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Multiple periodic solutions for resonant difference equations

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Abstract

In this paper, we study the existence of multiple periodic solutions for nonlinear second-order difference equations with resonance at origin. The approach is based on critical point theory, minimax methods, homological linking and Morse theory.

Keywords: difference equations; periodic solutions; homological linking; critical group; Morse theory

1 Introduction

In this paper, we consider the existence of multiple periodic solutions for the following nonlinear difference equations:

$$(P) \quad \begin{cases} -\triangle^2 u(k-1) = \lambda_m u(k) + f(k, u(k)), & k \in \mathbb{Z}[1, N], \\ u(0) = u(N), \quad u(1) = u(N+1), \end{cases}$$

where $N > 3$ is a fixed integer, $\triangle u(k) = u(k+1) - u(k)$, $\triangle^2 u(k) = \triangle(\triangle u(k))$, $f(k, \cdot) : \mathbb{R} \to \mathbb{R}$ is a differential function satisfying

$$f(k, 0) = 0, \quad k \in \mathbb{Z}[1, N], \tag{1.1}$$

and λ_m is the $m+1$th eigenvalue of the linear periodic boundary value problem

$$(P_0) \quad \begin{cases} -\triangle^2 u(k-1) = \mu u(k), & k \in \mathbb{Z}[1, N], \\ u(0) = u(N), \quad u(1) = u(N+1). \end{cases}$$

Since (1.1) implies that (P) possesses a trivial periodic solution $u \equiv 0$, we are interested in finding nontrivial periodic solutions for (P). It follows from [1] that all the eigenvalues of (P_0) are $\mu_k = 4\sin^2 \frac{k\pi}{N}$, $k \in \mathbb{Z}[0, N-1]$. Thus $\mu_0 = 0$, $\mu_j = \mu_{N-j}$ for $j \in \mathbb{Z}[1, N_1]$, where

$$N_1 = \begin{cases} \frac{N-1}{2} & \text{if } N \text{ is odd,} \\ \frac{N}{2} & \text{if } N \text{ is even.} \end{cases}$$

For the convenience of later use, we denote by $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_{N_1}$ the distinct eigenvalues of (P_0). Moreover, if N is odd then all eigenvalues of (P_0) are multiplicity two except λ_0,

and let

$$\phi_0, \phi_1, \psi_1, \ldots, \phi_{N_1}, \psi_{N_1}$$

be the corresponding orthonormal eigenvectors; if N is even then all eigenvalues are multiplicity two except λ_0 and λ_{N_1}, and let

$$\phi_0, \phi_1, \psi_1, \ldots, \phi_{N_1-1}, \psi_{N_1-1}, \phi_{N_1}$$

be the corresponding orthonormal eigenvectors.

Now we establish the variational framework associated with (P). Set

$$u = \big(u(1), u(2), \ldots, u(N)\big)^T, \qquad \mathbf{f}(u) = \big(f\big(1, u(1)\big), f\big(2, u(2)\big), \ldots, f\big(N, u(N)\big)\big)^T,$$

and

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ -1 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}_{N \times N}.$$

Then we can rewrite (P) and (P_0) as

$$Au = \lambda_m u + \mathbf{f}(u) \quad \text{and} \quad Au = \mu u,$$

respectively.

Let $E = \mathbb{R}^N$ with inner product $\langle u, v \rangle = \sum_{k=1}^{N} u(k)v(k)$ and norm $\|u\| = \sqrt{\langle u, u \rangle}$. Then

$$0 = \lambda_0 \|u\|^2 \leq \langle Au, u \rangle \leq \lambda_{N_1} \|u\|^2, \quad u \in E. \tag{1.2}$$

For $p \geq 1$, define $\|u\|_p = (\sum_{k=1}^{N} |u(k)|^p)^{1/p}$, then there exist positive numbers a_p, b_p such that

$$a_p \|u\| \leq \|u\|_p \leq b_p \|u\|, \quad \forall u \in E. \tag{1.3}$$

Define the functional $J : E \to \mathbb{R}$ by

$$J(u) = \frac{1}{2} \langle Au, u \rangle - \frac{1}{2} \lambda_m \|u\|^2 - \sum_{k=1}^{N} F\big(k, u(k)\big), \quad u \in E, \tag{1.4}$$

where $F(k, x) = \int_0^x f(k, s)\, ds$. Then $J \in C^2(E, \mathbb{R})$ with derivatives

$$J'(u) = Au - \lambda_m u - \mathbf{f}(u), \quad u \in E, \tag{1.5}$$

$$J''(u) = A - \lambda_m I_N - \mathbf{f}'(u), \quad u \in E, \tag{1.6}$$

where I_N is the identity matrix of order N, $\mathbf{f}'(u) = \mathrm{diag}\{f'(1, u(1)), \ldots, f'(N, u(N))\}$. Hence the periodic solutions of (P) are exactly the critical points of J or $-J$ in E.

We assume that the nonlinearity f satisfies the following conditions:

(f_0) $f'(k, 0) = 0$ for $k \in \mathbb{Z}[1, N]$.

