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# Meromorphic solutions of difference Painlevé IV equations

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**Abstract**

In this paper, we investigate the family of difference Painlevé IV equation $(w(z+1)+w(z))(w(z)+w(z-1)) = R(z,w)$, where $R$ is rational in $w$ and meromorphic in $z$. If the equation assumes an admissible meromorphic solution of hyper-order $\rho_2(w) < 1$, we fix the degree of $R(z,w)$, and we give some further discussions.

**MSC:** 39A10

**Keywords:** Painlevé difference equation; meromorphic solution; zeros

## 1 Introduction

Meromorphic solutions of complex difference equations have become a subject of great interest recently, due to the application of classical Nevanlinna theory in difference by Ablowitz *et al.* [1]. Halburd and Korhonen [2] studied the second order complex difference equation

$$w(z+1) + w(z-1) = R(z,w), \tag{1.1}$$

where $R(z,w)$ is rational in $w$ and meromorphic in $z$. They showed that if (1.1) has an admissible finite order meromorphic solution, then either $w$ satisfies a difference Riccati equation or (1.1) can be transformed into difference Painlevé I, II equations or a linear difference equation. The work on the family $w(z+1)w(z-1) = R(z,w)$, which includes the difference Painlevé III equations, was initiated in [3], with an additional assumption that the order of the poles of $w$ is bounded in a certain subcase of this family. Ronkainen [4] gave the full classification of $w(z+1)w(z-1) = R(z,w)$ in his dissertation without the assumption of the order of the poles of $w$, and in [4] his also studied the family $(w(z)w(z+1)-1)(w(z)w(z-1)-1) = R(z,w)$, which includes the difference Painlevé V equations. The family of difference Painlevé IV equations (see [5])

$$\big(w(z+1) + w(z)\big)\big(w(z) + w(z-1)\big) = R(z,w) \tag{1.2}$$

with constant coefficients was researched by Grammaticos *et al.* [6]. Korhonen [7] treated a subcase of (1.2) with $R(z,w)$ rational in $z$. In this paper, we will study (1.2) with $R(z,w)$ meromorphic in $z$.

Considering a meromorphic function $f(z)$ in the complex plane, we assume that the reader is familiar with the standard notations and results of Nevanlinna value distribution theory of meromorphic function (see, *e.g.*, [8–10]). In particular, we denote by $S(r,f)$ any quantity satisfying $S(r,f) = o(T(r,f))$ as $r$ tends to infinity outside of an exceptional set $E$ of finite logarithmic measure

$$\lim_{r\to\infty}\int_{E\cap[1,r)} dt/t < \infty.$$

A meromorphic function $a(z)$ is called a small function with respect to $f(z)$, if $T(r,a) = S(r,f)$. The family of all meromorphic functions that are small with respect to $f$ is denoted by $\mathcal{S}(f)$. A meromorphic solution $w$ of a difference equation is called *admissible* if all coefficients of the equation are in $\mathcal{S}(w)$.

We conclude this section by the following expatiation on the coefficients. While we only consider meromorphic solutions of equations with meromorphic coefficients, we might encounter a situation where the coefficients have finitely-sheeted branchings, and we will use the algebroid version of Nevanlinna theory (see for instance [11]), which studies meromorphic functions on a finitely-sheeted Riemann surface. Whenever the coefficients have branchings, $T(r,\cdot)$ denotes the corresponding Nevanlinna characteristic function of a finite-sheeted algebroid function. Since all algebroid functions we need to consider are small functions with respect to the meromorphic solution of (1.2), the change in notation only affects the error term which needs to be redefined in terms of the algebroid characteristic. The 'algebroid error term' will still be denoted by $S(r,\cdot)$ and it remains small with respect to the meromorphic solution of (1.2), we can still denote it by $S(r,w)$.

## 2 Some lemmas

The difference analog of the logarithmic derivative lemma, which was obtained independently by Halburd and Korhonen [12] and by Chiang and Feng [13], plays a key role in the value distribution of difference [14–16]. The original version is valid for functions of finite order, and it was generalized to hold for meromorphic functions with hyper-order less than one recently.

