha g(x) + \beta x. \tag{2.7}$$

Conversely, if $g\colon \mathbb{R} \to \mathbb{R}$ *is a solution of equation* (2.7), *then the function* $f\colon (0,+\infty) \to (0,+\infty)$ *given by* $f(x) = e^{g(\log x)}$ *satisfies* (1.1).

Equation (2.7) is a special case of the polynomial-like iterative inhomogeneous equation

$$\sum_{n=0}^{N} a_n f^n(x) = b, \tag{2.8}$$

where all a_n's and b are real numbers and f is an unknown self-mapping; here f^n denotes the nth iterate of f. For the theory of equation (2.8) and its generalizations, we refer the readers to [22–30]. The problem of finding all continuous solutions of equation (2.8) seems to be very difficult. It is completely solved in [31] for $N = 2$, but it is still open even in the case where $N = 3$ (see [32]). It turns out that the nature of continuous solutions of equation (2.8) depends on the behavior of complex roots $r_1, \ldots, r_N$ of its characteristic equation $\sum_{n=0}^{N} \alpha_n r^n = 0$. This characteristic equation is obtained by putting $f(x) = rx$ into (2.8) with $b = 0$; in this way we can determine all linear solutions of the homogeneous counterpart of equation (2.8), and then all affine solutions of equation (2.8). Therefore, to formulate our result on continuous solutions $f\colon (0,+\infty) \to (0,+\infty)$ of equation (1.1), denote by r_1 and r_2 the complex roots of the equation

$$r^2 - \alpha r - \beta = 0.$$

By our assumption, we have $r_1 + r_2 = \alpha \neq 0$ and $r_1 r_2 = -\beta \neq 0$.

Combining Lemma 2.5 with the results from [31], we get the following theorem.

Theorem 2.6

(i) *Assume* $r_1, r_2 \in \mathbb{C} \setminus \mathbb{R}$. *Then equation* (1.1) *has no continuous solution* $f\colon (0,+\infty) \to (0,+\infty)$.

(ii) *Assume* $r_1, r_2 \in \mathbb{R} \setminus \{1\}$. *Let* $\xi = \gamma^{\frac{1}{(r_1-1)(r_2-1)}}$ *and* $f_1, f_2\colon (0,+\infty) \to (0,+\infty)$ *be defined by*

$$f_1(x) = \gamma^{-\frac{1}{r_2-1}} x^{r_1}, \qquad f_2(x) = \gamma^{-\frac{1}{r_1-1}} x^{r_2}.$$

(ii$_1$) *Assume either* $1 < r_1 < r_2$ *or* $0 < r_1 < r_2 < 1$. *Let* $0 < x_0 < \xi < y_0$. *If* $f\colon (0,+\infty) \to (0,+\infty)$ *is a continuous solution of equation* (1.1), *then* $f(\xi) = \xi$, $f_0 = f|_{I(x_0,f(x_0))\cup I(y_0,f(y_0))}$ *is continuous,*

$$f_0(x_0) \in I\big(f_1(x_0), f_2(x_0)\big), \qquad f_0^2(x_0) = \gamma\big[f_0(x_0)\big]^{\alpha} x_0^{\beta}, \tag{2.9}$$

$$f_0(y_0) \in I\big(f_1(y_0), f_2(y_0)\big), \qquad f_0^2(y_0) = \gamma\big[f_0(y_0)\big]^{\alpha} y_0^{\beta} \tag{2.10}$$

and

$$\left(\frac{y}{x}\right)^{r_1} \le \frac{f_0(y)}{f_0(x)} \le \left(\frac{y}{x}\right)^{r_2} \tag{2.11}$$

for all $x, y \in I(x_0, f(x_0)) \cup I(y_0, f(y_0))$. *Conversely, every continuous function* $f_0 \colon I(x_0, f_0(x_0)) \cup I(y_0, f_0(y_0)) \to \mathbb{R}$ *such that* (2.9) *and* (2.10) *are satisfied and* (2.11) *holds for all* $x, y \in I(x_0, f_0(x_0)) \cup I(y_0, f_0(y_0))$ *can be uniquely extended to a continuous solution* $f \colon (0, +\infty) \to (0, +\infty)$ *of equation* (1.1).

