Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Dynamical behaviors of an SIRI epidemic model with nonlinear incidence and latent period

Peng Guo[1,2], Xinsong Yang[1] and Zhichun Yang[1*]

*Correspondence: yangzhch@126.com
[1]Department of Mathematics, Chongqing Normal University, No. 12 Tianchen Road, Shapingba District, Chongqing, 400047, China
Full list of author information is available at the end of the article

Abstract

In this paper, we study an SIRI epidemic model with nonlinear incidence rate and latent period, namely $\frac{kI(t-\tau)S}{1+\alpha I^h(t-\tau)}$, which describes the psychological effect of certain serious diseases on the community when the size of the set of infective individuals is getting larger. We first obtain the threshold dynamics on the global stability of the equilibria for the model without latent period, and then we analyze the stability and Hopf bifurcation for the model with the latent period. The results show the influence of nonlinear incidence rate and latent period on the dynamical behaviors of the SIRI model. The examples and its simulations are given to illustrate the obtained results.

Keywords: SIRI model; stability; nonlinear incidence rate; Hopf bifurcation

1 Introduction

An SIRI epidemiological model in a population was formulated by Tudor [1], which consists of a system with three compartments: susceptible, infective, and removed individuals, labeled by S, I, R. In this model, susceptibles become infectious, then they are removed with temporary immunity, and finally they become infectious again. The SIRI model is appropriate for some diseases such as human and bovine tuberculosis, and herpes [2–4], in which recovered individuals may revert back to the infective class due to reactivation of the latent infection or incomplete treatment. Some authors have studied this type of SIRI epidemic model to understand the principle of disease transmissions. Moreira and Wang [5] studied an SIRI model with general saturated incidence rate $\varphi(S)I$ and derived sufficient conditions for the global asymptotic stability of the disease-free and endemic equilibria of the model by using an elementary analysis of Lienard's equation and Lyapunov's direct method. In 2007, van den Driessche *et al.* [6] formulated an SEIRI model with standard incidence rate and distributed delay for a disease with an exposed (latent) period and relapse, and they investigated its threshold property by the obtained basic reproduction number. However, the qualitative analysis of an SIRI epidemic model with nonlinear incidence rates in the form of $\frac{kI^pS}{1+\alpha I^q}$ was not investigated.

In 1978, Capasso and Serio [7] introduced a saturated incidence rate $\frac{kI}{1+\alpha I}$ by research of the cholera epidemic spread in Bari. Ruan and Wang [8] studied an epidemic model with a particular nonlinear incident rate $\frac{kI^2S}{1+\alpha I^2}$ and provided a detailed qualitative analysis on the Hopf bifurcation and Bogdanov-Takens bifurcation for the model. Xiao and Ruan

Springer

©2014 Guo et al.; licensee Springer. This is an Open Access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/2.0), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

[9] studied nonlinear incidence rates of $\frac{kIS}{1+\alpha I^2}$. Yang and Xiao [10] investigated the general nonlinear incidence rate $\frac{kIS}{1+\alpha I^h}$. Recently, a more general incidence rate $\frac{kI^p S}{1+\alpha I^q}$ was investigated by Liu *et al.* [11], Derrick and van den Driessche [12], *etc.* In addition, for many infectious diseases, an individual once infected is unable to immediately infect another, and he undergoes a certain latent period before it can infect others. Mathematically, this corresponds to the delay effect of the infection (see also [10, 13, 14]). Therefore, it is interesting to investigate the nonlinear incidence rates and the latent period.

Our aim is to investigate an SIRI epidemic model with temporary immunity, nonlinear incidence rate, and latent period. The organization of this paper is as follows. In Section 2, we give the description of the SIRI model, and we discuss the existence of equilibria of the model by the obtained basic reproduction number $\mathscr{R}_0$. In Section 3, we obtain the threshold dynamics on stability for the model without the latent period (*i.e.*, $\tau = 0$). Furthermore, we study the stability and Hopf bifurcation for the model with $\tau > 0$ in Section 4. From these results, it can be seen that the nonlinear incidence rate and latent period influence the dynamical behaviors of the SIRI model. In Section 5, we give some conclusions and a discussion of the influence, and some illustrative examples and simulations also are provided.

