Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

# Positive solutions for impulsive fractional differential equations with generalized periodic boundary value conditions

Kaihong Zhao* and Ping Gong

*Correspondence:
zhaokaihongs@126.com
Department of Applied Mathematics, Kunming University of Science and Technology, Kunming, Yunnan 650093, China

**Abstract**

By constructing Green's function, we give the natural formulae of solutions for the following nonlinear impulsive fractional differential equation with generalized periodic boundary value conditions:

$$\begin{cases} {}^cD_t^q u(t) = f(t,u(t)), & t \in J' = J\backslash\{t_1,\dots,t_m\}, J = [0,1], \\ \Delta u(t_k) = I_k(u(t_k)), & \Delta u'(t_k) = J_k(u(t_k)), \quad k = 1,\dots,m, \\ au(0) - bu(1) = 0, & au'(0) - bu'(1) = 0, \end{cases}$$

where $1 < q < 2$ is a real number, ${}^cD_t^q$ is the standard Caputo differentiation. We present the properties of Green's function. Some sufficient conditions for the existence of single or multiple positive solutions of the above nonlinear fractional differential equation are established. Our analysis relies on a nonlinear alternative of the Schauder and Guo-Krasnosel'skii fixed point theorem on cones. As applications, some interesting examples are provided to illustrate the main results.

**MSC:** 34B10; 34B15; 34B37

**Keywords:** impulsive fractional differential equation; positive solutions; boundary value problems; fixed point theorem

## 1 Introduction

In recent years, the fractional order differential equation has aroused great attention due to both the further development of fractional order calculus theory and the important applications for the theory of fractional order calculus in the fields of science and engineering such as physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, Bode's analysis of feedback amplifiers, capacitor theory, electrical circuits, electron-analytical chemistry, biology, control theory, fitting of experimental data, and so forth. Many papers and books on fractional calculus differential equation have appeared recently. One can see [1–17] and the references therein.

In order to describe the dynamics of populations subject to abrupt changes as well as other phenomena such as harvesting, diseases, and so on, some authors have used an impulsive differential system to describe these kinds of phenomena since the last century. For the basic theory on impulsive differential equations, the reader can refer to the monographs of Bainov and Simeonov [18], Lakshmikantham *et al.* [19] and Benchohra *et al.* [20].

In this article, we consider the following nonlinear impulsive fractional differential equation with generalized periodic boundary value conditions (for short BVPs (1.1)):

$$\begin{cases} {}^cD_t^q u(t) = f(t,u(t)), & t \in J' = J\backslash\{t_1,\dots,t_m\}, J = [0,1], \\ \Delta u(t_k) = I_k(u(t_k)), & \Delta u'(t_k) = J_k(u(t_k)), \quad k = 1,\dots,m, \\ au(0) - bu(1) = 0, & au'(0) - bu'(1) = 0, \end{cases} \tag{1.1}$$

where $a$, $b$ are real constants with $a > b > 0$. ${}^cD_{0^+}^q$ is the Caputo fractional derivative of order $1 < q < 2$. $f : J \times \mathbb{R}^+ \to \mathbb{R}^+$ is jointly continuous. $I_k, J_k \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\mathbb{R}^+ = [0,+\infty)$. The impulsive point set $\{t_k\}_{k=1}^m$ satisfies $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$. $u(t_k^+) = \lim_{h\to 0^+} u(t_k + h)$ and $u(t_k^-) = \lim_{h\to 0^-} u(t_k + h)$ represent the right and left limits of $u(t)$ at the impulsive point $t = t_k$. Let us set $J_0 = [0,t_1]$, $J_k = (t_k, t_{k+1}]$, $1 \le k \le m$. The goal of this paper is to study the existence of single or multiple positive solutions for the impulsive BVPs (1.1) by a nonlinear alternative of the Schauder and Guo-Krasnosel'skii fixed point theorem on cones.

The rest of the paper is organized as follows. In Section 2, we present some useful definitions, lemmas and the properties of Green's function. In Section 3, we give some sufficient conditions for the existence of a single positive solution for BVPs (1.1). In Section 4, some sufficient criteria for the existence of multiple positive solutions for BVPs (1.1) are obtained. Finally, some examples are provided to illustrate our main results in Section 5.

## 2 Preliminaries

For the convenience of the reader, we present here the necessary definitions from fractional calculus theory. These definitions and properties can be found in the literature.

**Definition 2.1** (see [21, 22]) The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0,+\infty) \to \mathbb{R}$ is given by

$$I_{0+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds,$$

provided that the right-hand side is pointwise defined on $(0,+\infty)$.

**Definition 2.2** (see [21, 22]) The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $f : (0,+\infty) \to \mathbb{R}$ is given by

$${}^cD_{0+}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}}\,ds,$$

where $n - 1 < \alpha \le n$, provided that the right-hand side is pointwise defined on $(0,+\infty)$.

**Lemma 2.1** (see [21]) *Assume that $u \in C(0,1) \cap L(0,1)$ with a Caputo fractional derivative of order $q > 0$ that belongs to $u \in C^n[0,1]$, then*

$$I_{0+}^q D_{0+}^q u(t) = u(t) + c_0 + c_1 t + \cdots + c_{n-1} t^{n-1}$$

*for some $c_i \in \mathbb{R}$, $i = 0,1,2,\dots,n-1$ ($n = -[-q]$) and $[q]$ denotes the integer part of the real number $q$.*

**Lemma 2.2** (see [23]) *Let $E$ be a Banach space. Assume that $T : E \to E$ is a completely continuous operator and the set $V = \{u \in E \mid u = \mu Tu, 0 < \mu < 1\}$ is bounded. Then $T$ has a fixed point in $E$.*

**Lemma 2.3** (Schauder fixed point theorem, see [24]) *If $U$ is a close bounded convex subset of a Banach space $E$ and $T : U \to U$ is completely continuous, then $T$ has at least one fixed point in $U$.*

**Lemma 2.4** (see [25]) *Let $E$ be a Banach space, $P \subseteq E$ be a cone, and $\Omega_1$, $\Omega_2$ be two bounded open balls of $E$ centered at the origin with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Suppose that $A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ is a completely continuous operator such that either*

(i) *$\|Au\| \le \|u\|$, $u \in P \cap \partial\Omega_1$ and $\|Au\| \ge \|u\|$, $u \in P \cap \partial\Omega_2$, or*

(ii) *$\|Au\| \ge \|u\|$, $u \in P \cap \partial\Omega_1$ and $\|Au\| \le \|u\|$, $u \in P \cap \partial\Omega_2$*

*hold. Then $A$ has at least one fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.*

Now we present Green's function for a system associated with BVPs (1.1).

**Lemma 2.5** *Given $h \in C(J, \mathbb{R}^+)$ and $1 < q < 2$, the unique solution of*

$$\begin{cases} {}^cD_t^q u(t) = h(t), & t \in J', \\ \Delta u(t_k) = I_k(u(t_k)), \qquad \Delta u'(t_k) = J_k(u(t_k)), & k = 1, \dots, m, \\ au(0) - bu(1) = 0, \qquad au'(0) - bu'(1) = 0, & a > b > 0, \end{cases} \tag{2.1}$$

*is formulated by*

$$u(t) = \int_0^1 G_1(t,s)h(s)\,ds + \sum_{i=1}^m G_2(t,t_i)J_i\bigl(u(t_i)\bigr) + \sum_{i=1}^m G_3(t,t_i)I_i\bigl(u(t_i)\bigr), \quad t \in J,$$

*where*

$$G_1(t,s) = \begin{cases} \frac{(t-s)^{q-1}}{\Gamma(q)} + \frac{b(1-s)^{q-1}}{(a-b)\Gamma(q)} + \frac{b(q-1)t(1-s)^{q-2}}{(a-b)\Gamma(q)} + \frac{b^2(q-1)(1-s)^{q-2}}{(a-b)^2\Gamma(q)}, & 0 \le s \le t \le 1, \\ \frac{b(1-s)^{q-1}}{(a-b)\Gamma(q)} + \frac{b(q-1)t(1-s)^{q-2}}{(a-b)\Gamma(q)} + \frac{b^2(q-1)(1-s)^{q-2}}{(a-b)^2\Gamma(q)}, & 0 \le t \le s \le 1, \end{cases} \tag{2.2}$$

$$G_2(t,t_i) = \begin{cases} \frac{ab}{(a-b)^2} + \frac{a(t-t_i)}{a-b}, & 0 \le t_i < t \le 1, i = 1, 2, \dots, m, \\ \frac{ab}{(a-b)^2} + \frac{b(t-t_i)}{a-b}, & 0 \le t \le t_i \le 1, i = 1, 2, \dots, m, \end{cases} \tag{2.3}$$

$$G_3(t,t_i) = \begin{cases} \frac{a}{a-b}, & 0 \le t_i < t \le 1, i = 1, 2, \dots, m, \\ \frac{b}{a-b}, & 0 \le t \le t_i \le 1, i = 1, 2, \dots, m. \end{cases} \tag{2.4}$$