(f_0^+) There exists $\delta > 0$ such that

$$F(k, x) \geq 0, \quad k \in \mathbb{Z}[1, N], |x| \leq \delta.$$

(f_0^-) There exists $\delta > 0$ such that

$$F(k, x) \leq 0, \quad k \in \mathbb{Z}[1, N], |x| \leq \delta.$$

(f_∞^+) There exist $r > 0$ and $\theta > 2$ such that

$$0 \leq \theta F(k, x) \leq f(k, x)x, \quad k \in \mathbb{Z}[1, N], |x| > r.$$

(f_∞^-) There exist $r > 0$ and $\theta > 2$ such that

$$0 \geq \theta F(k, x) \geq f(k, x)x, \quad k \in \mathbb{Z}[1, N], |x| > r.$$

(f) For any a fixed number $\tau > \theta$, there exists $C > 0$ such that

$$\left|F(k, x)\right| \leq C\left(1 + |x|^\tau\right), \quad k \in \mathbb{Z}[1, N], x \in \mathbb{R}.$$

Therefore we regard the problem (P) as resonance at origin under the assumption (f_0).

Critical point theory has been widely used to study the existence of periodic solutions and solutions for nonlinear difference boundary value problems since the first result was established by using variational methods in 2003 (see [2]). Since then, by using critical point theory, minimax methods and Mores theory, the existence of solutions for non-resonant difference equations has been extensively investigated (see [3–9] and the references therein). As for resonant cases, Zhu and Yu [10] applied critical point theory to study the existence of positive solutions for a second order nonlinear discrete Dirichlet boundary value problem

$$\begin{cases} -\triangle^2 u(k-1) = f(k, u(k)), & k \in \mathbb{Z}[1, N], \\ u(0) = 0, \qquad u(N+1) = 0 \end{cases} \tag{1.7}$$

when nonlinearity f is odd and resonant at infinity. Zheng and Xiao [11] employed critical groups and the mountain pass theorem to study the existence of nontrivial solutions for (1.7) when the nonlinearity $f(k, u) = V'(u)$ and is resonant at infinity. Liu *et al.* [12] used Morse theory, critical point theory and minimax methods to study the existence of multiple solutions for (1.7) with resonance at both infinity and origin, one can refer to [13, 14].

However, we note that only a few papers concern the existence of periodic solutions for difference equations with resonance. In 2011, Zhang and Wang [15] used variational

methods and Morse theory to study the multiplicity of periodic solutions for (P) with double resonance between two consecutive eigenvalues at infinity. The main aim of this paper is to study the multiplicity of nonzero periodic solutions for (P) with resonance at origin. The approach is based on critical theory, Morse theory and homological linking.

The rest of this paper is organized as follows. In Section 2, we collect some useful preliminary results about Morse theory. In Section 3, we give some auxiliary results. Our main results and proofs will be given in Section 4.

2 Preliminaries about Morse theory

In this section, we recall some facts about Morse theory and critical groups [16, 17]. Let E be a real Hilbert space. We say that J satisfies the (PS) condition if every sequence $\{u_n\} \subset E$ such that $J(u_n)$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$ has a convergent subsequence.

Suppose that $J \in C^1(E, \mathbb{R})$ is a functional satisfying the (PS) condition. Let u_0 be an isolated critical point of J with $J(u_0) = c \in \mathbb{R}$, and let U be a neighborhood of u_0. The group

$$C_q(J, u_0) := H_q\big(J^c \cap U, \big(J^c \setminus \{u_0\}\big) \cap U\big), \quad q \in \mathbb{Z}$$

is called the qth critical group of J at u_0, where $J^c = \{u \in E \mid J(u) \le c\}$, $H_*(A, B)$ denotes the singular relative homology group of the topological pair (A, B) with coefficient field $\mathbb{F}$. Define

$$\mathcal{K} = \big\{u \in E \mid J'(u) = 0\big\}.$$

Assume that $\mathcal{K}$ is a finite number. Take $a < \inf J(\mathcal{K})$. The group

$$C_q(J, \infty) := H_q\big(E, J^a\big), \quad q \in \mathbb{Z}$$

is called the qth critical group of J at infinity (see [18]). The Morse type numbers of the pair (E, J^a) are defined by $M_q := \sum_{u \in \mathcal{K}} \dim C_q(J, u)$. Denote by $\beta_q := \dim C_q(J, \infty)$ the Betti numbers of the pair (E, J^a). By Morse theory, the relationship between M_q and β_q is described by

$$\sum_{j=0}^{q} (-1)^{q-j} M_j \ge \sum_{j=0}^{q} (-1)^{q-j} \beta_j \tag{2.1}$$

and

$$\sum_{q=0}^{\infty} (-1)^q M_q = \sum_{q=0}^{\infty} (-1)^q \beta_q. \tag{2.2}$$

From $M_q \ge \beta_q$, for each $q \in \mathbb{N}$, it follows that if $C_l(J, \infty) \not\cong 0$ for some $l \in \mathbb{N}$, then J must have a critical point u^* with $C_l(J, u^*) \not\cong 0$. If $\mathcal{K} = \{u^*\}$, then $C_q(J, \infty) \cong C_q(J, u^*)$ for all $q \in \mathbb{N}$. Thus if $C_l(J, \infty) \not\cong C_l(J, u^*)$ for some $l \in \mathbb{N}$, then J must have a new critical point.