2 Model description and preliminaries

In this paper, we consider the general incidence rate in an epidemiological model

$$g(I)S = \frac{kIS}{1+\alpha I^h},$$

where kI represents the infection force of the disease and $\frac{1}{1+\alpha I^h}$ represents the inhibition effect from the behavioral change of the susceptible individuals when their number increases or from the crowding effect of the infective individuals. Furthermore, assume that the latent period is a constant τ, that is, a susceptible becomes infective if he contacted the infected τ times ago. Therefore, delay can be incorporated into the nonlinear incidence function as follows:

$$g\big(I(t)\big)S = \frac{kI(t-\tau)S}{1+\alpha I^h(t-\tau)}.$$

Then we propose the following SIRI model with general incidence rate and latent period:

$$\begin{cases} \frac{dS(t)}{dt} = b - dS(t) - \frac{kS(t)I(t-\tau)}{1+\alpha I^h(t-\tau)}, \\ \frac{dI(t)}{dt} = \frac{kS(t)I(t-\tau)}{1+\alpha I^h(t-\tau)} - (d+\mu)I(t) + \gamma R(t), \\ \frac{dR(t)}{dt} = \mu I(t) - (d+\gamma)R(t), \end{cases} \tag{1}$$

where b is the birth rate of the population, d is the natural death rate of the population, k is the proportionality constant, μ is the rate at which infected individuals become temporarily immune, γ is the rate at which the recovered class revert to the infective class, α is the saturated parameter, and b, d, k, α, γ, μ, h are positive parameters.

For convenience, we first transform the three-dimensional Eq. (1) into its two-dimensional limit form. Summing up the three equations in Eq. (1) and denoting $N(t) = S(t) + I(t) + R(t)$, we have

$$\frac{dN(t)}{dt} = b - dN(t).$$

Clearly,

$$N(t) = \frac{1}{d}\left(b - \left(b - dN(t_0)\right)e^{-d(t-t_0)}\right) \quad \text{and} \quad \lim_{t\to\infty} N(t) = \frac{b}{d}. \tag{2}$$

Then we obtain the following limit equations of Eq. (1):

$$\begin{cases} \frac{dI(t)}{dt} = \frac{kI(t-\tau)}{1+\alpha I^h(t-\tau)}(\frac{b}{d} - I - R) - (d+\mu)I(t) + \gamma R(t), \\ \frac{dR(t)}{dt} = \mu I(t) - (d+\gamma)R(t). \end{cases} \tag{3}$$

Following the idea in [8, 11, 15], we shall mainly study the existence, uniqueness, and stability of equilibria and Hopf bifurcation of the limit Eq. (3) to obtain the dynamics of Eq. (1) (see also [9, 16, 17]).

As regards the biological meaning, we always assume that the initial value of Eq. (3) is nonnegative and has the form of

$$I(s) = \varphi_1(s) \geq 0, \qquad R(s) = \varphi_2(s) \geq 0, \tag{4}$$

where $(\varphi_1, \varphi_2) \in C$, and $C = C([-\tau, 0], \mathbb{R}^2_+)$ represents the continuous function space on $[-\tau, 0]$ with the norm $\|\varphi\| = \sup_{-\tau\leq\theta\leq 0} |\varphi(\theta)|$, $\varphi \in C$. From the literature on retarded differential equations or functional differential equations in [18, 19], there is a unique local solution of Eq. (3) for all allowable positive parameters.

Firstly, we have the following conclusions.