(ii$_2$) *Assume* $0 < r_1 < 1 < r_2$. *Let* $x_0 > 0$. *If* $f \colon (0, +\infty) \to (0, +\infty)$ *is a continuous solution of equation* (1.1), *then either* $f|_{(0,\xi]} \in \{f_1|_{(0,\xi]}, f_2|_{(0,\xi]}\}$ *and* $f|_{(\xi,+\infty)} \in \{f_1|_{(\xi,+\infty)}, f_2|_{(\xi,+\infty)}\}$ *or* $f_0 = f|_{I(x_0, f(x_0))}$ *is continuous,*

$$f_0(x_0) \notin I\big(f_1(x_0), f_2(x_0)\big), \qquad f_0^2(x_0) = \gamma \big[f_0(x_0)\big]^{\alpha} x_0^{\beta} \tag{2.12}$$

and (2.11) *holds for all* $x, y \in I(x_0, f(x_0))$. *Conversely, every continuous function* $f_0 \colon I(x_0, f_0(x_0)) \to \mathbb{R}$ *such that* (2.12) *is satisfied and* (2.11) *holds for all* $x, y \in I(x_0, f_0(x_0))$ *can be uniquely extended to a continuous solution* $f \colon (0, +\infty) \to (0, +\infty)$ *of equation* (1.1).

(ii$_3$) *Assume either* $r_1 < r_2 < -1$ *or* $-1 < r_1 < r_2 < 0$. *Let* $x_0 > \xi$. *If* $f \colon (0, +\infty) \to (0, +\infty)$ *is a continuous solution of equation* (1.1), *then* $f(\xi) = \xi$ *and* $f_0 = f|_{I(x_0, f^2(x_0))}$ *is continuous,*

$$f_0(x_0) \in I\big(f_1(x_0), f_2(x_0)\big), \qquad f_0^3(x_0) = \gamma \big[f_0^2(x_0)\big]^{\alpha} x_0^{\beta}, \tag{2.13}$$

and (2.11) *holds for all* $x, y \in I(x_0, f^2(x_0))$. *Conversely, every continuous function* $f_0 \colon I(x_0, f_0^2(x_0)) \to \mathbb{R}$ *such that* (2.13) *is satisfied and* (2.11) *holds for all* $x, y \in I(x_0, f_0^2(x_0))$ *can be uniquely extended to a continuous solution* $f \colon (0, +\infty) \to (0, +\infty)$ *of equation* (1.1).

(ii$_4$) *Assume either* $r_1 < r_2 = -1$ *or* $-1 = r_2 < r_1 < 0$. *Then every continuous solution* $f \colon (0, +\infty) \to (0, +\infty)$ *of equation* (1.1) *is of the form*

$$f(x) = \begin{cases} a^{r_1+1} \xi^{1-r_1} x^{r_1} & \text{for } x \in (0, a^{-1}\xi], \\ \xi^2 x^{-1} & \text{for } x \in (a^{-1}\xi, a\xi), \\ a^{-r_1-1} \xi^{1-r_1} x^{r_1} & \text{for } x \in [a\xi, +\infty) \end{cases}$$

with some $a \in [1, +\infty]$.

(ii$_5$) *Assume either* $r_1 < 0 < r_2$ *or* $r_1 < -1 < r_2 < 0$. *Then* f_1 *and* f_2 *are the only continuous solutions from* $(0, +\infty)$ *to* $(0, +\infty)$ *of equation* (1.1).