For some $k \in \mathbb{Z}$, define

$$\delta_{q,k}\mathbb{F} := \begin{cases} \mathbb{F}, & q = k, \\ 0, & q \ne k. \end{cases}$$

Then

$$C_q(J, u_0) \cong \begin{cases} \delta_{q,0}\mathbb{F} & \text{if } u_0 \text{ is an isolated local minimum of } J, \\ \delta_{q,l}\mathbb{F} & \text{if } u_0 \text{ is an isolated local maximum of } J \text{ and } l = \dim E < \infty. \end{cases}$$

Suppose that $J \in C^2(E, \mathbb{R})$ and $u_0 \in \mathcal{K}$. Then $J''(u_0)$ is a self-adjoint linear operator on E. The dimension of the largest negative space of $J''(u_0)$ is called the Morse index of J at u_0, and the dimension of the kernel of $J''(u_0)$ is called the nullity of J at u_0. We say that u_0 is nondegenerate if the nullity of J at u_0 is zero, *i.e.*, $J''(u_0)$ has a bounded inverse. For an isolated critical point, the following important result is valid.

Proposition 2.1 ([16, 17]) *Suppose that u_0 is an isolated critical point of $J \in C^2(E, \mathbb{R})$ with finite Morse index $\mu(u_0)$ and nullity $\nu(u_0)$.*

(i) *$C_q(J, u_0) \cong 0$ for $q \notin [\mu(u_0), \mu(u_0) + \nu(u_0)]$.*

(ii) *If u_0 is nondegenerate, then $C_q(J, u_0) \cong \delta_{q,\mu(u_0)}\mathbb{F}$.*

(iii) *If $C_l(J, u_0) \ncong 0$, then $C_q(J, u_0) \cong \delta_{q,l}\mathbb{F}$ for $l = \mu(u_0)$ or $l = \mu(u_0) + \nu(u_0)$.*

Proposition 2.2 ([19–23]) *Let 0 be an isolated critical point of $J \in C^2(E, \mathbb{R})$ with finite Morse index $\mu(0)$ and nullity $\nu(0)$. Assume that J has a local linking at 0 with respect to a direct sum decomposition $E = E^- \oplus E^+$, $l = \dim E^- < \infty$, i.e., there exists $\rho > 0$ such that*

$$J(z) \leq 0 \quad \text{for } z \in E^-, \|u\| \leq \rho,$$

$$J(u) \geq 0 \quad \text{for } u \in E^+, \|u\| \leq \rho.$$

Then $C_q(J, 0) \cong \delta_{q,l}\mathbb{F}$ for either $l = \mu(0)$ or $l = \mu(0) + \nu(0)$.

Proposition 2.3 ([16, 19]) *Let E be a real Banach space with $E = X \oplus Y$ and suppose that $l = \dim X$ is finite. Assume that $J \in C^2(E, \mathbb{R})$ satisfies the (PS) condition and*

(H_1) *there exist $\rho > 0$ and $\alpha > 0$ such that*

$$J(u) > \alpha, \quad u \in Y \cap B_\rho, \tag{2.3}$$

where $B_\rho = \{u \in E \mid \|u\| \leq \rho\}$,

(H_2) *there exist $R > \rho > 0$ and $\phi \in Y$ with $\|\phi\| = 1$ such that*

$$J(u) \leq \alpha, \quad u \in \partial Q, \tag{2.4}$$

where $Q = \{u = v + s\phi \mid v \in X, \|v\| \leq R, 0 \leq s \leq R\}$.

Then J has a critical point u_* with $J(u_*) = c_* \geq \alpha$ and

$$C_{l+1}(J, u_*) \ncong 0. \tag{2.5}$$

3 Auxiliary results

We first show that the functional J satisfies the (PS) condition.

Lemma 3.1 *Assume that f satisfies (f_∞^+) or (f_∞^-), then J defined by (1.4) satisfies the (PS) condition.*

Proof We only prove the case where (f_∞^+) holds; the other case can be proved similarly. Let $\{u_n\} \subset E$ be such that

$$|J(u_n)| \le C, \quad n \in \mathbb{N}, \qquad J'(u_n) \to 0 \quad \text{as } n \to \infty. \tag{3.1}$$

We only need to show that $\{u_n\}$ is bounded. Taking positive number $\alpha \in (1/\theta, 1/2)$, it follows from (3.1) that there exists $K \in \mathbb{N}$ such that

$$C + \alpha \|u_n\| \ge J(u_n) - \alpha \langle J'(u_n), u_n \rangle, \quad n > K.$$

By (f_∞^+), there exist $C_1, C_2 > 0$ such that

$$F(k, x) \ge C_1(|x|^\theta - 1), \quad x \in \mathbb{R}, k \in \mathbb{Z}[1, N], \tag{3.2}$$

$$|\theta F(k, x) - f(k, x)x| \le C_2, \quad |x| \le r, k \in \mathbb{Z}[1, N]. \tag{3.3}$$