Lemma 2.1 *The region* $S_\triangle = \{(I,R) | I \geq 0, R \geq 0, I + R \leq \frac{b}{d}\}$ *is an invariant set and also an absorbing set of Eq.* (3) *in the first quadrant.*

Proof From (3), we can see that on the line $I = 0$, $\frac{dI}{dt} > 0$, and on the line $R = 0$, $\frac{dR}{dt} > 0$. Hence, no orbit of Eq. (3) can exit from the first quadrant, with the boundary $R = 0$ and $I = 0$. From (2), we have

$$\lim_{t\to\infty} I(t) \leq \lim_{t\to\infty} N(t) = \frac{b}{d}, \qquad \lim_{t\to\infty} R(t) \leq \lim_{t\to\infty} N(t) = \frac{b}{d}.$$

Since

$$\begin{aligned} \left.\frac{d(I+R)}{dt}\right|_{I+R=\frac{b}{d}} &= \left.\frac{dI}{dt} + \frac{dR}{dt}\right|_{I+R=\frac{b}{d}} \\ &= \left.\frac{kI(t-\tau)}{1+\alpha I^h(t-\tau)}\left(\frac{b}{d} - I - R\right) - d(I+R)\right|_{I+R=\frac{b}{d}} \\ &= -b < 0, \end{aligned}$$

the orbit of Eq. (3) getting at the boundary $I + R = \frac{b}{d}$ must go into the interior of $S_\triangle$. Thus, the region $S_\triangle$ is an absorbing set of Eq. (3) in the first quadrant. This completes the proof. □

As is well known, the basic reproductive number is a key parameter to discuss the dynamical behaviors of epidemic model. According to the literature [20, 21], we get the basic reproductive number of model (3) as follows:

$$\mathscr{R}_0 = \frac{kb\gamma + kbd + \mu\gamma d}{d(\mu + d)(\gamma + d)}. \tag{5}$$

In terms of $\mathscr{R}_0$, we obtain the following result on the existence of the equilibrium for Eq. (3).

Lemma 2.2 (i) *If* $\mathscr{R}_0 \leq 1$, *then Eq.* (3) *has a unique equilibrium* $E_0 = (0, 0)$ *in the first quadrant.*

(ii) *If* $\mathscr{R}_0 > 1$, *then Eq.* (3) *has two equilibria in the first quadrant, which are* E_0 *and* $E_* = (I_*, R_*)$, *where* $I_*, R_* > 0$.

Proof Clearly, Eq. (3) always has an equilibrium $E_0 = (0, 0)$, and it has no other boundary equilibrium in the first quadrant from the second equation of (3). Furthermore, Eq. (3) has a positive equilibrium $E_* = (I_*, R_*)$ if and only if it satisfies

$$\begin{cases} \frac{kI}{1+\alpha I^h}(\frac{b}{d} - I - R) - (d + \mu)I + \gamma R = 0, \\ \mu I - (d + \gamma)R = 0. \end{cases} \tag{6}$$

It is equivalent to I_* being a positive solution of

$$\left(\frac{\gamma\mu}{\gamma + d} - (d + \mu)\right)\alpha I^h - \left(\frac{k\mu}{\gamma + d} + k\right)I = (d + \mu)(1 - \mathscr{R}_0). \tag{7}$$

When $\mathscr{R}_0 \leq 1$, Eq. (7) has no positive solution since $\frac{\gamma\mu}{\gamma+d} - (d + \mu) < 0$. So, the conclusion in case (i) holds.

To prove case (ii), set

$$\varphi(I) = \left(\frac{\gamma\mu}{\gamma + d} - (d + \mu)\right)\alpha I^h - \left(\frac{k\mu}{\gamma + d} + k\right)I.$$

Thus

$$\varphi'(I) = \left(\frac{\gamma\mu}{\gamma + d} - (d + \mu)\right)h\alpha I^{h-1} - \left(\frac{k\mu}{\gamma + d} + k\right).$$

When $\mathscr{R}_0 > 1$, we have $\varphi'(I) < 0$ and φ is strictly monotone decreasing for $I \geq 0$. Also,

$$\varphi\left(\frac{b}{d}\right) < (d + \mu) - \frac{kb}{d} - \frac{\gamma\mu}{\gamma + d} = (d + \mu)(1 - \mathscr{R}_0) < 0 = \varphi(0).$$

This implies that Eq. (7) has one unique positive solution and the conclusion in case (ii) holds. Therefore, we have completed the proof. □

As usual, E_0 and E_* are called the disease-free equilibrium (DFE) and the endemic equilibrium (EE) of Eq. (3), respectively. In what follows, we shall mainly discuss dynamical behaviors on the stability of two equilibria and Hopf bifurcation for Eq. (3) with or without the latent period.