**Lemma 2.1** ([17]) *Let $w$ be a nonconstant meromorphic function with hyper-order $\rho_2(w) = \rho_2 < 1$, $c \in \mathbb{C}$, and $\delta \in (0, 1-\rho_2)$. Then*

$$m\left(r, \frac{w(z+c)}{w(z)}\right) = o\left(\frac{T(r,w)}{r^{\delta}}\right)$$

*for all $r$ outside of a set of finite logarithmic measure.*

The Valiron-Mohon'ko identity [18, 19] is a useful tool to estimate the characteristic function of a rational function, the proof of which can be found in [9, Theorem 2.2.5].

**Lemma 2.2** *Let $w$ be a meromorphic function and $R(z,w)$ a function which is rational in $w$ and meromorphic in $z$. If all the coefficients of $R$ are small compared to $w$, then*

$$T\bigl(r, R(z,w)\bigr) = (\deg_w R)T(r,w) + S(r,w).$$

We also need the following lemma to detect the hyper-order of a meromorphic function to be at least one.

**Lemma 2.3** *Let $T : [0, \infty) \to [0, \infty)$ be a nondecreasing continuous function, and $s \in (0, \infty)$. If*

$$\limsup_{r\to\infty} \frac{\log\log T(r)}{\log r} = \rho_2 < 1,$$

*and $\delta \in (0, 1 - \rho_2)$, then*

$$T(r+s) = T(r) + o\left(\frac{T(r)}{r^\delta}\right),$$

*as $r$ tends to infinity outside of an exceptional set of finite logarithmic measure.*

**Proposition 2.4** *Let $w$, $c$, and $\delta$ be as in Lemma* 2.1. *We have*

$$T\bigl(r, w(z+c)\bigr) = T\bigl(r, w(z)\bigr) + S(r, w), \tag{2.1}$$

$$N\bigl(r, w(z+c)\bigr) = N\bigl(r, w(z)\bigr) + S(r, w). \tag{2.2}$$

*Proof* Noting that $N(r, w(z+c)) \le N(r + |c|, w(z))$, Lemma 2.3 yields

$$N\bigl(r, w(z+c)\bigr) \le N\bigl(r, w(z)\bigr) + o\left(\frac{N(r, w(z))}{r^\delta}\right),$$

$$\begin{aligned} N\bigl(r, w(z)\bigr) &= N\bigl(r, w(z+c-c)\bigr) \le N\bigl(r+|c|, w(z+c)\bigr) \\ &= N\bigl(r, w(z+c)\bigr) + o\left(\frac{N(r, w(z+c))}{r^\delta}\right). \end{aligned}$$

Hence we obtain $N(r, w(z+c)) = O(N(r, w(z)))$ and

$$N\bigl(r, w(z+c)\bigr) = N\bigl(r, w(z)\bigr) + o\left(\frac{N(r, w(z))}{r^\delta}\right),$$

(2.2) follows. On the other hand, Lemma 2.1 gives us

$$m\bigl(r, w(z+c)\bigr) \le m\bigl(r, w(z)\bigr) + m\left(r, \frac{w(z+c)}{w(z)}\right) = m\bigl(r, w(z)\bigr) + S(r, w) \tag{2.3}$$

and

$$m\bigl(r, w(z)\bigr) \le m\bigl(r, w(z+c)\bigr) + S\bigl(r, w(z+c)\bigr). \tag{2.4}$$

Combining with (2.3) and (2.2), we have

$$T\bigl(r, w(z+c)\bigr) \le T\bigl(r, w(z)\bigr) + S\bigl(r, w(z)\bigr),$$

which means that $T(r, w(z+c)) = O(T(r, w(z)))$, and we get from (2.3) and (2.4)

$$m\bigl(r, w(z+c)\bigr) = m\bigl(r, w(z)\bigr) + S\bigl(r, w(z)\bigr).$$

Then (2.1) follows from the last equation and (2.2). □

Without the order restriction, we have the following.

**Lemma 2.5** ([1, lemma 1]) *Given* $\epsilon > 0$ *and a meromorphic function* $w$, *we have*

$$T\bigl(r, w(z \pm 1)\bigr) \leq (1+\epsilon) T\bigl(r+1, w(z)\bigr) + O(1)$$

*for all* $r \geq 1/\epsilon$.

If a meromorphic function $f$ has a pole of the order $n$ at $z_0 \in \mathbb{C}$, we denote this by $f(z_0) = \infty^n$. Similarly, an $a$-point of the order $n$ is denoted by $f(z_0) = a + 0^n$.

It could always happen that the coefficients in (1.2) have poles or zeros when $w$ has a pole; whenever we meet these cases, we shall use the following result.