*Proof* Let $u$ be a general solution of (2.1) on each interval $(t_k, t_{k+1}]$ ($k = 0, 1, \dots, m$). Applying Lemma 2.1, Eq. (2.1) is translated into the following equivalent integral equation (2.5):

$$u(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s)\,ds - c_k - d_k t, \quad \forall t \in (t_k, t_{k+1}], \tag{2.5}$$

where $t_0 = 0$, $t_{m+1} = 1$. Then we have

$$u'(t) = \frac{1}{\Gamma(q-1)} \int_0^t (t-s)^{q-2} h(s)\,ds - d_k, \quad t \in (t_k, t_{k+1}].$$

In the light of the generalized periodic boundary value conditions of Eq. (2.1), we get

$$b\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}h(s)\,ds + ac_0 - bc_m - bd_m = 0, \tag{2.6}$$

$$b\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}h(s)\,ds + ad_0 - bd_m = 0. \tag{2.7}$$

Next, using the right impulsive condition of Eq. (2.1), we derive

$$c_{k-1} - c_k = I_k\big(u(t_k)\big) - J_k\big(u(t_k)\big)t_k, \tag{2.8}$$

$$d_{k-1} - d_k = J_k\big(u(t_k)\big). \tag{2.9}$$

By (2.7) and (2.9), we have

$$d_0 = \frac{-b}{a-b}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}h(s)\,ds - \frac{b}{a-b}\sum_{i=1}^m J_i\big(u(t_i)\big), \tag{2.10}$$

$$d_m = \frac{-b}{a-b}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}h(s)\,ds - \frac{a}{a-b}\sum_{i=1}^m J_i\big(u(t_i)\big). \tag{2.11}$$

By (2.9) we have

$$\begin{aligned} d_k &= d_0 - \sum_{i=1}^k J_i\big(u(t_i)\big) \\ &= \frac{-b}{a-b}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}h(s)\,ds - \frac{b}{a-b}\sum_{i=1}^m J_i\big(u(t_i)\big) - \sum_{i=1}^k J_i\big(u(t_i)\big). \end{aligned} \tag{2.12}$$

From (2.6), (2.8) and (2.11), we have

$$\begin{aligned} c_0 = {}& \frac{-b}{a-b}\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}h(s)\,ds - \frac{b^2}{(a-b)^2}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}h(s)\,ds \\ & - \frac{ab}{(a-b)^2}\sum_{i=1}^m J_i\big(u(t_i)\big) - \frac{b}{a-b}\sum_{i=1}^m \big(I_i\big(u(t_i)\big) - J_i\big(u(t_i)\big)t_i\big). \end{aligned} \tag{2.13}$$

According to (2.8), we obtain

$$\begin{aligned} c_k &= c_0 - \sum_{i=1}^k \big(I_i\big(u(t_i)\big) - J_i\big(u(t_i)\big)t_i\big) \\ &= \frac{-b}{a-b}\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}h(s)\,ds - \frac{b^2}{(a-b)^2}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}h(s)\,ds \\ &\quad - \frac{ab}{(a-b)^2}\sum_{i=1}^m J_i\big(u(t_i)\big) - \frac{b}{a-b}\sum_{i=1}^m I_i\big(u(t_i)\big) - \sum_{i=1}^k I_i\big(u(t_i)\big) \\ &\quad + \frac{b}{a-b}\sum_{i=1}^m J_i\big(u(t_i)\big)t_i + \sum_{i=1}^k J_i\big(u(t_i)\big)t_i. \end{aligned} \tag{2.14}$$

Hence, for $k = 1, 2, \ldots, m$, (2.12) and (2.14) imply

$$\begin{aligned} c_k + d_k t = {} & \frac{-b}{a-b}\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}h(s)\,ds - \frac{(ab-b^2)t+b^2}{(a-b)^2}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}h(s)\,ds \\ & - \frac{1}{(a-b)^2}\sum_{i=1}^m J_i\bigl(u(t_i)\bigr)\bigl[ab + b(a-b)(t-t_i)\bigr] - \frac{b}{a-b}\sum_{i=1}^m I_i\bigl(u(t_i)\bigr) \\ & - \sum_{i=1}^k I_i\bigl(u(t_i)\bigr) - \sum_{i=1}^k J_i\bigl(u(t_i)\bigr)(t-t_i). \end{aligned} \tag{2.15}$$

Now substituting (2.10) and (2.13) into (2.5), for $t \in J_0 = [0, t_1]$, we obtain

$$\begin{aligned} u(t) = {} & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}h(s)\,ds + \frac{b}{a-b}\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}h(s)\,ds \\ & + \frac{(ab-b^2)t+b^2}{(a-b)^2}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}h(s)\,ds \\ & + \frac{1}{(a-b)^2}\sum_{i=1}^m J_i\bigl(u(t_i)\bigr)\bigl[ab + b(a-b)(t-t_i)\bigr] + \frac{b}{a-b}\sum_{i=1}^m I_i\bigl(u(t_i)\bigr) \\ = {} & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}h(s)\,ds + \frac{b}{a-b}\left(\int_0^t + \int_t^1\right)\frac{(1-s)^{q-1}}{\Gamma(q)}h(s)\,ds \\ & + \frac{(ab-b^2)t+b^2}{(a-b)^2}\left(\int_0^t + \int_t^1\right)\frac{(1-s)^{q-2}}{\Gamma(q-1)}h(s)\,ds \\ & + \frac{1}{(a-b)^2}\sum_{i=1}^m J_i\bigl(u(t_i)\bigr)\bigl[ab + b(a-b)(t-t_i)\bigr] + \frac{b}{a-b}\sum_{i=1}^m I_i\bigl(u(t_i)\bigr) \\ = {} & \int_0^t \left[\frac{(t-s)^{q-1}}{\Gamma(q)} + \frac{b(1-s)^{q-1}}{(a-b)\Gamma(q)} + \frac{[(ab-b^2)t+b^2](1-s)^{q-2}}{(a-b)^2\Gamma(q-1)}\right]h(s)\,ds \\ & + \int_t^1 \left[\frac{b(1-s)^{q-1}}{(a-b)\Gamma(q)} + \frac{[(ab-b^2)t+b^2](1-s)^{q-2}}{(a-b)^2\Gamma(q-1)}\right]h(s)\,ds \\ & + \frac{1}{(a-b)^2}\sum_{i=1}^m J_i\bigl(u(t_i)\bigr)\bigl[ab + b(a-b)(t-t_i)\bigr] + \frac{b}{a-b}\sum_{i=1}^m I_i\bigl(u(t_i)\bigr) \\ = {} & \int_0^1 G_1(t,s)h(s)\,ds + \sum_{i=1}^m G_2(t,t_i)J_i\bigl(u(t_i)\bigr) + \sum_{i=1}^m G_3(t,t_i)I_i\bigl(u(t_i)\bigr), \end{aligned}$$

where $G_1(t,s)$, $G_2(t,t_i)$ and $G_3(t,t_i)$ are defined by (2.2)-(2.4).

Substituting (2.15) into (2.5), for $t \in J_k = (t_k, t_{k+1}]$, $k = 1, 2, \ldots, m$, we have

$$\begin{aligned} u(t) = {} & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}h(s)\,ds + \frac{b}{a-b}\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}h(s)\,ds \\ & + \frac{(ab-b^2)t+b^2}{(a-b)^2}\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}h(s)\,ds \\ & + \frac{1}{(a-b)^2}\sum_{i=1}^m J_i\bigl(u(t_i)\bigr)\bigl[ab + b(a-b)(t-t_i)\bigr] + \frac{b}{a-b}\sum_{i=1}^m I_i\bigl(u(t_i)\bigr) \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{i=1}^{k} I_i\big(u(t_i)\big) + \sum_{i=1}^{k} J_i\big(u(t_i)\big)(t-t_i) \\
&= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} h(s)\,ds + \frac{b}{a-b}\left(\int_0^t + \int_t^1\right)\frac{(1-s)^{q-1}}{\Gamma(q)} h(s)\,ds \\
&\quad + \frac{(ab-b^2)t+b^2}{(a-b)^2}\left(\int_0^t + \int_t^1\right)\frac{(1-s)^{q-2}}{\Gamma(q-1)} h(s)\,ds \\
&\quad + \frac{1}{(a-b)^2}\left(\sum_{i=1}^{k} + \sum_{i=k+1}^{m}\right) J_i\big(u(t_i)\big)\big[ab + b(a-b)(t-t_i)\big] \\
&\quad + \frac{b}{a-b}\left(\sum_{i=1}^{k} + \sum_{i=k+1}^{m}\right) I_i\big(u(t_i)\big) + \sum_{i=1}^{k} I_i\big(u(t_i)\big) + \sum_{i=1}^{k} J_i\big(u(t_i)\big)(t-t_i) \\
&= \int_0^t \left[\frac{(t-s)^{q-1}}{\Gamma(q)} + \frac{b(1-s)^{q-1}}{(a-b)\Gamma(q)} + \frac{[(ab-b^2)t+b^2](1-s)^{q-2}}{(a-b)^2\Gamma(q-1)}\right] h(s)\,ds \\
&\quad + \int_t^1 \left[\frac{b(1-s)^{q-1}}{(a-b)\Gamma(q)} + \frac{[(ab-b^2)t+b^2](1-s)^{q-2}}{(a-b)^2\Gamma(q-1)}\right] h(s)\,ds \\
&\quad + \sum_{i=1}^{k}\left[\frac{ab}{(a-b)^2} + \frac{b(a-b)+(a-b)^2}{(a-b)^2}(t-t_i)\right] J_i\big(u(t_i)\big) \\
&\quad + \sum_{i=k+1}^{m} \frac{ab + b(a-b)(t-t_i)}{(a-b)^2} J_i\big(u(t_i)\big) + \sum_{i=1}^{k}\left[\frac{b}{a-b} + 1\right] I_i\big(u(t_i)\big) \\
&\quad + \sum_{i=k+1}^{m} \frac{b}{a-b} I_i\big(u(t_i)\big) \\
&= \int_0^t \left[\frac{(t-s)^{q-1}}{\Gamma(q)} + \frac{b(1-s)^{q-1}}{(a-b)\Gamma(q)} + \frac{[(ab-b^2)t+b^2](1-s)^{q-2}}{(a-b)^2\Gamma(q-1)}\right] h(s)\,ds \\
&\quad + \int_t^1 \left[\frac{b(1-s)^{q-1}}{(a-b)\Gamma(q)} + \frac{[(ab-b^2)t+b^2](1-s)^{q-2}}{(a-b)^2\Gamma(q-1)}\right] h(s)\,ds \\
&\quad + \sum_{i=1}^{k} \frac{ab + a(a-b)(t-t_i)}{(a-b)^2} J_i\big(u(t_i)\big) + \sum_{i=k+1}^{m} \frac{ab + b(a-b)(t-t_i)}{(a-b)^2} J_i\big(u(t_i)\big) \\
&\quad + \sum_{i=1}^{k} \frac{a}{a-b} I_i\big(u(t_i)\big) + \sum_{i=k+1}^{m} \frac{b}{a-b} I_i\big(u(t_i)\big) \\
&= \int_0^1 G_1(t,s)h(s)\,ds + \sum_{i=1}^{m} G_2(t,t_i) J_i\big(u(t_i)\big) + \sum_{i=1}^{m} G_3(t,t_i) I_i\big(u(t_i)\big),
\end{aligned}
$$