(iii) *Assume* $r_2 = 1 \neq r_1$. *Let* $f_3 \colon (0, +\infty) \to (0, +\infty)$ *be defined by*

$$f_3(x) = \gamma^{-\frac{1}{r_1 - 1}} x.$$

(iii$_1$) *Assume* $r_1 > 1 \neq \gamma$. *If* $f \colon (0, +\infty) \to (0, +\infty)$ *is a continuous solution of equation* (1.1), *then* $f_0 = f|_{I(x_0, f(x_0))}$ *is continuous,*

$$f_0(x_0) \le f_3(x_0), \qquad f_0^2(x_0) = \gamma \big[f_0(x_0)\big]^{\alpha} x_0^{\beta} \tag{2.14}$$

and

$$\frac{y}{x} \leq \frac{f_0(y)}{f_0(x)} \leq \left(\frac{y}{x}\right)^{r_1} \tag{2.15}$$

for all $x, y \in I(x_0, f(x_0))$. *Conversely, every continuous function* $f_0 \colon I(x_0, f_0(x_0)) \to \mathbb{R}$ *such that* (2.14) *is satisfied and* (2.15) *holds for all* $x, y \in I(x_0, f_0(x_0))$ *can be uniquely extended to a continuous solution* $f \colon (0, +\infty) \to (0, +\infty)$ *of equation* (1.1).

(iii$_2$) *Assume* $0 < r_1 < 1 \neq \gamma$. *If* $f \colon (0, +\infty) \to (0, +\infty)$ *is a continuous solution of equation* (1.1), *then* $f_0 = f|_{I(x_0, f(x_0))}$ *is continuous,*

$$f_0(x_0) \geq f_3(x_0), \qquad f_0^2(x_0) = \gamma \left[f_0(x_0)\right]^{\alpha} x_0^{\beta} \tag{2.16}$$

and

$$\left(\frac{y}{x}\right)^{r_1} \leq \frac{f_0(y)}{f_0(x)} \leq \frac{y}{x} \tag{2.17}$$

for all $x, y \in I(x_0, f(x_0))$. *Conversely, every continuous function* $f_0 \colon I(x_0, f_0(x_0)) \to \mathbb{R}$ *such that* (2.16) *is satisfied and* (2.17) *holds for all* $x, y \in I(x_0, f_0(x_0))$ *can be uniquely extended to a continuous solution* $f \colon (0, +\infty) \to (0, +\infty)$ *of equation* (1.1).

(iii$_3$) *Assume* $r_1 > 0$ *and* $\gamma = 1$. *Then every continuous solution* $f \colon (0, +\infty) \to (0, +\infty)$ *of equation* (1.1) *is of the form*

$$f(x) = \begin{cases} a^{1-r_1} x^{r_1} & \text{for } x \in (0, a], \\ x & \text{for } x \in (a, b), \\ b^{1-r_1} x^{r_1} & \text{for } x \in [b, +\infty) \end{cases}$$

with some $0 \leq a \leq b \leq +\infty$.

(iii$_4$) *Assume* $r_1 < 0$ *and* $\gamma \neq 1$. *Then* f_3 *is the unique continuous solution from* $(0, +\infty)$ *to* $(0, +\infty)$ *of equation* (1.1).

(iii$_5$) *Assume* $r_1 < 0$ *and* $\gamma = 1$. *Then* f_3 *is a continuous solution of equation* (1.1) *and every other continuous solution* $f \colon (0, +\infty) \to (0, +\infty)$ *of equation* (1.1) *is of the form*

$$f(x) = a x^{r_1}$$

with some $a \in (0, +\infty)$.

(iv) *Assume* $r_1 = r_2$.

(iv$_1$) *Assume* $r_1 \neq 1$. *Then the formula*

$$f(x) = \gamma^{-\frac{1}{r_1 - 1}} x^{r_1}$$

defines the unique continuous solution $f \colon (0, +\infty) \to (0, +\infty)$ *of equation* (1.1).

(iv$_2$) *Assume* $r_1 = 1 \neq \gamma$. *Then equation* (1.1) *has no continuous solution* $f: (0, +\infty) \to (0, +\infty)$.

(iv$_3$) *Assume* $r_1 = \gamma = 1$. *Then every continuous solution* $f: (0, +\infty) \to (0, +\infty)$ *of equation* (1.1) *is of the form*

$$f(x) = ax$$

with some $a \in (0, +\infty)$.

Competing interests
The author declares that he has no competing interests.

Author's contributions
The author wrote the whole article, read and approved the final manuscript.

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