**Lemma 2.6** ([2]) *Let* $w(z)$ *be a meromorphic function with more than* $S(r,w)$ *poles* (*or* $c$*-points*, $c \in \mathbb{C}$) *counting multiplicities, and let* $a_1, a_2, \ldots, a_n \in \mathcal{S}(w)$. *Assume moreover that none of the functions* $a_i$ *is identically zero. Denote by* $z_j$ *the poles and zeros of the functions* $a_i$ (*where* $j$ *is in some index set*), *and let*

$$m_j := \max_{1 \leq j \leq n} \bigl\{ l_i \in \mathbb{N} \mid a_i(z_j) = \infty^{l_i} \text{ or } a_i(z_j) = 0^{l_i} \bigr\}$$

*be the maximal order of zeros and poles of the functions* $a_i$ *at* $z_j$. *Then for any* $\epsilon > 0$ *there are at most* $S(r,w)$ *points* $z_j$ *such that* $w(z_j) = \infty^{k_i}$ (*or* $w(z_j) = c + 0^{k_j}$) *where* $m_j \geq \epsilon k_j$.

The next result on the Nevanlinna characteristic is essential in the study of the family (1.2), for the proof, see, *e.g.*, [6].

**Lemma 2.7** *Let* $f$, $h$, *and* $g$ *be three meromorphic functions. Then*

$$T(r, fg + gh + hf) \leq T(r,f) + T(r,g) + T(r,h) + O(1).$$

## 3 Main results

**Definition 3.1** ([7]) Let $c_j \in \mathbb{C}$ for $j = 1, \ldots, n$ and let $I$ be a finite set of multi-indexes $\lambda = (\lambda_0, \ldots, \lambda_n)$. A difference polynomial of $w$ is defined as

$$\begin{aligned} P(z,w) &= P\bigl(z, w(z), w(z+c_1), \ldots, w(z+c_n)\bigr) \\ &= \sum_{\lambda \in I} a_\lambda(z) w(z)^{\lambda_0} w(z+c_1)^{\lambda_1} \cdots w(z+c_n)^{\lambda_n}, \end{aligned} \tag{3.1}$$

where $a_\lambda(z) \in \mathcal{S}(w)$. Let $d_\lambda = \lambda_0 + \cdots + \lambda_n$. The degree of $P$ is defined by

$$\deg_w P = \max_{\lambda \in I} d_\lambda.$$

$P$ is said to be homogeneous if $d_\lambda$ of each term in the sum (3.1) is nonzero and the same for all $\lambda \in I$. The order of a zero of $P(z, x, x_1, \ldots, x_n)$, as a function of $x_0$ at $x_0 = 0$, is denoted by $\operatorname{ord}_0(P)$. Let $\Lambda_0$ be the maximum power of $w(z)$ in $P(z,w)$, and let $\Lambda_i$ be the maximum power of $w(z+c_i)$ in $P(z, w(z))$. Obviously, $\Lambda_i \geq 1$ $(i = 0, \ldots, n)$. Denote $\Lambda = \sum_{i=0}^{n} \Lambda_i$.

A difference version of the Clunie lemma, developed by Halburd and Korhonen [12] and by Laine and Yang [20], works on such difference polynomials, say $P(z,w)$, having only one term of maximal degree in the sum (3.1). If $P(z,w)$ is homogeneous, the above version does not work. We will establish a similar version of the Clunie lemma, for the other new type of version, see [7].

**Theorem 3.2** *Suppose that $w(z)$ is a meromorphic solution of*

$$H(z,w)P(z,w) = Q(z,w)$$

*with hyper-order $\rho_2(w) < 1$, where $P(z,w)$ is a homogeneous difference polynomial defined as in* (3.1), *$H(z,w)$ and $Q(z,w)$ are polynomials in $w(z)$ with no common factors. All the coefficients of $P(z,w)$, $H(z,w)$, and $Q(z,w)$ are in $\mathcal{S}(w)$. If*

$$\max\{\deg_w Q, \deg_w H\} = \Lambda > \deg_w P, \tag{3.2}$$

*then $m(r,w) = S(r,w)$.*

**Theorem 3.3** *Suppose that $w(z)$ is an admissible meromorphic solution of* (1.2) *with hyper-order $\rho_2(w) < 1$, where $R(z,w) = \frac{P(z,w)}{Q(z,w)}$, $P(z,w)$, and $Q(z,w)$ are polynomials in $w$ with degrees $p$ and $q$, respectively. Then*:

(i) *$p \leq 4$ and $q \leq 2$*;

(ii) *If $q = 2$, then $p = 4$. The coefficients of the highest degree of $P$ and $Q$ are identical*;

(iii) *If $q = 1$, then $p \leq 3$.*

**Theorem 3.4** *Suppose that $w(z)$ is an admissible meromorphic solution of*

$$L(w) = \frac{(w-h_1)(w-h_2)(w-h_3)(w-h_4)}{(w-a_1)(w-a_2)} \tag{3.3}$$

*with hyper-order $\rho_2(w) < 1$, where $L(w) = (\overline{w}+w)(w+\underline{w})$, neither of $a_i$ and $h_j$ vanishes identically. Then either $w$ satisfies a difference Riccati equation*

$$\overline{w} = \frac{\alpha w + \beta}{w + \gamma},$$

*where $\alpha, \beta, \gamma \in \mathcal{S}(w)$ are algebroid functions, or one of the following holds*:

(1) $b_3 = \underline{a}_1 + \overline{a}_1 = \underline{a}_2 + \overline{a}_2$;

(2) $b_3 = \underline{a}_1 + \overline{a}_2 = \underline{a}_2 + \overline{a}_1$,

*where $b_3 = a_1 + a_2 - H_3$ and $H_3 = \sum_{j=1}^{4} h_j$. For each $n = 1,2,3,4$, $h_n + \overline{h}_k = 0$ or $h_n + \underline{h}_k = 0$ for some $k \in \{1,2,3,4\}$.*

**Remark 1** In the present paper, we use similar notations to [4].

## 4 Proofs of theorems

*Proof of Theorem* 3.2 Since $H(z,w)$ and $Q(z,w)$ are polynomials in $w(z)$ with no common factors, Lemma 2.2 gives us

$$\begin{aligned} T\bigl(r,P(z,w)\bigr) &= T\left(r,\frac{Q(z,w)}{H(z,w)}\right) = \max\{\deg_w Q, \deg_w H\}T(r,w) + S(r,w) \\ &= \Lambda T(r,w) + S(r,w). \end{aligned} \tag{4.1}$$

On the other hand, by the definition of $P(z,w)$, we get from (2.2)

$$\begin{aligned} N\bigl(r,P(z,w)\bigr) &\leq \Lambda_0 N\bigl(r,w(z)\bigr) + \sum_{i=1}^{n} \Lambda_i N\bigl(r,w(z+c_i)\bigr) + S(r,w) \\ &= \Lambda N\bigl(r,w(z)\bigr) + S(r,w) \end{aligned}$$

and from Lemma 2.1

$$\begin{aligned} m\bigl(r,P(z,w)\bigr) &\leq \deg_w P m\bigl(r,w(z)\bigr) + m\left(r,\frac{P(z,w)}{w^{\deg_w P}}\right) \\ &\leq \deg_w P m\bigl(r,w(z)\bigr) + S(r,w). \end{aligned}$$

Combining (4.1) and the last two inequalities, we obtain

$$\Lambda T(r,w) = \Lambda N(r,w) + \deg_w P m(r,w) + S(r,w),$$

which is $(\Lambda - \deg_w P)m(r,w) = S(r,w)$, then $m(r,w) = S(r,w)$ from (3.2). □

*Proof of Theorem* 3.3 We restrict $p$ and $q$ duo to the reasoning by Grammaticos *et al.* [6], where $P(z,w)$ and $Q(z,w)$ are polynomials in $w$ with constant coefficients.

Applying Lemma 2.2 to (1.2), we get from Lemma 2.5

$$\begin{aligned} \deg_w R T(r,w) &\leq 2T(r,w) + T(r,\overline{w}) + T(r,\underline{w}) + S(r,w) \\ &\leq (4+2\epsilon)T(r,w) + S(r,w). \end{aligned}$$

Then $\max\{p,q\} \leq 4$. Rewriting (1.2) gives

$$\overline{w}\underline{w} + \overline{w}w + w\underline{w} = \frac{P(z,w) - w^2 Q(z,w)}{Q(z,w)} =: K(z,w). \tag{4.2}$$