where $G_1(t,s)$, $G_2(t,t_i)$ and $G_3(t,t_i)$ are defined by (2.2)-(2.4). The proof is complete. □

**Lemma 2.6** *Let* $0 < b < a < +\infty$, *then Green's functions* $G_1(t,s)$, $G_2(t,t_i)$ *and* $G_3(t,t_i)$ *defined by* (2.2), (2.3) *and* (2.4) *are continuous and satisfy the following*:

(i) $G_1(t,s) \in C(J \times J, \mathbb{R}^+)$, $G_2(t,t_i), G_3(t,t_i) \in C(J \times J, \mathbb{R}^+)$, *and* $G_1(t,s), G_2(t,t_i), G_3(t,t_i) > 0$ *for all* $t, t_i, s \in (0,1)$, *where* $J = [0,1]$.

(ii) *The functions* $G_1(t,s)$, $G_2(t,t_i)$ *and* $G_3(t,t_i)$ *have the following properties*:

$$\frac{b}{a}M(s) \le G_1(t,s) \le M(s), \quad \forall t \in J, s \in (0,1), \tag{2.16}$$

$$\frac{b^2}{(a-b)^2} \le G_2(t,t_i) \le \frac{a^2}{(a-b)^2}, \quad \forall t, t_i \in J, \tag{2.17}$$

$$\frac{b}{a-b} \le G_3(t,t_i) \le \frac{a}{a-b}, \quad \forall t, t_i \in J, \tag{2.18}$$

*where*

$$M(s) = \frac{a[(1-s)a-(2-s-q)b](1-s)^{q-2}}{(a-b)^2\Gamma(q)} > 0, \quad s \in [0,1). \tag{2.19}$$

*Proof* From the expressions of $G_1(t,s)$, $G_2(t,t_i)$ and $G_3(t,t_i)$, it is obvious that $G_1(t,s)$, $G_2(t,t_i), G_3(t,t_i) \in C(J \times J, \mathbb{R}^+)$ and $G_1(t,s), G_2(t,t_i), G_3(t,t_i) > 0$ for all $t, t_i, s \in (0,1)$. Next, we will prove (ii). From the definition of $G_1(t,s)$, we can know that, for given $s \in (0,1)$, $G_1(t,s)$ is increasing with respect to $t$ for $t \in J$. We let

$$g_1(t,s) = \frac{(a-b)^2(t-s)^{q-1} + [(a-b)(1-s) + [(a-b)t+b](q-1)]b(1-s)^{q-2}}{(a-b)^2\Gamma(q)}, \quad t \in [s,1],$$

$$g_2(t,s) = \frac{[(a-b)(1-s) + [(a-b)t+b](q-1)]b(1-s)^{q-2}}{(a-b)^2\Gamma(q)}, \quad t \in [0,s].$$

Hence, we derive

$$\begin{aligned}
\min_{t\in[0,1]} G_1(t,s) &= \min\Big\{\min_{t\in[s,1]} g_1(t,s), \min_{t\in[0,s]} g_2(t,s)\Big\} = \min\big\{g_1(s,s), g_2(0,s)\big\} = g_2(0,s) \\
&= \frac{[(a-b)(1-s)+b(q-1)]b(1-s)^{q-2}}{(a-b)^2\Gamma(q)} \\
&= \frac{b[(1-s)a-(2-s-q)b](1-s)^{q-2}}{(a-b)^2\Gamma(q)} \triangleq m(s), \quad s \in [0,1), \\
\max_{t\in[0,1]} G_1(t,s) &= \max\Big\{\max_{t\in[s,1]} g_1(t,s), \max_{t\in[0,s]} g_2(t,s)\Big\} = \max\big\{g_1(1,s), g_2(s,s)\big\} = g_1(1,s) \\
&= \frac{[(a-b)(1-s)+b(q-1)]a(1-s)^{q-2}}{(a-b)^2\Gamma(q)} \\
&= \frac{a[(1-s)a-(2-s-q)b](1-s)^{q-2}}{(a-b)^2\Gamma(q)} \triangleq M(s), \quad s \in [0,1).
\end{aligned}$$

Thus, we have

$$\frac{b}{a}M(s) = m(s) \le G_1(t,s) \le M(s).$$

It is obvious that

$$\frac{b^2}{(a-b)^2} = G_2(0,1) \le G_2(t,t_i) \le G_2(1,0) = \frac{a^2}{(a-b)^2}, \qquad \frac{b}{a-b} \le G_3(t,t_i) \le \frac{a}{a-b}.$$

The proof is completed. □

## 3 Existence of single positive solutions

In this section, we discuss the existence of positive solutions for BVP (1.1).

Let $E = \{u(t) : u(t) \in C(J)\}$ denote a real Banach space with the norm $\|\cdot\|$ defined by $\|u\| = \max_{t\in J} |u(t)|$. Let

$$\begin{aligned} PC(J) &= \{u \in E \mid u : J \to \mathbb{R}^+, u \in C(J'), u(t_k^-) \text{ and } u(t_k^+) \\ &\qquad \text{exist with } u(t_k^-) = u(t_k), 1 \le k \le m\}, \\ K &= \left\{u \in PC(J) : u(t) \ge \frac{b^2}{a^2}\|u\|, t \in J\right\}, \qquad (3.1) \\ K_r &= \{u \in K : \|u\| < r\}, \quad \partial K_r = \{u \in K : \|u\| = r\}. \qquad (3.2) \end{aligned}$$

Obviously, $PC(J) \subset E$ is a Banach space with the norm $\|u\| = \max_{t\in J} |u(t)|$. $K \subset PC(J)$ is a positive cone.

In the following, we need the assumptions and some notations as follows:

($B_1$) $0 < b < a < 1$, $0 < \sigma_1, \sigma_2 < +\infty$, where $\sigma_1 = \int_0^1 M(s)\,ds$, $\sigma_2 = \frac{b^3}{a^3}\int_0^1 M(s)\,ds$.
($B_2$) $f \in C(J \times \mathbb{R}^+, \mathbb{R}^+)$ and $f(t, 0) = 0$ for all $t \in J$.
($B_3$) $I_k(u(t_k)), J_k(u(t_k)) \in C(\mathbb{R}^+, \mathbb{R}^+)$, $k = 1, 2, \ldots, m$.

Let

$$\begin{aligned} \overline{N} &= \max\left\{\frac{a^2}{(a-b)^2}\sum_{i=1}^m J_i(u(t_i)), \frac{a}{a-b}\sum_{i=1}^m I_i(u(t_i))\right\}, \\ f^\delta &= \limsup_{u\to\delta} \max_{t\in J} \frac{f(t,u)}{u}, \qquad f_\delta = \liminf_{u\to\delta} \min_{t\in J} \frac{f(t,u)}{u}, \end{aligned}$$

where $\delta$ denotes 0 or $+\infty$. In addition, we introduce the following weight functions:

$$\begin{aligned} \Phi(r) &= \max\left\{f(t, u(t)) : (t, u) \in [0,1] \times \left[\frac{b^2}{a^2}r, r\right]\right\}, \\ \phi(r) &= \min\left\{f(t, u(t)) : (t, u) \in [0,1] \times \left[\frac{b^2}{a^2}r, r\right]\right\}. \end{aligned}$$

From Lemma 2.4, we can obtain the following lemma.