3 Threshold dynamics for the case $\tau = 0$

In this section, we shall study the stability and topological structure of the DFE and EE of Eq. (3) without the latent period, that is,

$$\begin{cases} \frac{dI}{dt} = \frac{kI}{1+\alpha I^h}(\frac{b}{d} - I - R) - (d+\mu)I + \gamma R, \\ \frac{dR}{dt} = \mu I - (d+\gamma)R. \end{cases} \tag{8}$$

We first give the local stability and topological structure of the equilibria.

Lemma 3.1 (i) *If $\mathscr{R}_0 < 1$, then the DFE E_0 of Eq.* (8) *is a locally asymptotically stable hyperbolic node.*

(ii) *If $\mathscr{R}_0 = 1$, then DFE E_0 of Eq.* (8) *is a saddle-node and is locally asymptotically stable in the first quadrant.*

(iii) *If $\mathscr{R}_0 > 1$, then the DFE E_0 of Eq.* (8) *is an unstable saddle and the EE E_* is a locally asymptotically stable hyperbolic node.*

Proof The Jacobian matrix of Eq. (8) at $E_0 = (0,0)$ is

$$M_0 = \begin{bmatrix} \frac{kb}{d} - (d+\mu) & \gamma \\ \mu & -(d+\gamma) \end{bmatrix}.$$

When $\mathscr{R}_0 < 1$, we compute the determinant and trace of M_0

$$\det(M_0) = (d+\mu)(d+\gamma)(1-\mathscr{R}_0) > 0,$$
$$\operatorname{tr}(M_0) = (d+\mu)(\mathscr{R}_0 - 1) - \frac{\mu\gamma}{d+\gamma} - (d+\gamma) < 0.$$

Then the DFE E_0 of Eq. (8) is locally asymptotically stable if $\mathscr{R}_0 < 1$.

When $\mathscr{R}_0 = 1$, $\det(M_0) = 0$ and $\operatorname{tr}(M_0) < 0$, which implies that the DFE E_0 is locally stable since one eigenvalue of M_0 is equal to 0 and the other is less than 0. Furthermore, we construct the Lyapunov function $V(I,R) = I + \frac{\gamma}{\gamma+d}R$. From $\mathscr{R}_0 = 1$, we have

$$\left.\frac{dV(I(t),R(t))}{dt}\right|_{(8)} = -\frac{kI(t)}{1+\alpha I^h(t)}\bigl(I(t)+R(t)\bigr) - \frac{\alpha kbI^{h+1}(t)}{d(1+\alpha I^h(t))} < 0$$

and

$$\left\{\bigl(I(t),R(t)\bigr) \,\middle|\, \left.\frac{dV(I(t),R(t))}{dt}\right|_{(8)} = 0, \text{for all } t \geq 0\right\},$$

and this has a unique point $E_0 = (0,0)$. It follows from the Lasalle invariant principle that the DFE E_0 is asymptotically stable when $\mathscr{R}_0 = 1$.

When $\mathscr{R}_0 > 1$, we have $\det(M_0) < 0$, which implies E_0 of Eq. (8) is an unstable saddle. Furthermore, Eq. (8) has an unique EE $E_* = (I_*, R_*)$. We compute the Jacobian matrix at E_* as follows:

$$M_* = \begin{bmatrix} \frac{k+(1-h)k\alpha I_*^h}{(1+\alpha I_*^h)^2}(\frac{b}{d} - I_* - R_*) - \frac{kI_*}{1+\alpha I_*^h} - (d+\mu) & -\frac{kI_*}{1+\alpha I_*^h} + \gamma \\ \mu & -(d+\gamma) \end{bmatrix}.$$