Since $P(z,w)$ and $Q(z,w)$ have no common factors, the right side of (4.2) is irreducible. Applying Lemma 2.2 now to (4.2), we get from Lemma 2.5 and Lemma 2.7

$$\begin{aligned} \deg_w K T(r,w) &\leq T(r,w) + T(r,\overline{w}) + T(r,\underline{w}) + S(r,w) \\ &\leq (3+2\epsilon)T(r,w) + S(r,w), \end{aligned}$$

so $\deg_w K \leq 3$. Thus $q \leq 3$. If $q = 3$, the degree of $P - w^2 Q$ that was denoted by $k$ would be 5 since $p \leq 4$, a contradiction. Hence, $q \leq 2$, $p \leq 4$, and $k \leq 3$.

If $q = 2$, since the degree of $P - w^2 Q$, $k \leq 3$, we have $p = 4$ and the coefficients of the highest degree of $P$ and $Q$ are identical. □

*Proof of Theorem* 3.4 We get from (3.3)

$$L(w)Q(z,w) = P(z,w),$$

where $L(w) = \overline{w}\underline{w} + \overline{w}w + w\underline{w} + w^2$, $Q(z,w) = (w-a_1)(w-a_2)$, and $P(z,w) = (w-h_1)(w-h_2)(w-h_3)(w-h_4)$. Then $\deg_w P = 4$, $\deg_w Q = 2$, $\deg_w L = 2$, and $\Lambda$ of $L(w)$ equals 4. It follows from Theorem 3.2 that

$$m(r,w) = S(r,w) \tag{4.3}$$

and

$$m\big(r,L(w)\big) \le m\left(r,\frac{L(w)}{w^2}\right) + m\big(r,w^2\big) + S(r,w) = S(r,w),$$

thus $N(r,L(w)) = T(r,L(w)) + S(r,w) = 4T(r,w) + S(r,w)$. In a similar way to the proof of [4, Lemma 4.2], we have

$$N\left(r,\frac{1}{w-a_m}\right) = T(r,w) + S(r,w) \tag{4.4}$$

for $m = 1,2$. From Lemma 2.6, given $\epsilon > 0$, there are at most $S(r,w)$ points $z_j$ where $Q(z_j,w) = 0^{k_j}$, but where $L(w)$ has a pole of order greater that $(1+\epsilon)k_j$ or less than $(1-\epsilon)k_j$ due to poles or zeros of $P(z_j,w)$. The combined effect of all such points can be included in the error term, and so we only consider the rest of the zeros of $Q$ in what follows.

For a point $z_j$ where $w(z_j) = a_m(z_j)$, define

$$L(z_j,w) = (\ldots, z_j - 1, z_j, z_j + 1, \ldots)$$

to be the longest possible list of points such that each $z_j + 2n \in L(z_j,w)$ is a zero of $w - a_m$, $m = 1,2$, and each $z_j + 2n + 1 \in L(z_j,w)$ is a pole of $w$.

Suppose that $w$ has more than $S(r,w)$ poles that are not contained in any sequence $L(z_j,w)$. Let $N^*(r,w)$ be the integrated counting function of such poles; by assumption we have $N^*(r,w) > CT(r,w)$ for some $C > 0$ in a set of infinite logarithmic measure. By (3.3), $N^*(r,L(w)) = 2N^*(r,w) + S(r,w)$, and so we get

$$\begin{aligned} 4T(r,w) &= \big(N\big(r,L(w)\big) - N^*\big(r,L(w)\big)\big) + N^*\big(r,L(w)\big) + S(r,w) \\ &\le 4\big(N(r+1,w) - N^*(r+1,w)\big) + 2N^*(r+1,w) + S(r,w) \\ &\le (4-2C)T(r+1,w) + S(r,w), \end{aligned}$$

which implies that $\rho_2(w) \ge 1$ by Lemma 2.3. Therefore all except at most $S(r,w)$ poles of $w$ are in some sequence $L(z_j,w)$.

We will call the total number of zeros of $w - a_m$ in $L(z_j,w)$ divided by the total number of poles of $w$ (both counting multiplicities) the $a_m$/pole ratio of the sequence.