**Lemma 3.1** *Suppose that $f(t,u)$ is continuous, then $u \in PC(J)$ is a solution of BVPs* (1.1) *if and only if $u \in PC(J)$ is a solution of the integral equation*

$$u(t) = \int_0^1 G_1(t,s)f(s, u(s))\,ds + \sum_{i=1}^m G_2(t,t_i)J_i(u(t_i)) + \sum_{i=1}^m G_3(t,t_i)I_i(u(t_i)), \quad \forall t \in J.$$

Define $T : PC(J) \to PC(J)$ to be the operator defined as

$$(Tu)(t) = \int_0^1 G_1(t,s)f(s, u(s))\,ds + \sum_{i=1}^m G_2(t,t_i)J_i(u(t_i)) + \sum_{i=1}^m G_3(t,t_i)I_i(u(t_i)). \qquad (3.3)$$

Then, by Lemma 3.1, the existence of solutions for BVPs (1.1) is translated into the existence of the fixed point for $u = Tu$, where $T$ is given by (3.3). Thus, the fixed point of the operator $T$ coincides with the solution of problem (1.1).

**Lemma 3.2** *Assume that* $(\mathrm{B}_1)$-$(\mathrm{B}_3)$ *hold, then* $T : PC(J) \to PC(J)$ *and* $T : K \to K$ *defined by* (3.3) *are completely continuous.*

*Proof* Firstly, we shall show that $T : PC(J) \to PC(J)$ is completely continuous through three steps.

Step 1. Let $u \in PC(J)$, in view of the nonnegativity and continuity of functions $G_1(t,s)$, $G_2(t,t_i)$, $G_3(t,t_i)$, $f(t,u(t))$, $I_k$, $J_k$ and $a > b > 0$, we conclude that $T : PC(J) \to PC(J)$ is continuous.

Step 2. We will prove that $T$ maps bounded sets into bounded sets. Indeed, it is enough to show that for any $r > 0$ there exists a positive constant $L$ such that, for each $u \in \Omega_r = \{u \in PC(J) : \|u\| \le r\}$, $\|Tu\| \le L$ when $|f(t,u)| \le l_1$, $|J_k| \le l_2$, $|I_k| \le l_3$, where $l_i$ $(i = 1,2,3)$ are some fixed positive constants. In fact, for each $t \in J_k$, $u \in \Omega_r$, $k = 0,1,2,\dots,m$, by Lemma 2.5, we have

$$\begin{aligned}
\bigl|(Tu)(t)\bigr| &\le \int_0^1 \bigl|G_1(t,s)f\bigl(s,u(s)\bigr)\bigr|\,ds + \sum_{i=1}^m \bigl|G_2(t,t_i)J_i\bigl(u(t_i)\bigr)\bigr| + \sum_{i=1}^m \bigl|G_3(t,t_i)I_i\bigl(u(t_i)\bigr)\bigr| \\
&\le \sigma_1 l_1 + \frac{a^2 m l_2}{(a-b)^2} + \frac{a m l_3}{a-b} \triangleq L,
\end{aligned}$$

which imply that $\|Tu\| \le L$.

Step 3. $T$ is equicontinuous. In fact, since $G_1(t,s)$, $G_2(t,t_i)$, $G_3(t,t_i)$ are continuous on $J \times J$, they are uniformly continuous on $J \times J$. Thus, for fixed $s \in J$ and for any $\varepsilon > 0$, there exists a constant $\delta > 0$ such that for any $t_1, t_2 \in J_k$ with $|t_1 - t_2| < \delta$, $0 \le k \le m$, we have

$$\begin{aligned}
&\bigl|G_1(t_1,s) - G_2(t_2,s)\bigr| < \frac{\varepsilon}{3l_1}, \qquad \bigl|G_2(t_1,t_i) - G_2(t_2,t_i)\bigr| < \frac{\varepsilon}{3ml_2}, \\
&\bigl|G_3(t_1,t_i) - G_3(t_2,t_i)\bigr| < \frac{\varepsilon}{3ml_3}.
\end{aligned}$$

Then

$$\begin{aligned}
&\bigl|Tu(t_2) - Tu(t_1)\bigr| \\
&\quad = \biggl|\int_0^1 \bigl(G_1(t_2,s) - G_1(t_1,s)\bigr)f\bigl(s,u(s)\bigr)\,ds + \sum_{i=1}^m \bigl(G_2(t_2,t_i) - G_2(t_1,t_i)\bigr)J_i\bigl(u(t_i)\bigr) \\
&\qquad + \sum_{i=1}^m \bigl(G_3(t_2,t_i) - G_3(t_1,t_i)\bigr)I_i\bigl(u(t_i)\bigr)\biggr| \\
&\quad \le l_1 \int_0^1 \bigl|G_1(t_2,s) - G_2(t_1,s)\bigr|\,ds + ml_2\bigl|G_2(t_2,t_i) - G_2(t_1,t_i)\bigr| \\
&\qquad + ml_3\bigl|G_3(t_2,t_i) - G_3(t_1,t_i)\bigr| \\
&\quad < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,
\end{aligned}$$

which means that $T(\Omega_r)$ is equicontinuous on all the subintervals $t \in J_k$, $k = 0,1,\dots,m$. Thus, by means of the Arzela-Ascoli theorem, we have that $T : PC(J) \to PC(J)$ is completely continuous.

Next, we will show that $T: K \to K$ is completely continuous. Indeed, for each $t \in J_k$, every $u \in C(J_k, \mathbb{R}^+)$, $k = 0, 1, 2, \ldots, m$, Lemma 2.5 implies that

$$(Tu)(t) \geq \frac{b}{a} \int_0^1 M(s) f\bigl(s, u(s)\bigr)\, ds + \frac{b^2}{(a-b)^2} \sum_{i=1}^{m} J_i\bigl(u(t_i)\bigr) + \frac{b}{a-b} \sum_{i=1}^{m} I_i\bigl(u(t_i)\bigr).$$

On the other hand,

$$\|Tu\| = \max_{t \in J_k} (Tu)(t) \leq \int_0^1 M(s) f\bigl(s, u(s)\bigr)\, ds + \frac{a^2}{(a-b)^2} \sum_{i=1}^{m} J_i\bigl(u(t_i)\bigr) + \frac{a}{a-b} \sum_{i=1}^{m} I_i\bigl(u(t_i)\bigr).$$

Thus,

$$\begin{aligned}
\frac{b^2}{a^2} \|Tu\| &= \frac{b^2}{a^2} \max_{t \in J_k} (Tu)(t) \\
&\leq \frac{b^2}{a^2} \int_0^1 M(s) f\bigl(s, u(s)\bigr)\, ds + \frac{b^2}{(a-b)^2} \sum_{i=1}^{m} J_i\bigl(u(t_i)\bigr) + \frac{b^2}{a(a-b)} \sum_{i=1}^{m} I_i\bigl(u(t_i)\bigr) \\
&\leq (Tu)(t).
\end{aligned}$$

So $(Tu)(t) \geq \frac{b^2}{a^2} \|Tu\|$ for every $u \in C(J, \mathbb{R}^+)$, which implies $T(K) \subset K$. Similar to the above arguments, we can easily conclude that $T: K \to K$ is a completely continuous operator. The proof is complete. □

**Theorem 3.1** *Assume that* $(\mathrm{B}_1)$-$(\mathrm{B}_3)$ *hold, and suppose that the following assumptions hold*:

- $(\mathrm{A}_1)$ *There exists a constant* $L_1 > 0$ *such that* $|f(t,u) - f(t,v)| \leq L_1 |u - v|$ *for each* $t \in J$ *and all* $u, v \in \mathbb{R}^+$.
- $(\mathrm{A}_2)$ *There exists a constant* $L_2 > 0$ *such that* $|J_k(u) - J_k(v)| \leq L_2 |u - v|$ *for all* $u, v \in \mathbb{R}^+$, $k = 1, 2, \ldots, m$.
- $(\mathrm{A}_3)$ *There exists a constant* $L_3 > 0$ *such that* $|I_k(u) - I_k(v)| \leq L_3 |u - v|$ *for all* $u, v \in \mathbb{R}^+$, $k = 1, 2, \ldots, m$.

*If* $\rho = \sigma_1 L_1 + \frac{ma^2 L_2}{(a-b)^2} + \frac{maL_3}{a-b} < 1$, *then problem* (1.1) *has a unique solution in* $K_\rho$.