Let $\Lambda_1 = \frac{1}{(1+\alpha I_*^h)^2}$. Since $(I_*, R_*) \in S_\triangle$ satisfies Eq. (6), we have

$$\begin{aligned}
\det(M_*) &= \Lambda_1\biggl[-(d+\gamma)\bigl(1+\alpha I_*^h\bigr)k\biggl(\frac{b}{d} - I_* - R_*\biggr) + kh\alpha I_*^h(d+\gamma)\biggl(\frac{b}{d} - I_* - R_*\biggr) \\
&\quad + (d+\gamma)kI_*\bigl(1+\alpha I_*^h\bigr) + (d+\mu)(d+\gamma)\bigl(1+\alpha I_*^h\bigr)^2 \\
&\quad + \mu k I_*\bigl(1+\alpha I_*^h\bigr) - \mu\gamma\bigl(1+\alpha I_*^h\bigr)^2\biggr] \\
&= -(d+\gamma)\biggl[\frac{k}{1+\alpha I_*^h}\biggl(\frac{b}{d} - I_* - R_*\biggr) - (d+\mu) + \frac{\mu\gamma}{d+\gamma}\biggr] \\
&\quad + \Lambda_1\biggl[kh\alpha I_*^h(d+\gamma)\biggl(\frac{b}{d} - I_* - R_*\biggr) + (d+\mu+\gamma)kI_*\bigl(1+\alpha I_*^h\bigr)\biggr] \\
&= \Lambda_1\biggl[kh\alpha I_*^h(d+\gamma)\biggl(\frac{b}{d} - I_* - R_*\biggr) + (d+\mu+\gamma)kI_*\bigl(1+\alpha I_*^h\bigr)\biggr] \\
&> 0
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{tr}(M_*) &= \Lambda_1\biggl(\bigl(k + (1-h)k\alpha I_*^h\bigr)\biggl(\frac{b}{d} - I_* - R_*\biggr) - kI_*\bigl(1+\alpha I_*^h\bigr) \\
&\quad - (d+\mu)\bigl(1+\alpha I_*^h\bigr)^2 - (d+\gamma)\bigl(1+\alpha I_*^h\bigr)^2\biggr) \\
&= \Lambda_1\biggl(\bigl(1+\alpha I_*^h\bigr)\biggl(k\biggl(\frac{b}{d} - I_* - R_*\biggr) - (d+\mu)\bigl(1+\alpha I_*^h\bigr)\biggr) \\
&\quad - hk\alpha I_*^h\biggl(\frac{b}{d} - I_* - R_*\biggr) - (d+\gamma)\bigl(1+\alpha I_*^h\bigr)^2 - kI_*\bigl(1+\alpha I_*^h\bigr)\biggr) \\
&= \Lambda_1\biggl(-\frac{\mu\gamma}{d+\gamma}I_*\bigl(1+\alpha I_*^h\bigr) - hk\alpha I_*^h\biggl(\frac{b}{d} - I_* - R_*\biggr) \\
&\quad - (d+\gamma)\bigl(1+\alpha I_*^h\bigr)^2 - kI_*\bigl(1+\alpha I_*^h\bigr)\biggr) \\
&< 0.
\end{aligned}$$

This implies that the endemic equilibrium E_* is a locally asymptotically stable hyperbolic node. The proof is completed. □

Furthermore, we discuss the topological structure of the trajectories of Eq. (8).

Lemma 3.2 *Equation* (8) *has neither a nontrivial periodic orbit nor a singular closed trajectory including a finite number of equilibria in* $S_\triangle$.

Proof Let

$$\begin{cases} \frac{dI}{dt} = \frac{kI}{1+\alpha I^h}\bigl(\frac{b}{d} - I - R\bigr) - (d+\mu)I + \gamma R \triangleq P(I,R), \\ \frac{dR}{dt} = \mu I - (d+\gamma)R \triangleq Q(I,R). \end{cases}$$