Consider a sequence $L(z_j,w)$ that contains only one zero of $w - a_m$. Then there are one or two poles in that sequence. With one pole we would have $w(z_j) = a_m(z_j) + 0^{kj}$ and $w(z_j+1) = \infty^{m_j}$ or $w(z_j-1) = \infty^{m_j}$, where $(1-\epsilon)k_j < m_j$. If there are two poles, the situation is the same except that now we have $L(w)(z_j) = \infty^{m_j}$. In any case, in such a sequence the $a_m$/pole ratio is at most $\frac{1}{1-\epsilon} =: \beta < 2$. We suppose that there are more than $S(r,w)z_j$, $L(z_j,w)$ contains only one zero of $w - a_m$. Then there are more than $S(r,w)$ sequences $L(z_j,w)$ such that $N(r,\frac{1}{Q(z,w)}) \le \beta N(r+1,w)$. Hence, we have

$$N\big(r,L(w)\big) = N(r,R) = 2N(r,w) + N\left(r,\frac{1}{Q(z,w)}\right) + S(r,w) < 4T(r,w),$$

which is a contradiction, thus all except at most $S(r,w)$ sequences $L(z_j,w)$ contain at least two zeros of $w-a_m$. This means that there must be at least $T(r,w)+S(r,w)$ points $z_j$ such that $w(z_j+1)=\infty$ and one of the following holds:

$$w(z_j)=a_2(z_j) \quad \text{and} \quad w(z_j+2)=a_1(z_j+2), \tag{4.5}$$

$$w(z_j)=a_1(z_j) \quad \text{and} \quad w(z_j+2)=a_2(z_j+2), \tag{4.6}$$

$$w(z_j)=a_1(z_j) \quad \text{and} \quad w(z_j+2)=a_1(z_j+2), \tag{4.7}$$

$$w(z_j)=a_2(z_j) \quad \text{and} \quad w(z_j+2)=a_2(z_j+2). \tag{4.8}$$

Since $N(r,1/(w-a_m))=T(r,w)+S(r,w)$ holds for both choices of $m=1,2$, exactly one of the following is true:

(i) Both (4.7) and (4.8) hold $\gtrdot S(r,w)$.
(ii) Both (4.5) and (4.6) hold $\gtrdot S(r,w)$, (4.7) and (4.8) hold $\lessdot S(r,w)$.
(iii) Relation (4.6) holds $\gtrdot S(r,w)$, (4.5), (4.7), and (4.8) hold $\lessdot S(r,w)$.
(iv) Relation (4.5) holds $\gtrdot S(r,w)$, (4.6)-(4.8) hold $\lessdot S(r,w)$.

In what follows, we will derive some consequences separately for the conditions (i)-(iv). We rewrite (3.3) as

$$\overline{w}w+\overline{w}\underline{w}+w\underline{w}=\frac{b_3w^3+b_2w^2+b_1w+b_0}{(w-a_1)(w-a_2)}, \tag{4.9}$$

where $b_3=a_1+a_2-H_3$ and $H_3=\sum_{j=1}^{4}h_j$.

Case (i) holds. Now we have both (4.7) and (4.8) hold $\gtrdot S(r,w)$. If (4.7) holds $\gtrdot S(r,w)$, we get from (4.9)

$$\begin{aligned} &w(z_j+2)w(z_j)+w(z_j+2)w(z_j+1)+w(z_j+1)w(z_j) \\ &\quad=\left(\frac{b_3w^3+b_2w^2+b_1w+a_0}{(w-a_1)(w-a_2)}\right)(z_j+1), \end{aligned}$$

which means $a_1(z_j)+a_1(z_j+2)=b_3(z_j+1)$ holds at more than $S(r,w)$ points $z_j$, then $\underline{a}_1+\overline{a}_1=b_3$. Similarly, if (4.8) holds $\gtrdot S(r,w)$, we get $\underline{a}_2+\overline{a}_2=b_3$. In this case, we have

$$b_3=\underline{a}_1+\overline{a}_1=\underline{a}_2+\overline{a}_2. \tag{4.10}$$

Case (ii) holds. In the same way as above, we have

$$b_3=\underline{a}_1+\overline{a}_2=\underline{a}_2+\overline{a}_1. \tag{4.11}$$