*Proof* Let the operator $T: K_\rho \to K_\rho$ be defined by (3.3). For all $u, v \in K_\rho$, from Lemma 2.5, we obtain

$$\begin{aligned}
&\bigl|(Tu)(t) - (Tv)(t)\bigr| \\
&\quad\leq \int_0^1 G_1(t,s) \bigl|f\bigl(s, u(s)\bigr) - f\bigl(s, v(s)\bigr)\bigr|\, ds \\
&\qquad + \sum_{i=1}^{m} G_2(t, t_i) \bigl|J_i\bigl(u(t_i)\bigr) - J_i\bigl(v(t_i)\bigr)\bigr| + \sum_{i=1}^{m} G_3(t, t_i) \bigl|I_i\bigl(u(t_i)\bigr) - I_i\bigl(v(t_i)\bigr)\bigr| \\
&\quad\leq \sigma_1 L_1 \|u - v\| + \frac{ma^2 L_2}{(a-b)^2} \|u - v\| + \frac{maL_3}{a-b} \|u - v\| = \rho \|u - v\|,
\end{aligned}$$

where $\rho = \sigma_1 L_1 + \frac{ma^2 L_2}{(a-b)^2} + \frac{maL_3}{a-b} < 1$. Consequently, $T$ is a contraction mapping. Moreover, from Lemma 3.2, $T$ is completely continuous. Therefore, by the Banach contraction map principle, the operator $T$ has a unique fixed point in $K_\rho$ which is the unique positive solution of system (1.1). This completes the proof. □

**Theorem 3.2** *Assume that* ($B_1$)-($B_3$) *hold, and suppose that the following assumptions hold*:

($A_4$) *There exists a constant* $N_1 > 0$ *such that* $|f(t,u)| \leq N_1$ *for each* $t \in J$ *and all* $u \in \mathbb{R}^+$.
($A_5$) *There exists a constant* $N_2 > 0$ *such that* $|J_k(u)| \leq N_2$ *for all* $u \in \mathbb{R}^+$, $k = 1,2,\ldots,m$.
($A_6$) *There exists a constant* $N_3 > 0$ *such that* $|I_k(u)| \leq N_3$ *for all* $u \in \mathbb{R}^+$, $k = 1,2,\ldots,m$.

*Then BVPs* (1.1) *have at least one positive solution in* $PC(J)$.

*Proof* Let $T : PC(J) \to PC(J)$ be cone preserving completely continuous that is defined by (3.3). According to Lemma 2.2, now it remains to show that the set

$$\Omega = \{u \in PC(J) \mid u = \lambda Tu \text{ for some } 0 < \lambda < 1\} \tag{3.4}$$

is bounded.

Let $u \in \Omega$, then $u = \lambda Tu$ for some $0 < \lambda < 1$. Thus, by Lemma 2.5, for each $t \in J_k$, $k = 0,1,\ldots,m$, we have

$$\begin{aligned}
|u(t)| &= |\lambda Tu| \\
&\leq \int_0^1 \bigl|G_1(t,s)f(s,u(s))\bigr|\,ds + \sum_{i=1}^m \bigl|G_2(t,t_i)J_i(u(t_i))\bigr| + \sum_{i=1}^m \bigl|G_3(t,t_i)I_i(u(t_i))\bigr| \\
&\leq \sigma_1 N_1 + \frac{a^2 mN_2}{(a-b)^2} + \frac{amN_3}{a-b}.
\end{aligned}$$

Thus, for every $t \in J$, we have $\|u(t)\| \leq \sigma_1 N_1 + \frac{a^2 mN_2}{(a-b)^2} + \frac{amN_3}{a-b}$, which indicates that the set $\Omega$ is bounded. According to Lemma 2.2, $T$ has a fixed point $u \in PC(J)$. Therefore, BVPs (1.1) have at least one positive solution. The proof is complete. □

In the following, we present an existence result when the nonlinearity and the impulse functions have sublinear growth.

**Theorem 3.3** *Assume that* ($B_1$)-($B_3$) *hold and suppose that the following assumptions hold*:

($A_7$) *There exist* $a_1 \in PC(J)$, $b_1 > 0$ *and* $\alpha \in [0,1)$ *such that* $|f(t,u)| \leq a_1(t) + b_1|u|^\alpha$ *for each* $t \in J$ *and all* $u \in \mathbb{R}^+$.
($A_8$) *There exist constants* $a_2, b_2 > 0$ *and* $\alpha \in [0,1)$ *such that* $|J_k(u)| \leq a_2 + b_2|u|^\alpha$ *for all* $u \in \mathbb{R}^+$, $k = 1,2,\ldots,m$.
($A_9$) *There exist constants* $a_3, b_3 > 0$ *and* $\alpha \in [0,1)$ *such that* $|I_k(u)| \leq a_3 + b_3|u|^\alpha$ *for all* $u \in \mathbb{R}^+$, $k = 1,2,\ldots,m$.

($A_{10}$) $b^* < 1$, $a^* + b^* \geq 1$, *where* $a^* = p\sigma_1 + \frac{a^2 m a_2}{(a-b)^2} + \frac{a m a_3}{a-b}$, $b^* = b_1\sigma_1 + \frac{a^2 m b_2}{(a-b)^2} + \frac{a m b_3}{a-b}$.

*Then BVPs* (1.1) *have at least one positive solution in* $PC(J)$.

*Proof* Let $T : PC(J) \to PC(J)$ and $\Omega$ be defined by (3.3) and (3.4), respectively. Denote $p = \max_{t\in J} |a_1(t)|$. If $u \in \Omega$, then for $t \in J$ we have

$$\begin{aligned}
\bigl|u(t)\bigr| &= |\lambda Tu| \\
&\leq \int_0^1 \bigl|G_1(t,s)\bigl(a_1(s) + b_1\bigl|u(s)\bigr|^\alpha\bigr)\bigr|\,ds + \sum_{i=1}^m \bigl|G_2(t,t_i)\bigl(a_2 + b_2|u|^\alpha\bigr)\bigr| \\
&\quad + \sum_{i=1}^m \bigl|G_3(t,t_i)\bigl(a_3 + b_3|u|^\alpha\bigr)\bigr| \\
&\leq p\int_0^1 M(s)\,ds + b_1\|u\|^\alpha \int_0^1 M(s)\,ds + \frac{a^2 m(a_2 + b_2\|u\|^\alpha)}{(a-b)^2} + \frac{am(a_3 + b_3\|u\|^\alpha)}{a-b} \\
&= a^* + b^*\|u\|^\alpha,
\end{aligned}$$

which imply that $\|u\| \leq a^* + b^*\|u\|^\alpha$. When $0 < \|u\| \leq 1$, then $\|u\| \leq a^* + b^*$. When $\|u\| > 1$, then $\|u\| \leq \frac{a^*}{1-b^*}$. Taking $C = \max\{a^* + b^*, \frac{a^*}{1-b^*},\}$, we have $\|u\| \leq C$ for any solution of (3.4). This shows that the set $\Omega$ is bounded. According to Lemma 2.2, $T$ has at least one fixed point in $PC(J)$. Therefore, BVPs (1.1) have at least one positive solution in $PC(J)$. The proof is complete. □

**Theorem 3.4** *Assume that* ($B_1$)-($B_3$) *hold. And suppose that one of the following conditions is satisfied*:

($H_1$) $f^\infty < \frac{1}{\sigma_1}$ (*particularly*, $f^\infty = 0$).
($H_2$) *There exists a constant* $M > 0$ *such that* $f(t,u) \leq \frac{M}{\sigma_1}$ *for* $t \in J$, $u \in [M, +\infty)$.
($H_3$) *There exists a constant* $N > 0$ *such that* $\Phi(N) \leq \frac{N}{3\sigma_1}$ *for* $t \in J$, $u \in [\frac{b^2}{a^2}N, N]$.

*Then BVPs* (1.1) *have at least one positive solution*.

*Proof* Case 1. Considering $f^\infty < \frac{1}{\sigma_1}$, there exists $\overline{r}_1 > 0$ such that $f(t,u) \leq (f^\infty + \varepsilon_1)u$ for all $u \in (\overline{r}_1, +\infty)$, $t \in J$, where $\varepsilon_1$ satisfies $\sigma_1(f^\infty + \varepsilon_1) \leq 1$.

Choose $r_1 > \max\{\overline{r}_1, 2\overline{N}(1 - \sigma_1(f^\infty + \varepsilon_1))^{-1}\}$, let $u \in \Omega_1 \triangleq K_{r_1}$. We can easily know that $\Omega_1$ is a close bounded convex subset of a Banach space $PC(J)$. Then, for $t \in J$, $u \in \Omega_1$, in view of the nonnegativity and continuity of functions $G_1(t,s)$, $G_2(t,t_i)$, $G_3(t,t_i)$, $f(t,u(t))$, $I_k$, $J_k$ and $a > b > 0$, we conclude that $Tu \in P$, $Tu \geq 0$, $t \in J$. By Lemma 2.5, we can obtain the following inequality:

$$\begin{aligned}
\frac{b^2}{a^2}\|Tu\| &= \frac{b^2}{a^2}\max_{t\in J}(Tu)(t) \\
&\leq \frac{b^2}{a^2}\Biggl[\int_0^1 M(s)f\bigl(s,u(s)\bigr)\,ds + \frac{a^2}{(a-b)^2}\sum_{i=1}^m J_i\bigl(u(t_i)\bigr) + \frac{a}{a-b}\sum_{i=1}^m I_i\bigl(u(t_i)\bigr)\Biggr] \\
&\leq \frac{b^2}{a^2}\Biggl[\int_0^1 M(s)f\bigl(s,u(s)\bigr)\,ds + \frac{a^2}{(a-b)^2}\sum_{i=1}^m J_i\bigl(u(t_i)\bigr) + \frac{a}{a-b}\sum_{i=1}^m I_i\bigl(u(t_i)\bigr)\Biggr]
\end{aligned}$$

$$\begin{aligned}&\leq \frac{b}{a}\int_0^1 M(s)f\big(s,u(s)\big)\,ds + \frac{b^2}{(a-b)^2}\sum_{i=1}^m J_i\big(u(t_i)\big) + \frac{b}{a-b}\sum_{i=1}^m I_i\big(u(t_i)\big)\\&\leq (Tu)(t),\quad t\in J.\end{aligned}$$

Thus $Tu \in K$.