Select

$$0 < \alpha_1 < \min\biggl\{\alpha, \frac{d^h}{hb^h}\biggr\}.$$

Thus, we have $\alpha_1 - \alpha < 0$ and $h\alpha_1 \frac{b^h}{d^h} < 1$. Let

$$D(I,R) = \frac{1 + \alpha_1 I^h}{kI}.$$

Then

$$\begin{aligned}
&\frac{\partial(DP)}{\partial I} + \frac{\partial(DQ)}{\partial R} \\
&\quad = \frac{(\alpha_1 - \alpha)hI^{h-1}}{(1+\alpha I^h)^2}\left(\frac{b}{d} - I - R\right) - \frac{1+\alpha_1 I^h}{1+\alpha I^h} - \frac{(d+\mu)h\alpha_1 I^{h-1}}{k} \\
&\qquad + \gamma R \frac{k(h\alpha_1 I^h - 1) - k\alpha_1 I^h}{(kI)^2} - \frac{(1+\alpha_1 I^h)(d+\gamma)}{kI} \\
&\quad \le \frac{(\alpha_1 - \alpha)hI^{h-1}}{(1+\alpha I^h)^2}\left(\frac{b}{d} - I - R\right) - \frac{1+\alpha_1 I^h}{1+\alpha I^h} - \frac{(d+\mu)h\alpha_1 I^{h-1}}{k} \\
&\qquad + \gamma R \frac{k(h\alpha_1 \frac{b^h}{d^h} - 1) - kp\alpha_1 I^h}{(kI)^2} - \frac{(1+\alpha_1 I^h)(d+\gamma)}{kI} \\
&\quad < 0.
\end{aligned}$$

By the Bendixson-Dulac criterion [22], we know that Eq. (8) does not have nontrivial periodic orbits or a singular closed trajectory which contains a finite number of equilibria. The proof is completed. □

Then we show that there is no positive half-trajectory close to E_0 in $S_\triangle$ if $\mathscr{R}_0 > 1$.

Lemma 3.3 *The two stable manifolds of saddle E_0 are not in $S_\triangle$ if $\mathscr{R}_0 > 1$.*

Proof Suppose there is a stable manifold L_P^+ in $S_\triangle$, then the L_P^- stay in the $S_\triangle$ or traverse through $S_\triangle$. Assuming the former case, we know that the L_P^- satisfy one of the following cases (see [22]):

(i) $A_P = \{E_*\}$,

(ii) $A_P = \{E_0\}$, A_P is a closed orbit, or A_P is a singular closed trajectory,

where A_P is negative limit set of L_P. Case (i) and Lemma 3.1 lead to a contradiction, and case (ii) and Lemma 3.2 lead to a contradiction.

If the L_P^- ran out of the region $S_\triangle$, it is divided into two parts to $S_\triangle$ (if the two stable manifolds are all in $S_\triangle$, and all passed through this region $S_\triangle$, they are divided into three parts of $S_\triangle$) since $I = 0$ or $R = 0$ is not an orbit of Eq. (8). Then the equilibrium E_* is not in one part. In this part, we arbitrarily select a point $Q \neq O$ and Q is not in any of the stable manifolds, then the positive half orbit through the point Q will run out of this part or Ω_Q is a closed orbit or a singular closed trajectory, where Ω_Q is a positive limit set of L_Q. This conflicts with the topological structure of a saddle or Lemma 2.1 or Lemma 3.2. This completes the proof. □

On the basis of Lemmas 3.1, 3.2, and 3.3, we have the following.

Theorem 3.1 (i) *If $\mathscr{R}_0 < 1$, then Eq.* (8) *has a unique DFE E_0, which is a globally asymptotically stable hyperbolic node in the first quadrant.*

(ii) *If* $\mathscr{R}_0 = 1$, *then Eq.* (8) *has a unique DFE* E_0, *which is a saddle-node and is globally asymptotically stable in the first quadrant*.

(iii) *If* $\mathscr{R}_0 > 1$, *then Eq.* (8) *has one unique DFE* E_0 *and a unique EE* E_*, *and* E_0 *is a unstable saddle and* E_* *is a globally asymptotically stable hyperbolic node in the first quadrant*.