Hence, by (1.2), (f_∞^+), (3.2) and (3.3), we have

$$
\begin{aligned}
C + \alpha \|u_n\| &\ge J(u_n) - \alpha \langle J'(u_n), u_n \rangle \\
&= \left(\frac{1}{2} - \alpha\right) \langle A u_n, u_n \rangle - \left(\frac{1}{2} - \alpha\right) \lambda_m \|u_n\|^2 \\
&\quad - \sum_{k=1}^{N} \left[F(k, u_n(k)) - \alpha f(k, u_n(k)) u_n(k) \right] \\
&\ge \left(\alpha - \frac{1}{2}\right) \lambda_m \|u_n\|^2 - \sum_{|u_n(k)| > r} \left[F(k, u_n(k)) - \alpha f(k, u_n(k)) u_n(k) \right] \\
&\quad - \sum_{|u_n(k)| \le r} \left[F(k, u_n(k)) - \alpha f(k, u_n(k)) u_n(k) \right] \\
&\ge \left(\alpha - \frac{1}{2}\right) \lambda_m \|u_n\|^2 + (\theta \alpha - 1) \sum_{k=1}^{N} F(k, u_n(k)) \\
&\quad - \alpha \sum_{|u_n(k)| \le r} \left[\theta F(k, u_n(k)) - f(k, u_n(k)) u_n(k) \right] \\
&\ge \left(\alpha - \frac{1}{2}\right) \lambda_m \|u_n\|^2 + (\theta \alpha - 1) C_1 (\|u_n\|_\theta^\theta - N) - N C_2 \\
&\ge \left(\alpha - \frac{1}{2}\right) \lambda_m \|u_n\|^2 + (\theta \alpha - 1) C_1 a_\theta^\theta \|u_n\|^\theta \\
&\quad - N\left[(\theta \alpha - 1) C_1 + C_2 \right], \quad n > K.
\end{aligned}
$$

Since $\theta > 2$ and $\alpha \in (\frac{1}{\theta}, \frac{1}{2})$, we get that $\{u_n\}$ is bounded. $\qquad \square$

Remark 3.1 By the proof of Lemma 3.1, we have that if f satisfies (f_∞^+) or (f_∞^-), then $-J$ satisfies the (PS) condition.

For $k \in \mathbb{Z}[0, N_1 - 1]$, defining

$$E_k = \operatorname{span}\{\phi_0, \phi_1, \psi_1, \ldots, \phi_k, \psi_k\}, \qquad E = E_k \oplus E_k^\perp, \qquad \omega_k = \dim E_k,$$

then $\omega_k = 2k + 1$ and

$$E_k^\perp = \begin{cases} \operatorname{span}\{\phi_{k+1}, \psi_{k+1}, \ldots, \phi_{N_1}, \psi_{N_1}\} & \text{while } N \text{ is odd,} \\ \operatorname{span}\{\phi_{k+1}, \psi_{k+1}, \ldots, \phi_{N_1-1}, \psi_{N_1-1}, \phi_{N_1}\} & \text{while } N \text{ is even.} \end{cases}$$

Now we construct a linking with respect to the decomposition $E = E_m \oplus E_m^\perp$ or $E = E_{m-1}^\perp \oplus E_{m-1}$.

Lemma 3.2 *Suppose that f satisfies (f_0) and (f). Then*

(i) *for any fixed $m \in \mathbb{Z}[0, N_1 - 1]$, there exist $\rho_m > 0$ and $\alpha_m > 0$ such that*

$$J(u) > \alpha_m \quad \text{for } u \in E_m^\perp \text{ with } \|u\| = \rho_m; \tag{3.4}$$

(ii) *for any fixed $m \in \mathbb{Z}[1, N_1]$, there exist $\widetilde{\rho}_m > 0$ and $\widetilde{\alpha}_m > 0$ such that*

$$-J(u) > \widetilde{\alpha}_m \quad \text{for } u \in E_{m-1} \text{ with } \|u\| = \widetilde{\rho}_m. \tag{3.5}$$

Proof By (f_0) and (f), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\left| F(k, x) \right| \leq \frac{1}{2} \varepsilon x^2 + C_\varepsilon |x|^\tau, \quad k \in \mathbb{Z}[1, N], x \in \mathbb{R}. \tag{3.6}$$

(i) For any $u \in E_m^\perp$, it follows from (3.6) that

$$\begin{aligned}
J(u) &= \frac{1}{2} \langle Au, u \rangle - \frac{1}{2} \lambda_m \|u\|^2 - \sum_{k=1}^N F\big(k, u(k)\big) \\
&\geq \frac{1}{2} (\lambda_{m+1} - \lambda_m) \|u\|^2 - \sum_{k=1}^N \left(\frac{1}{2} \varepsilon |u(k)|^2 + C_\varepsilon |u(k)|^\tau \right) \\
&\geq \frac{1}{2} (\lambda_{m+1} - \lambda_m - \varepsilon) \|u\|^2 - C_\varepsilon b_\tau^\tau \|u\|^\tau \\
&= \frac{1}{2} (\lambda_{m+1} - \lambda_m - \varepsilon) \|u\|^2 - \widetilde{C}_\varepsilon \|u\|^\tau.
\end{aligned}$$

Taking $\varepsilon = \frac{\lambda_{m+1} - \lambda_m}{2}$, we then have

$$J(u) \geq \frac{1}{2} \varepsilon \|u\|^2 - \widetilde{C}_\varepsilon \|u\|^\tau.$$

Because $\tau > 2$, the function $h(\rho) = \frac{1}{2} \varepsilon \rho^2 - \widetilde{C}_\varepsilon \rho^\tau$ defined on $[0, \infty)$ achieves its maximum

$$h_{\max} = \frac{\tau - 2}{2} \big(\tau^{-1} \varepsilon\big)^{\frac{\tau}{\tau - 2}} \widetilde{C}_\varepsilon^{\frac{2}{2 - \tau}}$$

at

$$\rho_m = \left(\varepsilon^{-1}\tau\widetilde{C}_\varepsilon\right)^{\frac{1}{2-\tau}}.$$

Hence we get (3.4), where $\alpha_m := \frac{3}{4}h_{\max}$.