Next, we prove $\|Tu\| \leq r_1$. Indeed, for $t\in J$, $u\in\partial K_{r_1}$, we get

$$\begin{aligned}\|Tu\| &= \max_{t\in J}(Tu)(t)\\&\leq \int_0^1 M(s)f\big(s,u(s)\big)\,ds + \frac{a^2}{(a-b)^2}\sum_{i=1}^m J_i\big(u(t_i)\big) + \frac{a}{a-b}\sum_{i=1}^m I_i\big(u(t_i)\big)\\&\leq \int_0^1 M(s)\big(f^\infty+\varepsilon_1\big)u(s)\,ds + \frac{a^2}{(a-b)^2}\sum_{i=1}^m J_i\big(u(t_i)\big) + \frac{a}{a-b}\sum_{i=1}^m I_i\big(u(t_i)\big)\\&\leq \sigma_1\big(f^\infty+\varepsilon_1\big)\|u\| + 2\overline{N}\\&< \sigma_1\big(f^\infty+\varepsilon_1\big)r_1 + r_1 - \sigma_1\big(f^\infty+\varepsilon_1\big)r_1 = r_1.\end{aligned}$$

Therefore, $T(\Omega_1)\subset\Omega_1$. From Lemma 3.2, we have that $T:\Omega_1\to\Omega_1$ is completely continuous. Thus BVPs (1.1) have at least a positive solution by Lemma 2.3.

Case 2. Condition ($H_2$) holds. Let $u\in\Omega_2 \triangleq K_d$, where $d>0$ satisfies $d\geq 1+M+\sigma_1\max_{t\in J,u\in[0,M]}f(t,u)+2\overline{N}$. By the ways of Case 1, we can also get $Tu\in K$. Now we prove $\|Tu\|\leq d$. In fact,

$$\begin{aligned}\|Tu\| &= \max_{t\in J}(Tu)(t)\\&\leq \int_0^1 M(s)f\big(s,u(s)\big)\,ds + \frac{a^2}{(a-b)^2}\sum_{i=1}^m J_i\big(u(t_i)\big) + \frac{a}{a-b}\sum_{i=1}^m I_i\big(u(t_i)\big)\\&\leq \int_{s\in J,u(s)>M} M(s)f\big(s,u(s)\big)\,ds + \int_{s\in J,0\leq u(s)\leq M} M(s)f\big(s,u(s)\big)\,ds + 2\overline{N}\\&\leq \int_0^1 M(s)\frac{M}{\sigma_1}\,ds + \int_0^1 M(s)\,ds \max_{t\in J,u\in[0,M]} f(t,u) + 2\overline{N}\\&= M + \sigma_1\max_{t\in J,u\in[0,M]} f(t,u) + 2\overline{N} < d.\end{aligned}$$

Therefore, $T(\Omega_2)\subset\Omega_2$. From Lemma 3.2 we have that $T:\Omega_2\to\Omega_2$ is completely continuous. Thus BVPs (1.1) have at least a positive solution by Lemma 2.3.

Case 3. Condition ($H_3$) holds. Let $u\in\Omega_3\triangleq K_N$, where $N>0$ satisfies $N\geq 3\overline{N}$, we get $\frac{b^2}{a^2}\|u\|\leq u(t)\leq\|u\|$. By the ways of Case 1, we can also get $Tu\in K$. Now we prove $\|Tu\|\leq N$. By assumption ($H_3$), we have

$$f(t,u)\leq\Phi(N)\leq\frac{N}{3\sigma_1},\quad \forall t\in J, u\in\left[\frac{b^2}{a^2}N,N\right].$$

In view of Lemma 2.6, we have

$$\begin{aligned}
\|Tu\| &= \max_{t\in J}(Tu)(t) \\
&\le \int_0^1 M(s)f\bigl(s,u(s)\bigr)\,ds + \frac{a^2}{(a-b)^2}\sum_{i=1}^m J_i\bigl(u(t_i)\bigr) + \frac{a}{a-b}\sum_{i=1}^m I_i\bigl(u(t_i)\bigr) \\
&\le \int_0^1 M(s)\frac{N}{3\sigma_1}\,ds + 2\overline{N} \\
&\le \frac{N}{3} + \frac{2N}{3} = N.
\end{aligned}$$

Therefore, $T(\Omega_3) \subset \Omega_3$. From Lemma 3.2 we have $T : \Omega_3 \to \Omega_3$ is completely continuous. Thus BVPs (1.1) have at least a positive solution by Lemma 2.3. We complete the proof of Theorem 3.4. □

## 4 Existence of multiple positive solutions

In this section, we discuss the multiplicity of positive solutions for BVPs (1.1) by the Guo-Krasnoselskii fixed point theorem.

**Theorem 4.1** *Assume that* $(B_1)$-$(B_3)$ *hold, and suppose that the following two conditions are satisfied*:

$(H_4)$ $f_0 > \frac{1}{\sigma_2}$ *and* $f_\infty > \frac{1}{\sigma_2}$ (*particularly*, $f_0 = f_\infty = \infty$).
$(H_5)$ *There exists a constant* $c \ge 3\overline{N}$ *such that* $\Phi(c) < \frac{c}{3\sigma_1}$ *for* $t \in J$, $u \in [\frac{b^2}{a^2}c, c]$.

*Then for BVPs* (1.1) *there exist at least two positive solutions* $u_1$, $u_2$, *which satisfy*

$$0 < \|u_1\| < c < \|u_2\|. \tag{4.1}$$

*Proof* Choose $r$, $R$ with $0 < r < c < R$. Considering $f_0 > \frac{1}{\sigma_2}$, there exists $r > 0$ such that $f(t,u) \ge (f_0 - \varepsilon_2)u$ for $t \in J$, $u \in [0,r]$, where $\varepsilon_2 > 0$ satisfies $(f_0 - \varepsilon_2)\sigma_2 \ge 1$. Then, for $u \in \partial K_r$, $t \in J$, we have

$$\begin{aligned}
\|Tu\| &= \max_{t\in J}(Tu)(t) \\
&\ge \frac{b}{a}\int_0^1 M(s)f\bigl(s,u(s)\bigr)\,ds + \frac{b^2}{(a-b)^2}\sum_{i=1}^m J_i\bigl(u(t_i)\bigr) + \frac{b}{a-b}\sum_{i=1}^m I_i\bigl(u(t_i)\bigr) \\
&\ge \frac{b}{a}\int_0^1 M(s)f\bigl(s,u(s)\bigr)\,ds \ge \frac{b}{a}\int_0^1 M(s)(f_0-\varepsilon_2)u(s)\,ds \\
&\ge \frac{b}{a}\int_0^1 M(s)(f_0-\varepsilon_2)\frac{b^2}{a^2}\|u\|\,ds \\
&= (f_0-\varepsilon_2)\sigma_2\|u\| \ge \|u\|.
\end{aligned}$$

Therefore,

$$\|Tu\| \ge \|u\|, \quad u \in \partial K_r. \tag{4.2}$$

Considering $f_\infty > \frac{1}{\sigma_2}$, there exists $R > 0$ such that $f(t,u) \geq (f_\infty - \varepsilon_3)u$ for $t \in J$, $u \in [R,\infty)$, where $\varepsilon_3 > 0$ satisfies $(f_\infty - \varepsilon_3)\sigma_2 \geq 1$. Then, for $u \in \partial K_R$, $t \in J$, we have

$$\begin{aligned}
\|Tu\| &= \max_{t\in J}(Tu)(t) \\
&\geq \frac{b}{a}\int_0^1 M(s)f\bigl(s,u(s)\bigr)\,ds + \frac{b^2}{(a-b)^2}\sum_{i=1}^m J_i\bigl(u(t_i)\bigr) + \frac{b}{a-b}\sum_{i=1}^m I_i\bigl(u(t_i)\bigr) \\
&\geq \frac{b}{a}\int_0^1 M(s)f\bigl(s,u(s)\bigr)\,ds \geq \frac{b}{a}\int_0^1 M(s)(f_\infty - \varepsilon_3)u(s)\,ds \\
&\geq \frac{b}{a}\int_0^1 M(s)(f_\infty - \varepsilon_3)\frac{b^2}{a^2}\|u\|\,ds = (f_\infty - \varepsilon_3)\sigma_2\|u\| \geq \|u\|.
\end{aligned}$$