Proof If $\mathscr{R}_0 \leq 1$, $S_\triangle$ has a unique disease-free equilibrium. Obviously, $S_\triangle$ is a bounded absorbing region which contains exactly one critical point E_0. For any point P in $S_\triangle$, according to [22], its Omega-limit set satisfies $\Omega_P = \{E_0\}$ or Ω_P is a closed orbit or Ω_P is a singular closed trajectory. Lemma 3.2 essentially means that there is no closed orbit or singular closed trajectory in $S_\triangle$. So we have $\Omega_P = \{E_0\}$. This proves that E_0 is a global attractor in $S_\triangle$.

From Lemma 3.1 and Lemma 3.3, E_0 is an unstable saddle in $S_\triangle$ and its two stable manifolds are not in $S_\triangle$ if $\mathscr{R}_0 > 1$. By a similar proof, we have $\Omega_P = \{E_*\}$, where P is an arbitrary point in $S_\triangle - \{E_0\}$. This proves that E_* is a global attractor in $S_\triangle$.

It follows from Lemma 2.1 that $S_\triangle$ is a absorbing set in the first quadrant. Thus, E_0 in case (i) and case (ii) and E_* in case (iii) is a global attractor in the first quadrant.

Combining with the local stability in Lemma 3.1, we obtain the conclusion. This completes the proof. □

From the above theorem, we can see that $\mathscr{R}_0$ plays a threshold role in determining the stability of Eq. (8).

4 Dynamical behaviors for the case $\tau > 0$

In this section, we shall study the stability and Hopf bifurcation of the equilibria for Eq. (3) with the latent period $\tau > 0$.

The following result shows that $\mathscr{R}_0$ is a threshold value for the DFE.

Theorem 4.1 *If* $\mathscr{R}_0 \leq 1$, *the DFE* E_0 *of Eq.* (3) *is globally asymptotically stable*. *If* $\mathscr{R}_0 > 1$, *the DFE is unstable*.

Proof We firstly give the local stability of DFE $E_0 = (0, 0)$. Linearize Eq. (3) at E_0 and obtain the characteristic equation

$$\det\left(\lambda E - A_1 - e^{-\lambda\tau} B_1\right) = 0,$$

where E is the unit matrix and

$$A_1 = \begin{bmatrix} -(d+\mu) & \gamma \\ \mu & -(d+\gamma) \end{bmatrix}, \qquad B_1 = \begin{bmatrix} k\frac{b}{d} & 0 \\ 0 & 0 \end{bmatrix}.$$

Then we have

$$\left(\lambda + d + \mu - \frac{kb}{d} e^{-\lambda\tau}\right)(\lambda + d + \gamma) - \mu\gamma = 0,$$

i.e.

$$\lambda^2 + (2d + \gamma + \mu)\lambda + d^2 + (\mu + \gamma)d = \frac{kb}{d} e^{-\lambda\tau}(\lambda + d + \gamma). \tag{9}$$

From Theorem 3.1, the real parts of all roots of Eq. (9) are negative if $\mathscr{R}_0 < 1$ and $\tau = 0$. Thus, all roots of Eq. (9) have negative real parts if $0 < \tau \ll 1$. Assume Eq. (9) has a pair conjugate imaginary roots $\lambda = \pm\omega i$ if $\tau = \tau_0$, where ω is a positive real number. Then we have

$$\begin{aligned}
&-\omega^2 + i(2d + \gamma + \mu)\omega + d^2 + (\mu + \gamma)d \\
&\quad= \frac{kb}{d}(\cos\omega\tau_0 - i\sin\omega\tau_0)(i\omega + d + \gamma) \\
&\quad= \left(\frac{kb}{d}\omega\sin\omega\tau_0 + \frac{kb}{d}\cos\omega\tau_0\right) + i\left(\frac{kb}{d}\omega\cos\omega\tau_0 - \frac{kb}{d}(d+\gamma)\sin\omega\tau_0\right).
\end{aligned}$$

Thus, we have

$$\begin{cases} \frac{kb}{d}\omega\sin\omega\tau_0 + \frac{kb}{d}(d+\gamma)\cos\omega\tau_0 = -\omega^2 + d^2 + (\mu+\gamma)d, \\ -\frac{kb}{d}(d+\gamma)\sin\omega\tau_0 + \frac{kb}{d}\omega\cos\omega\tau_0 = \omega(2d + \gamma + \mu). \end{cases}$$