(ii) For any $u \in E_{m-1}$, by (3.6), we obtain

$$
\begin{aligned}
J(u) &= \frac{1}{2}\langle Au, u\rangle - \frac{1}{2}\lambda_m\|u\|^2 - \sum_{k=1}^{N} F(k, u(k)) \\
&\le \frac{1}{2}(\lambda_{m-1} - \lambda_m)\|u\|^2 + \sum_{k=1}^{N}\left(\frac{1}{2}\varepsilon\big|u(k)\big|^2 + C_\varepsilon\big|u(k)\big|^\tau\right) \\
&\le \frac{1}{2}(\lambda_{m-1} - \lambda_m + \varepsilon)\|u\|^2 + C_\varepsilon b_\tau^\tau\|u\|^\tau \\
&= \frac{1}{2}(\lambda_{m-1} - \lambda_m + \varepsilon)\|u\|^2 + \widetilde{C}_\varepsilon\|u\|^\tau.
\end{aligned}
$$

Taking $\varepsilon = \frac{\lambda_m - \lambda_{m-1}}{2}$, we then obtain

$$J(u) \le -\frac{1}{2}\varepsilon\|u\|^2 + \widetilde{C}_\varepsilon\|u\|^\tau.$$

As $\tau > 2$, the function $h(\rho) = -\frac{1}{2}\varepsilon\rho^2 + \hat{C}_\varepsilon\rho^\tau$ defined on $[0, \infty)$ achieves its minimum

$$h_{\min} = -\frac{\tau-2}{2}\left(\tau^{-1}\varepsilon\right)^{\frac{\tau}{\tau-2}}\widetilde{C}_\varepsilon^{\frac{2}{2-\tau}}$$

at

$$\widetilde{\rho}_m = \left(\varepsilon^{-1}\tau\widetilde{C}_\varepsilon\right)^{\frac{1}{2-\tau}}.$$

Therefore, we obtain (3.5), where $\widetilde{\alpha}_m := \frac{3}{4}h_{\min}$. □

Now we define

$$
\begin{aligned}
&B_m = \left\{u \in E_m^\perp, \|u\| = \rho_m\right\}, && E^{m+1} = E_m \oplus \text{span}\{\phi_{m+1}\}, && m \in \mathbb{Z}[0, N_1 - 1], \\
&\widetilde{B}_m = \left\{u \in E_{m-1}, \|u\| = \widetilde{\rho}_m\right\}, && \widetilde{E}^{m+1} = \text{span}\{\phi_{m-1}\} \oplus E_{m-1}^\perp, && m \in \mathbb{Z}[1, N_1], \\
&F^-(k, x) = \max\{-F(k, x), 0\}, && F^+(k, x) = \max\{F(k, x), 0\}, \\
&M^- = \sup_{(k,x)\in\mathbb{Z}[1,N]\times\mathbb{R}} F^-(k, x), && M^+ = \sup_{(k,x)\in\mathbb{Z}[1,N]\times\mathbb{R}} F^+(k, x).
\end{aligned}
$$

Lemma 3.3 (i) *Suppose that f satisfies (f_∞^+). Then, for any fixed $m \in \mathbb{Z}[0, N_1 - 1]$, there exist $\sigma > 0$ and $R > 0$ such that when $M^- \le \sigma$,*

$$J(u) \le \alpha_m \quad \text{for } u \in \partial Q_m, \tag{3.7}$$

where $Q_m = \{u \in E^{m+1} \mid \|u\| \le R, u = v_m + t\phi_{m+1}, v_m \in E_m, t \ge 0\}$.

(ii) *Suppose that f satisfies* (f_∞^-). *Then, for any fixed* $m \in \mathbb{Z}[1, N_1]$, *there exist* $\sigma > 0$ *and* $R > 0$ *such that when* $M^+ \leq \sigma$,

$$-J(u) \leq \widetilde{\alpha}_m, \quad \textit{for } u \in \partial\widetilde{Q}_m,$$

where $\widetilde{Q}_m = \{u \in \widetilde{E}^{m+1} \mid \|u\| \leq R, u = t\phi_{m-1} + v_{m-1}, v_{m-1} \in E_{m-1}, t \geq 0\}$.