So

$$\|Tu\| \geq \|u\|, \quad u \in \partial K_R. \tag{4.3}$$

On the other hand, by assumption $(\mathrm{H}_5)$, we have

$$f(t,u) \leq \Phi(c) < \frac{c}{3\sigma_1}, \quad \text{for } t \in J, u \in \left[\frac{b^2}{a^2}c, c\right].$$

For $u \in \partial K_c$, where $c > 0$ satisfies $c \geq 3\overline{N}$. In view of Lemma 2.6, we have

$$\begin{aligned}
\|Tu\| &= \max_{t\in J}(Tu)(t) \\
&\leq \int_0^1 M(s)f\bigl(s,u(s)\bigr)\,ds + \frac{a^2}{(a-b)^2}\sum_{i=1}^m J_i\bigl(u(t_i)\bigr) + \frac{a}{a-b}\sum_{i=1}^m I_i\bigl(u(t_i)\bigr) \\
&< \int_0^1 M(s)\frac{c}{3\sigma_1}\,ds + 2\overline{N} \leq \frac{c}{3} + \frac{2c}{3} = c = \|u\|.
\end{aligned}$$

Therefore,

$$\|Tu\| < \|u\|, \quad u \in \partial K_c. \tag{4.4}$$

Thus, applying Lemma 2.4 to (4.2)-(4.4) yields that $T$ has the fixed point $u_1 \in K \cap (\overline{K}_c \setminus K_r)$ and the fixed point $u_2 \in K \cap (\overline{K}_R \setminus K_c)$. Thus it follows that problem (1.1) has at least two positive solutions $u_1$ and $u_2$. Noticing (4.4), we have $\|u_1\| \neq c$ and $\|u_2\| \neq c$. Therefore (4.1) holds. The proof is complete. □

**Theorem 4.2** *Assume that* $(\mathrm{B}_1)$-$(\mathrm{B}_3)$ *hold. Further suppose that there exist three positive numbers* $\xi_i$ $(i = 1,2,3)$ *with* $3\overline{N} \leq \xi_1 < \xi_2 < \xi_3$ *such that one of the following conditions is satisfied*:

$(\mathrm{H}_6)$ $\phi(\xi_1) \geq \frac{\xi_1}{\sigma_2}$, $\Phi(\xi_2) \leq \frac{\xi_2}{3\sigma_1}$, $\phi(\xi_3) \geq \frac{\xi_3}{\sigma_2}$.
$(\mathrm{H}_7)$ $\Phi(\xi_1) \leq \frac{\xi_1}{3\sigma_1}$, $\phi(\xi_2) > \frac{\xi_2}{\sigma_2}$, $\Phi(\xi_3) \leq \frac{\xi_3}{3\sigma_1}$.

*Then BVPs* (1.1) *have at least two positive solutions* $u_1$, $u_2$ *with*

$$\xi_1 \leq \|u_1\| < \xi_2 < \|u_2\| \leq \xi_3. \tag{4.5}$$

*Proof* Because the proofs are similar, we prove only case ($H_6$). Considering $\phi(\xi_1) \geq \frac{\xi_1}{\sigma_2}$, we have $f(t,u) \geq \phi(\xi_1) \geq \frac{\xi_1}{\sigma_2}$ for $t \in J$, $u \in [\frac{b^2}{a^2}\xi_1, \xi_1]$. Then, for $u \in \partial K_{\xi_1}$, $t \in J$, we have

$$\begin{aligned}
\|Tu\| &= \max_{t\in J}(Tu)(t) \\
&\geq \frac{b}{a}\int_0^1 M(s)f\bigl(s,u(s)\bigr)\,ds + \frac{b^2}{(a-b)^2}\sum_{i=1}^m J_i\bigl(u(t_i)\bigr) + \frac{b}{a-b}\sum_{i=1}^m I_i\bigl(u(t_i)\bigr) \\
&\geq \frac{b}{a}\int_0^1 M(s)f\bigl(s,u(s)\bigr)\,ds \geq \frac{b}{a}\int_0^1 M(s)\frac{\xi_1}{\sigma_2}\,ds \\
&\geq \frac{b^3}{a^3}\int_0^1 M(s)\,ds\frac{\xi_1}{\sigma_2} = \xi_1 = \|u\|.
\end{aligned}$$

Therefore,

$$\|Tu\| \geq \|u\|, \quad u \in \partial K_{\xi_1}. \tag{4.6}$$

Considering $\Phi(\xi_2) \leq \frac{\xi_2}{3\sigma_1}$, we have $f(t,u) \leq \Phi(\xi_2) \leq \frac{\xi_2}{3\sigma_1}$ for $t \in J$, $u \in [\frac{b^2}{a^2}\xi_2, \xi_2]$. Then, for $u \in \partial K_{\xi_2}$, $t \in J$, we derive

$$\begin{aligned}
\|Tu\| &= \max_{t\in J}(Tu)(t) \\
&\leq \int_0^1 M(s)f\bigl(s,u(s)\bigr)\,ds + \frac{a^2}{(a-b)^2}\sum_{i=1}^m J_i\bigl(u(t_i)\bigr) + \frac{a}{a-b}\sum_{i=1}^m I_i\bigl(u(t_i)\bigr) \\
&\leq \int_0^1 M(s)\frac{\xi_2}{3\sigma_1}\,ds + 2\overline{N} \leq \frac{\xi_2}{3} + \frac{2\xi_1}{3} < \frac{\xi_2}{3} + \frac{2\xi_2}{3} = \xi_2 = \|u\|.
\end{aligned}$$

So,

$$\|Tu\| < \|u\|, \quad u \in \partial K_{\xi_2}. \tag{4.7}$$

Considering $\phi(\xi_3) \geq \frac{\xi_3}{\sigma_2}$, we have $f(t,u) \geq \phi(\xi_3) \geq \frac{\xi_3}{\sigma_2}$ for $t \in J$, $u \in [\frac{b^2}{a^2}\xi_3, \xi_3]$. Then, for $u \in \partial K_{\xi_3}$, $t \in J$, we have

$$\begin{aligned}
\|Tu\| &= \max_{t\in J}(Tu)(t) \\
&\geq \frac{b}{a}\int_0^1 M(s)f\bigl(s,u(s)\bigr)\,ds + \frac{b^2}{(a-b)^2}\sum_{i=1}^m J_i\bigl(u(t_i)\bigr) + \frac{b}{a-b}\sum_{i=1}^m I_i\bigl(u(t_i)\bigr) \\
&\geq \frac{b}{a}\int_0^1 M(s)f\bigl(s,u(s)\bigr)\,ds \geq \frac{b}{a}\int_0^1 M(s)\frac{\xi_3}{\sigma_2}\,ds \geq \frac{b^3}{a^3}\int_0^1 M(s)\,ds\frac{\xi_3}{\sigma_2} = \xi_3 = \|u\|.
\end{aligned}$$

Therefore,

$$\|Tu\| \geq \|u\|, \quad u \in \partial K_{\xi_3}. \tag{4.8}$$

Thus, applying Lemma 2.4 to (4.6)-(4.8) yields that $T$ has the fixed point $u_1 \in K \cap (\overline{K}_{\xi_2} \setminus K_{\xi_1})$ and the fixed point $u_2 \in K \cap (\overline{K}_{\xi_3} \setminus K_{\xi_2})$. Thus it follows that BVPs (1.1) have at least two positive solutions $u_1$ and $u_2$. Noticing (4.7), we have $\|u_1\| \neq \xi_2$ and $\|u_2\| \neq \xi_2$. Therefore (4.5) holds. The proof is complete. □

Similar to the above proof, we can obtain the general theorem.

**Theorem 4.3** *Assume that* $(\mathrm{B}_1)$-$(\mathrm{B}_3)$ *hold. Suppose that there exist* $n+1$ *positive numbers* $\xi_i$ $(i = 1,2,\dots,n+1)$ *with* $3\overline{N} \leq \xi_1 < \xi_2 < \cdots < \xi_{n+1}$ *such that one of the following conditions is satisfied*:

$(\mathrm{H}_8)$ $\phi(\xi_{2j-1}) > \frac{\xi_{2j-1}}{\sigma_2}$, $\Phi(\xi_{2j}) < \frac{\xi_{2j}}{3\sigma_1}$, $j = 1,2,\dots,[\frac{n+2}{2}]$;
$(\mathrm{H}_9)$ $\Phi(\xi_{2j-1}) < \frac{\xi_{2j-1}}{3\sigma_1}$, $\phi(\xi_{2j}) > \frac{\xi_{2j}}{\sigma_2}$, $j = 1,2,\dots,[\frac{n+2}{2}]$.

*Then BVPs* (1.1) *have at least n positive solutions* $u_i$ $(i = 1,2,\dots,n)$ *with*

$$\xi_i < \|u_i\| < \xi_{i+1}. \tag{4.9}$$

## 5 Illustrative examples

**Example 5.1** Consider the BVPs of impulsive nonlinear fractional order differential equations:

$$\begin{cases} {}^cD_t^q u(t) = f(t,u(t)), & t \in J, t \neq \frac{1}{2}, \\ \Delta u(\frac{1}{2}) = I(u(\frac{1}{2})), & \Delta u'(\frac{1}{2}) = J(u(\frac{1}{2})), \\ au(0) - bu(1) = 0, & au'(0) - bu'(1) = 0. \end{cases} \tag{5.1}$$

If we let $q = \frac{3}{2}$, $a = 2$, $b = 1$, $f(t,u) = \frac{\Gamma(\frac{3}{2})\cos t}{(t+2\sqrt{5})^2}\frac{u(t)}{1+u(t)}$, $(t,u) \in [0,1] \times [0,\infty)$, $I(u) = \frac{u}{5+u}$, $J(u) = \frac{u}{10+u}$, $u \in [0,\infty)$.