Cases (iii) and (iv) hold. Assume that (iii) holds. As we know, $L(z_j,w)$ contains at least two zeros of $w-a_m$. From (4.6), there are exactly one zero of $w-a_1$ and one zero of $w-a_2$, otherwise, (4.5), (4.7), and (4.8) will hold $\gtrdot S(r,w)$. Then all except at most $S(r,w)$ poles of $w$ must be contained in sequences of the form

$$\left(\infty^{l_{j-}},a_1+0^{k_{j-}},\infty^{m_j},a_2+0^{k_{j+}},\infty^{l_{j+}}\right),$$

where if $l_{j\pm} < 0$, the corresponding endpoint of the sequence is a zero of order $|l_{j\pm}|$, and if $l_{j\pm} = 0$, it is some nonzero finite value. By Lemma 2.6,

$$(1-\epsilon)k_{j\pm} < m_j + \max\{l_{j\pm}, 0\} < (1+\epsilon)k_{j\pm} \tag{4.12}$$

holds for both choices of the $\pm$ sign. Denote

$$U = (w - a_1)(\overline{w} - \overline{a}_2). \tag{4.13}$$

Next, we will show that $U$ is a small function with respect to $w$. From (4.3), we get $m(r, U) = S(r, w)$. From the definition of $U$ and the fact that all but at most $S(r, w)$ poles of $w$ are in sequences of the above form, it follows that if $U$ has more than $S(r, w)$ poles, then there are more than $S(r, w)$ sequences where $l_{j\pm} > 0$.

For the sequences with $l_{j-} > 0$, we may assume that $l_{j-}/m_j \geq s > 0$ for all such sequences, otherwise the poles with $l_{j-} > 0$ will only have a small effect (at most $S(r, w)$) on $N(r, U)$. The $a_m$/pole ratio for the sequences in consideration is

$$\frac{k_{j-} + k_{j+}}{m_j + l_{j-} + \max\{l_{j+}, 0\}} < \frac{1}{1-\epsilon} \frac{2m_j + l_{j-} + \max\{l_{j+}, 0\}}{m_j + l_{j-} + \max\{l_{j+}, 0\}}$$

using (4.12). Take $d$ such that

$$\frac{1 + s/2}{1 + s} < d < 1.$$

Then $d \in (1/2, 1)$. For a fixed $j$, there exists an $\epsilon_j$ satisfying

$$\epsilon_j < 1 - \frac{2m_j + l_{j-} + \max\{l_{j+}, 0\}}{2d(m_j + l_{j-} + \max\{l_{j+}, 0\})}.$$

Define $\epsilon = \inf_j \epsilon_j$. Since

$$\begin{aligned} & 2d\big(m_j + l_{j-} + \max\{l_{j+}, 0\}\big) - 2m_j - l_{j-} - \max\{l_{j+}, 0\} \\ &\quad \geq (2d-2)m_j + (2d-1)l_j \\ &\quad \geq m_j\big(2d - 2 + (2d-1)s\big) \\ &\quad > 0. \end{aligned}$$

Noting that $m_j \geq 1$, then $\epsilon$ is well defined. Thus we conclude that if $l_{j-} > 0$, then in such sequences the $a_m$/pole ratio is at most some $2d < 2$. We will get a contradiction as the above. If $l_{j+} > 0$, we will get a contradiction similarly. Therefore, $U \in S(w)$, and so (3.3) becomes the Riccati difference equation

$$\overline{w} = \frac{\overline{a}_2 w - U - a_1\overline{a}_2}{w - a_1}.$$

The same reasoning works for case (iv) as we exchange the roles of $a_1$ and $a_2$.

In the case that $a_1 = a_2$, the proof is similar. The condition (i) is the only possibility. □

*Restriction of* $h_n$. We claim that for each $n = 1, 2, 3, 4$,

$$h_n + \overline{h}_k = 0 \quad \text{or} \quad h_n + \underline{h}_k = 0 \tag{4.14}$$

for some $k \in \{1, 2, 3, 4\}$. Following the same reasoning as in the proof of [4, Lemma 4.2], we have

$$N\left(r, \frac{1}{w - h_n}\right) = T(r, w) + S(r, w), \tag{4.15}$$

provided $h_n$ is not a solution of (3.3), *i.e.*, $h_n$ does not satisfy $h_n + \overline{h}_n = 0$. In the following, we may assume that $w - h_n$ has a large number of zeros, or (4.14) holds as desired.

Consider the points $z_j$ where $w(z_j) = h_n(z_j) + 0^{k_j}$. From (3.3) we get

$$w(z_j) + w(z_j \pm 1) = 0$$

for all except at most $S(r, w)$ points $z_j$ and for either or both choices of $\pm$.