Proof We only prove (i); the proof of (ii) is similar. For any $u \in E^{m+1}$, it follows from (f_∞^+) and (3.2) that

$$J(u) = \frac{1}{2}\langle Au, u\rangle - \frac{1}{2}\lambda_m\|u\|^2 - \sum_{k=1}^{N} F(k, u(k))$$

$$\leq \frac{1}{2}(\lambda_{m+1} - \lambda_m)\|u\|^2 - \sum_{k=1}^{N} C_1(|u(k)|^\theta - 1)$$

$$\leq \frac{1}{2}(\lambda_{m+1} - \lambda_m)\|u\|^2 - C_1 a_\theta^\theta \|u\|^\theta + NC_1.$$

As $\theta > 2$, we get

$$J(u) \to -\infty, \quad u \in E^{m+1}, \|u\| \to \infty.$$

Therefore, there exists $R > \rho_m$ such that

$$J(u) \leq 0, \quad u \in E^{m+1}, \|u\| = R. \tag{3.8}$$

For $u \in E_m$,

$$J(u) = \frac{1}{2}\langle Au, u\rangle - \frac{1}{2}\lambda_m\|u\|^2 - \sum_{k=1}^{N} F(k, u(k))$$

$$\leq - \sum_{F(k,u(k))\leq 0} F(k, u(k))$$

$$\leq NM^-.$$

Taking $\sigma = \frac{\alpha_m}{N}$, when $M^- \leq \sigma$,

$$J(u) \leq \alpha_m \quad \text{for } u \in E_m \text{ with } \|u\| \leq R. \tag{3.9}$$

By (3.8) and (3.9), we get (3.7). □

4 Main results and proofs

In this section, we give our main results and proofs. First, we compute the critical groups of J at both infinity and the origin.

Lemma 4.1 *Assume that f satisfies* (f_0). *Then*
  (i) $C_q(J, \infty) \cong \delta_{q,N}\mathbb{F}$ *iff satisfies* (f_∞^+).
  (ii) $C_q(-J, \infty) \cong \delta_{q,N}\mathbb{F}$ *iff satisfies* (f_∞^-).

Proof We only prove (i); (ii) can be proved similarly. Fix $\lambda_m \in \mathbb{R}$. Defining $B = \{u \in E \mid \|u\| \leq 1\}$, we then have $\partial B = \{u \in E \mid \|u\| = 1\}$. For any $u \in \partial B$ and $t > 0$, it follows from (3.2) and (1.3) that

$$
\begin{aligned}
J(tu) &= \frac{1}{2} t^2 \langle Au, u \rangle - \frac{1}{2} \lambda_m t^2 - \sum_{k=1}^{N} F(k, tu(k)) \\
&\leq \frac{1}{2} (\lambda_{N_1} - \lambda_m) t^2 - \sum_{k=1}^{N} C_1 \left(|tu(k)|^\theta - 1 \right) \\
&\leq \frac{1}{2} (\lambda_{N_1} - \lambda_m) t^2 - C_1 a_\theta^\theta t^\theta + NC_1.
\end{aligned}
$$

As $\theta > 2$, this implies that

$$
J(tu) \to -\infty \quad \text{as } t \to +\infty. \tag{4.1}
$$

On the other hand, by (f_∞^+), we have

$$
\begin{aligned}
\frac{dJ(tu)}{dt} &= t \langle Au, u \rangle - \lambda_m t - \sum_{k=1}^{N} f(k, tu(k)) u(k) \\
&= \frac{1}{t} \left(2J(tu) + \sum_{k=1}^{N} \left(2F(k, tu(k)) - f(k, tu(k)) tu(k) \right) \right) \\
&\leq \frac{1}{t} \left(2J(tu) + \sum_{|tu(k)| \leq r} \left(2F(k, tu(k)) - f(k, tu(k)) tu(k) \right) \right) \\
&\leq \frac{1}{t} \left(2J(tu) + NM_1 \right),
\end{aligned}
$$

where $M_1 := \max_{\{|x| \leq r, k \in \mathbb{Z}[1,N]\}} |2F(k, x) - f(k, x)x|$. Hence, for any fixed number a with $2a < -NM_1$, we obtain that

$$
J(tu) \leq a \quad \text{implies} \quad \frac{dJ(tu)}{dt} < 0. \tag{4.2}
$$

By $J(0) = 0$, (4.1) and (4.2), we get that for any $u \in \partial B$, there exists a unique $\xi(u) > 0$ such that

$$
J(\xi(u)u) = a, \quad u \in \partial B. \tag{4.3}
$$

By (4.3) and the implicit function theorem, we obtain that $\xi \in C(\partial B, \mathbb{R})$. Define $h : E \setminus \{0\} \to \mathbb{R}$ as

$$
h(u) = \begin{cases} 1 & \text{if } J(u) \leq a, \\ \|u\|^{-1} \xi(\|u\|^{-1} u) & \text{if } J(u) > a, u \neq 0. \end{cases}
$$

Then $h \in C(E \setminus \{0\}, \mathbb{R})$. Now define $\eta : [0,1] \times E \setminus \{0\} \to E \setminus \{0\}$ as

$$
\eta(t, u) = (1 - t)u + th(u)u.
$$

Clearly, η is continuous, and for any $u \in E \setminus \{0\}$ with $J(u) > a$, it follows from (4.3) that

$$J\big(\eta(1, u)\big) = J\big(\xi\big(\|u\|^{-1} u\big)\|u\|^{-1} u\big) = a.$$