For $u, v \in [0,\infty)$, $t \in [0,1]$,

$$\begin{aligned} \bigl|f(t,u) - f(t,v)\bigr| &\leq \left|\frac{\Gamma(\frac{3}{2})\cos t}{(t+2\sqrt{5})^2}\right|\left|\frac{u-v}{(1+u)(1+v)}\right| \leq \frac{\Gamma(\frac{3}{2})}{20}|u-v|, \\ \bigl|I(u) - I(v)\bigr| &\leq \frac{5}{(5+u)(5+v)}|u-v| \leq \frac{1}{5}|u-v|, \\ \bigl|J(u) - J(v)\bigr| &\leq \frac{10}{(10+u)(10+v)}|u-v| \leq \frac{1}{10}|u-v|. \end{aligned}$$

Clearly, $L_1 = \frac{\Gamma(\frac{3}{2})}{20}$, $L_2 = \frac{1}{10}$, $L_3 = \frac{1}{5}$. Therefore,

$$\rho = \sigma_1 L_1 + \frac{ma^2L_2}{(a-b)^2} + \frac{maL_3}{a-b} = \frac{10}{3\Gamma(\frac{3}{2})}\frac{\Gamma(\frac{3}{2})}{20} + \frac{2}{5} + \frac{2}{5} = \frac{29}{30} < 1.$$

Thus, all the assumptions of Theorem 3.1 are satisfied. Hence, BVPs (5.1) have a unique solution on $[0,1]$.

In addition, in this case, let $N_1 = \frac{\Gamma(\frac{3}{2})}{20}$, $N_2 = N_3 = 1$. It is clear that $|f(t,u)| \leq N_1$, $|J_k(u)| \leq N_2$, $|I_k(u)| \leq N_3$. Thus, BVPs (5.1) have at least one solution on $[0,1]$ by Theorem 3.2.

**Example 5.2** Consider the BVPs of impulsive nonlinear fractional order differential equations:

$$\begin{cases} {}^cD_t^q u(t)) = f(t,u(t), & t \in J, t \neq \frac{1}{2}, \\ \Delta u(\frac{1}{2}) = I(u(\frac{1}{2})), & \Delta u'(\frac{1}{2}) = J(u(\frac{1}{2})), \\ au(0) - bu(1) = 0, & au'(0) - bu'(1) = 0. \end{cases} \tag{5.2}$$

Let $a = 2$, $b = 1$, $q = \frac{3}{2}$, $f(t,u) = |\frac{u(t)\ln u(t)}{5(1+t^2)}|$, $I(u) = J(u) = \frac{1}{16(1+u)}$. It is easy to see that $(\mathrm{H}_4)$ holds. By a simple computation, we have

$$f_0 = \liminf_{u\to 0} \min_{t\in[0,1]} \left|\frac{u\ln u}{5(1+t^2)u}\right| = \liminf_{u\to 0} \frac{|\ln u|}{10} = +\infty,$$

$$f_\infty = \liminf_{u\to\infty} \min_{t\in[0,1]} \left|\frac{u\ln u}{5(1+t^2)u}\right| = \liminf_{u\to\infty} \frac{|\ln u|}{10} = +\infty.$$

Take $c = 1$, it is clear that $3\overline{N} < \frac{3}{4} < c$. For $\frac{1}{4} \leq u \leq 1$, $f(t,u) = \frac{-u\ln u}{5(1+t^2)}$, we can obtain that $f(t,u)$ arrives at maximum at $u = \frac{1}{e} \in [\frac{1}{4},1]$, $t = 0$. Thus, we have

$$\Phi(1) = \max_{t\in[0,1],u\in[\frac{1}{4},1]} f(t,u) = f\left(0,\frac{1}{e}\right) = \frac{1}{5e} \approx 0.0736 < \frac{1}{3\sigma_1} = \frac{\sqrt{\pi}}{20} \approx 0.0886.$$

Thus it follows that BVPs (5.2) have at least two positive solutions $u_1$, $u_2$ with $0 < \|u_1\| < 1 < \|u_2\|$ by Theorem 4.1.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

**Acknowledgements**
The authors thank the referees for their valuable comments and suggestions for the improvement of the manuscript. This work is supported by the National Natural Sciences Foundation of Peoples Republic of China under Grant No. 11161025 and Yunnan Province Natural Scientific Research Fund Project under Grant No. 2011FZ058.

Received: 20 June 2014 Accepted: 9 September 2014 Published: 29 Sep 2014

**References**
1. Lakshmikantham, V, Leela, S: Nagumo-type uniqueness result for fractional differential equations. Nonlinear Anal. **71**, 2886-2889 (2009)
2. Chen, F, Zhou, Y: Attractivity of fractional functional differential equations. Comput. Math. Appl. **62**, 1359-1369 (2011)
3. Chang, Y, Nieto, J: Some new existence results for fractional differential inclusions with boundary conditions. Math. Comput. Model. **49**, 605-609 (2009)
4. Kilbas, AA, Trujillo, JJ: Differential equations of fractional order: methods, results and problems-I. Appl. Anal. **78**, 153-192 (2001)
5. Kilbas, AA, Trujillo, JJ: Differential equations of fractional order: methods, results and problems-II. Appl. Anal. **81**, 435-493 (2002)
6. Bai, Z: On positive solutions of a nonlocal fractional boundary value problem. Nonlinear Anal. **72**, 916-924 (2010)
7. Zhang, X, Liu, L, Wu, Y: Multiple positive solutions of a singular fractional differential equation with negatively perturbed term. Math. Comput. Model. **55**(3), 1263-1274 (2012)
8. Zhao, Y, Sun, S, Han, Z, et al.: The existence of multiple positive solutions for boundary value problems of nonlinear fractional differential equations. Commun. Nonlinear Sci. Numer. Simul. **16**(4), 2086-2097 (2011)
9. Bai, Z, Qiu, T: Existence of positive solution for singular fractional differential equation. Appl. Math. Comput. **215**(7), 2761-2767 (2009)
10. Zhao, Y, Sun, S, Han, Z, et al.: Positive solutions for boundary value problems of nonlinear fractional differential equations. Appl. Math. Comput. **217**(16), 6950-6958 (2011)
11. Goodrich, C: Existence of a positive solution to a class of fractional differential equations. Appl. Math. Lett. **23**, 1050-1055 (2010)

12. Salem, H: On the existence of continuous solutions for a singular system of nonlinear fractional differential equations. Appl. Math. Comput. **198**, 445-452 (2008)
13. Bai, Z, Lv, H: Positive solutions for boundary value problem of nonlinear fractional differential equation. J. Math. Anal. Appl. **311**, 495-505 (2005)
14. Zhao, Y, Chen, H, Zhang, Q: Existence and multiplicity of positive solutions for nonhomogeneous boundary value problems with fractional *q*-derivatives. Bound. Value Probl. **2013**, Article ID 103 (2013)
15. Li, X, Chen, F, Li, X: Generalized anti-periodic boundary value problems of impulsive fractional differential equations. Commun. Nonlinear Sci. Numer. Simul. **18**, 28-41 (2013)
16. Tian, Y, Bai, Z: Impulsive boundary value problem for differential equations with fractional order. Differ. Equ. Dyn. Syst. **21**(3), 253-260 (2013)
17. Zhao, KH, Gong, P: Existence of positive solutions for a class of higher-order Caputo fractional differential equation. Qual. Theory Dyn. Syst. (2014). doi:10.1007/s12346-014-0121-0
18. Bainov, DD, Simeonov, PS: Impulsive Differential Equations: Periodic Solutions and Applications. Longman, New York (1993)
19. Lakshmikantham, V, Bainov, DD, Simeonov, PS: Theory of Impulsive Differential Equations. World Scientific, Singapore (1989)
20. Benchohra, M, Henderson, J, Ntouyas, SK: Impulsive Differential Equations and Inclusions, vol. 2. Hindawi Publishing Corporation, New York (2006)
21. Kilbas, A, Srivastava, H, Trujillo, J: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam (2006)
22. Podlubny, I: Fractional Differential Equations. Academic Press, New York (1993)
23. Sun, JX: Nonlinear Functional Analysis and Its Application. Science Press, Beijing (2008) (in Chinese)
24. Hale, JK: Theory of Functional Differential Equations. Springer, New York (1977)
25. Guo, D, Lakshmikantham, V, Liu, X: Nonlinear Integral Equations in Abstract Spaces. Mathematics and Its Applications, vol. 373. Kluwer Academic, Dordrecht (1996)

10.1186/1687-1847-2014-255
**Cite this article as:** Zhao and Gong: Positive solutions for impulsive fractional differential equations with generalized periodic boundary value conditions. *Advances in Difference Equations* 2014, 2014:255