We assume the condition (i) or (ii) is true, otherwise, $w$ satisfies a Riccati difference equation. We discuss the following three cases:

Case 1. More than $S(r, w)$ points $z_j$ satisfy

$$w(z_j) + w(z_j + 1) = 0^{m_{j+}}.$$

Then $w(z_j + 1) = -h_n(z_j) + 0^{m_{j+}}$. By

$$\bigl(w(z_j + 2) + w(z_j + 1)\bigr)\bigl(w(z_j + 1) + w(z_j)\bigr) = \frac{P(z_j + 1)}{Q(z_j + 1)},$$

we have $w(z_j + 2) = \infty$. If $-h_n \not\equiv \overline{a}_m$ for $m = 1, 2$, then $-h_n(z_j) \neq a_m(z_j + 1)$. It follows that $w(z_j + 3) = a_m(z_j + 3)$ (note the conditions (i) and (ii)). Then from

$$\bigl(w(z_j + 3) + w(z_j + 2)\bigr)\bigl(w(z_j + 2) + w(z_j + 1)\bigr) = \frac{P(z_j + 2)}{Q(z_j + 2)},$$

we get $a_m(z_j + 3) - h_n(z_j) = b(z_j + 2)$, which means

$$\overline{\overline{\overline{a}}}_m - h_n = \overline{\overline{b}}.$$

Combining the above equation with (4.10) or (4.11), we have $-h_n = \overline{a}_m$ for $m = 1$ or $m = 2$, which is a contradiction.

Thus

$$-h_n = \overline{a}_m \tag{4.16}$$

holds for $m = 1$ or $m = 2$.

Case 2. More than $S(r, w)$ points $z_j$ satisfy

$$w(z_j) + w(z_j - 1) = 0^{m_{j-}}.$$

By a similar reasoning to the above, we obtain

$$-h_n = \underline{a}_m \tag{4.17}$$

holds for $m = 1$ or $m = 2$.

Case 3. More than $S(r,w)$ points $z_j$ satisfy

$$w(z_j) + w(z_j + 1) = 0^{m_{j+}}.$$

Also, more than $S(r,w)$ points $z_j$ satisfy

$$w(z_j) + w(z_j - 1) = 0^{m_{j-}}.$$

In this case, we have both (4.16) and (4.17).

Note the above three cases, all except at most $S(r,w)$ points $z_j$ are contained in one of forms the sequences of

$$\bigl(h_n(z_j) + 0^{k_j}, -h_n(z_j) + 0^{m_{j+}}, \infty^{s_{j+}}, a_m(z_j + 3), \varphi\bigr), \tag{4.18}$$

$$\bigl(\varphi, a_m(z_j - 3), \infty^{s_{j-}}, -h_n(z_j) + 0^{m_{j-}}, h_n(z_j) + 0^{k_j}\bigr), \tag{4.19}$$

$$\bigl(h_n(z_j) + 0^{t_j}, a_m(z_j + 1) + 0^{m_j}, \infty^{s_j}, a_m(z_j + 3) + 0^{l_j}, h_n(z_j + 4) + 0^{r_j}\bigr), \tag{4.20}$$

where $s_{j+} \approx k_j + m_{j+}$, $s_{j-} \approx k_j + m_{j-}$, $s_j \approx t_j + m_j \approx l_j + r_j$, and $\varphi$ is a pole or a finite value but not the zero of $w - h_n$. In fact, if $\varphi$ is the zero of $w - h_n$, it will be a starting point of another sequence.

If (4.20) holds for more than $S(r,w)$ points, there are more than $S(r,w)$ sequences $L(z_j, w)$ such that $N(r, \frac{1}{Q(z,w)}) \leq \beta N(r + 1, w)$, where $\beta < 2$. This is a contradiction. Hence, (4.20) holds at most $S(r,w)$ points. Then almost all the zeros of $w - h_n$ are in the sequences (4.18) and (4.19).

However, noting that $s_{j+} \approx k_j + m_{j+}$ and $s_{j-} \approx k_j + m_{j-}$, we have $N(r + 2, w) > (1 + \epsilon)N(r, \frac{1}{w-h_n}) + S(r,w) = (1 + \epsilon)T(r,w) + S(r,w)$, which contradicts with Lemma 2.3.

**Competing interests**
The author declares that he has no competing interests.

**Author's contributions**
The author drafted the manuscript and read and approved the final manuscript.

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