Therefore

$$\eta(1, u) \in J^a, \quad \forall u \in E \setminus \{0\}, \qquad \eta(t, u) = u, \quad \forall t \in [0, 1], u \in J^a.$$

So J^a is a strong deformation retract of $E \setminus \{0\}$. Notice that $\dim E = N$. Hence we have

$$C_q(J, \infty) := H_q\big(E, J^a\big) \cong H_q\big(E, E \setminus \{0\}\big) \cong H(B, \partial B) \cong \delta_{q,N} \mathbb{F}. \qquad \square$$

Lemma 4.2 *Assume that f satisfies (f_0).*

(i) *If f satisfies (f_0^-), then for $m \in \mathbb{Z}[1, N_1]$,*

$$C_q(J, 0) \cong \delta_{q,2m-1} \mathbb{F}, \qquad C_q(-J, 0) \cong \delta_{q,N-2m+1} \mathbb{F}.$$

(ii) *If f satisfies (f_0^+), then for $m \in \mathbb{Z}[0, N_1 - 1]$,*

$$C_q(J, 0) \cong \delta_{q,2m+1} \mathbb{F}, \qquad C_q(-J, 0) \cong \delta_{q,N-2m-1} \mathbb{F}.$$

Proof By (f_0) and (1.6), we have $J''(0) = A - \lambda_m I_N$.

It follows from (f_0) that $u = 0$ is a degenerate critical point of J with Morse index $\mu(0) = \omega_{m-1}$ and nullity $\nu(0)$. This implies that $u = 0$ is a degenerate critical point of $-J$ with Morse index $\widetilde{\mu}(0) = N - \omega_m$ and nullity $\widetilde{\nu}(0) = \nu(0)$.

(i) By (f_0^-), we can verify that J has a local linking structure at 0 with respect to $E = E_{m-1} \oplus E_{m-1}^\perp$ (see [12]). That implies $-J$ has a local linking structure at 0 with respect to $E = E_{m-1}^\perp \oplus E_{m-1}$. Notice that $\omega_{m-1} = 2m - 1$ and $\widetilde{\mu}(0) + \widetilde{\nu}(0) = N - 2m + 1$ for $m \in \mathbb{Z}[1, N_1]$. Using Proposition 2.2, we get

$$C_q(J, 0) \cong \delta_{q,2m-1} \mathbb{F}, \qquad C_q(-J, 0) \cong \delta_{q,N-2m+1} \mathbb{F}.$$

(ii) Similarly, by (f_0^+), we can verify that J has a local linking structure at 0 with respect to $E = E_m \oplus E_m^\perp$ (see [12]). That implies $-J$ has a local linking structure at 0 with respect to $E = E_m^\perp \oplus E_m$. Notice that $\mu(0) + \nu(0) = \omega_m = 2m + 1$ for $m \in \mathbb{Z}[0, N_1 - 1]$. By Proposition 2.2, we have

$$C_q(J, 0) \cong \delta_{q,2m+1} \mathbb{F}, \qquad C_q(-J, 0) \cong \delta_{q,N-2m-1} \mathbb{F}. \qquad \square$$

Next, we give our main results.

Theorem 4.1 *Let f satisfy (f_0) and (f).*

(i) *If f satisfies (f_∞^+), then for any fixed $m \in \mathbb{Z}[0, N_1 - 1]$, there exists $\sigma > 0$ such that when $M^- \leq \sigma$, (P) has at least one nonzero periodic solution u_1 satisfying*

$$C_{2m+2}(J, u_1) \ncong 0. \tag{4.4}$$

(ii) *If f satisfies (f_∞^-), then for any fixed $m \in \mathbb{Z}[1, N_1]$, there exists $\sigma > 0$ such that when $M^+ \le \sigma$, (P) has at least one nonzero periodic solution u_1 satisfying*

$$C_{N-2m+2}(-J, u_1) \not\cong 0. \tag{4.5}$$

Theorem 4.2 *Let f satisfy (f_0) and (f).*

(i) *If f satisfies (f_0^-) and (f_∞^+), then, for any fixed $m \in \mathbb{Z}[1, N_1 - 2]$, there exists $\sigma > 0$ such that when $M^- \le \sigma$, (P) has at least three nonzero periodic solutions.*

(ii) *If f satisfies (f_0^+) and (f_∞^-), then, for any fixed $m \in \mathbb{Z}[2, N_1 - 1]$, there exists $\sigma > 0$ such that when $M^+ \le \sigma$, (P) has at least three nonzero periodic solutions.*

Finally, we prove our main results.

Proof of Theorem 4.1 We only prove (i); the proof of (ii) is similar. It follows from (f_∞^+) and Lemma 3.1 that J satisfies the (PS) condition. By Lemmas 3.2 and 3.3, J satisfies (H_1) and (H_2). This implies that B_m and ∂Q_m homologically link with respect to the direct sum decomposition $E = E_m \oplus E_m^\perp$ (see Example 3 of Chapter II in [16]). Notice that $\dim E_m = 2m + 1$. Applying Proposition 2.3, we get that J has a critical point u_1 such that $J(u_1) = c_1 \ge \alpha_m$ and (4.4). Moreover, it follows from (f_0) that $J(0) = 0$. Hence $u_1 \ne 0$. □

Proof of Theorem 4.2 We only prove (i); (ii) can be proved similarly. It follows from Lemma 4.1(i) that

$$C_q(J, \infty) \cong \delta_{q,N} \mathbb{F}. \tag{4.6}$$

Using Lemma 4.2(i), we have

$$C_q(J, 0) \cong \delta_{q,2m-1} \mathbb{F}. \tag{4.7}$$

By Theorem 4.1, we know that there exists $\sigma > 0$ such that when $M^- \le \sigma$, (P) has at least one nonzero periodic solution u_1 satisfying (4.4).

By Proposition 2.1(i), we have

$$C_q(J, u_1) \cong 0 \quad \text{for } q \notin \big[\mu(u_1), \mu(u_1) + \nu(u_1)\big].$$

Combining with (4.4), we get that $2m + 2 \in [\mu(u_1), \mu(u_1) + \nu(u_1)]$. Note that $\nu(u_1) = \dim \ker J''(u_1) \le 2$.

(1) If $\nu(u_1) = 0$, then by Proposition 2.1(ii) and (4.4),

$$C_q(J, u_1) \cong \delta_{q,2m+2} \mathbb{F}.$$

(2) If $\nu(u_1) = 1$, then $2m + 2 = \mu(u_1)$ or $2m + 2 = \mu(u_1) + \nu(u_1)$. By Proposition 2.1(iii),

$$C_q(J, u_1) \cong \delta_{q,2m+2} \mathbb{F}.$$

(3) If $\nu(u_1) = 2$, then $2m + 2 = \mu(u_1)$ or $2m + 2 = \mu(u_1) + \nu(u_1)$ or $C_q(J, u_1) \cong 0$ for $q \neq \mu(u_1) + 1$. It follows from Proposition 2.1(iii) and (4.4) that

$$C_q(J, u_1) \cong \begin{cases} \delta_{q,2m+2}\mathbb{F} & \text{if } 2m + 2 = \mu(u_1) \text{ or } 2m + 2 = \mu(u_1) + \nu(u_1), \\ 0 & \forall q \neq 2m + 2. \end{cases}$$

Thus, we conclude that

$$C_q(J, u_1) \cong \begin{cases} \delta_{q,2m+2}\mathbb{F} & \text{if } \nu(u_1) = 0, 1, \\ 0 & \forall q \neq 2m + 2, \text{if } \nu(u_1) = 2. \end{cases} \tag{4.8}$$

Assume that $\mathcal{K} = \{0, u_1\}$. Then the $2m$th Morse inequality (2.1) is expressed as $-1 > 0$. This is impossible. Thus J must have another nonzero critical point u_2. By Morse theory, we have either

$$C_{2m}(J, u_2) \not\cong 0 \tag{4.9}$$

or

$$C_{2m-2}(J, u_2) \not\cong 0. \tag{4.10}$$

Suppose that (4.9) holds. Then it follows from (4.9) and Proposition 2.1(i), (iii) that

$$C_q(J, u_2) \cong \begin{cases} \delta_{q,2m}\mathbb{F} & \text{if } \nu(u_2) = 0, 1, \\ 0 & \forall q \neq 2m, \text{if } \nu(u_2) = 2. \end{cases} \tag{4.11}$$

Noticing that $m \in \mathbb{Z}[1, N_1 - 2]$, we have that

$$\begin{cases} 2m + 3 \leq N & \text{while } N \text{ is odd,} \\ 2m + 3 < N & \text{while } N \text{ is even.} \end{cases} \tag{4.12}$$

Assume that $\mathcal{K} = \{0, u_1, u_2\}$. We will divide the consideration into four cases.

(1) $\nu(u_1) = 0, 1$ and $\nu(u_2) = 0, 1$. By (4.6)-(4.8), (4.11) and (4.12), we get

$$\begin{cases} (2.2) \text{ is described as } 1 = -1 & \text{while } N \text{ is odd,} \\ \text{the } 2m + 3\text{th } (2.1) \text{ is expressed as } -1 \geq 0 & \text{while } N \text{ is even.} \end{cases} \tag{4.13}$$

These are impossible.

(2) $\nu(u_1) = 2$ and $\nu(u_2) = 2$. By the $2m$th and $2m + 1$th Morse inequalities, we get that

$$\text{rank } C_{2m}(J, u_2) = 1. \tag{4.14}$$

It then follows from (4.6)-(4.8), (4.11), (4.12) and (4.14) that

$$\begin{cases} (2.2) \text{ is expressed as } \text{rank } C_{2m+2}(J, u_1) = -1 & \text{while } N \text{ is odd,} \\ \text{the } 2m + 3\text{th } (2.1) \text{ is expressed as } -\text{rank } C_{2m+2}(J, u_1) \geq 0 & \text{while } N \text{ is even.} \end{cases} \tag{4.15}$$

These contradict (4.8).

(3) $\nu(u_1) = 0, 1$ and $\nu(u_2) = 2$. We have (4.14) and (4.13). These are impossible.

(4) $\nu(u_1) = 2$ and $\nu(u_2) = 0, 1$. Then we have (4.15), which contradicts (4.8).

Hence, J must have a third nonzero critical point u_3. The proof of the case where (4.10) holds is similar. This completes the proof. □

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