By a simple calculation, we have

$$(\omega)^4 + D_1\omega^2 + D_2 = 0, \tag{10}$$

where

$$\begin{aligned}
D_1 &= \left(\left(\frac{kb}{d}\right)^2 - 2\bigl(d^2 + (\mu+\gamma)d\bigr) + (2d + \mu + \gamma)^2\right), \\
D_2 &= \bigl(d^2 + (\mu+\gamma)d\bigr)^2 - \left(\frac{kb}{d}(d+\gamma)\right)^2.
\end{aligned}$$

If $\mathscr{R}_0 < 1$, then $D_1 > 0$, $D_2 > 0$. Accordingly, Eq. (10) does not have any real root and all roots of Eq. (3) have negative real parts for any $\tau > 0$ if $\mathscr{R}_0 < 1$.

By a similar analysis, we derive that Eq. (9) has a unique zero root and all other roots with negative real parts for any $\tau > 0$ if $\mathscr{R}_0 = 1$. Thus, E_0 is locally asymptotically stable if $\mathscr{R}_0 \le 1$ for any $\tau > 0$.

If $\mathscr{R}_0 > 1$, Eq. (9) has a root with the positive real part for any $\tau > 0$. Thus, the DFE is unstable.

Next, we show that the DFE is globally asymptotically stable if $\mathscr{R}_0 \le 1$. Construct the Lyapunov functional

$$V\bigl(I(t), R(t)\bigr) = I(t) + \frac{\gamma}{\gamma + d}R(t) + \frac{kb}{d}\int_{t-\tau}^{t} \frac{I(u)}{1 + \alpha I^h(u)}\,du. \tag{11}$$

Clearly, $V(0,0) = 0$ and $V(I(t), R(t)) > 0$ in the interior of R_+^2. Since $\mathscr{R}_0 \le 1$, we have

$$\begin{aligned}
\left.\frac{dV(I(t),R(t))}{dt}\right|_{(3)} &= \frac{kI(t-\tau)}{1 + \alpha I^h(t-\tau)}\left(\frac{b}{d} - I(t) - R(t)\right) - (d+\mu)I(t) \\
&\quad + \frac{\gamma\mu}{\gamma + d}I + \frac{kb}{d}\frac{I(t)}{1 + \alpha I^h(t)} - \frac{kb}{d}\frac{I(t-\tau)}{1 + \alpha I^h(t-\tau)} \\
&= -\frac{kI(t-\tau)}{1 + \alpha I^h(t-\tau)}\bigl(I(t) + R(t)\bigr)
\end{aligned}$$

$$\begin{aligned}
&+ \left(\frac{kb}{d} + \frac{\gamma\mu}{\gamma+d} - (d+\mu)\right) I(t) - \frac{\alpha k b I^{h+1}(t)}{d(1+\alpha I^h(t))} \\
= &-\frac{kI(t-\tau)}{1+\alpha I^h(t-\tau)}\big(I(t)+R(t)\big) - \frac{\alpha k b I^{h+1}(t)}{d(1+\alpha I^h(t))} \\
&- (d+\mu)(1-\mathscr{R}_0) \\
< &\ 0.
\end{aligned}$$

Furthermore, the set

$$\left\{\big(I(t),R(t)\big) \,\middle|\, \frac{dV(I(t),R(t))}{dt}\bigg|_{(3)} = 0, \text{for all } t \geq 0\right\}$$

has a unique point $E_0 = (0,0)$. It follows from the Lyapunov-Lasalle invariant principle in [18, 23] that the DFE E_0 is globally asymptotically stable. This completes the proof. □

When $\mathscr{R}_0 > 1$, the system has a unique endemic equilibrium $E_* = (I_*, R_*)$. In the following, we shall analyze the stability of E^*, which is dependent on the parameters h and τ. To do this, we calculate the linearization of Eq. (3) at E_* and obtain the characteristic equation

$$\det\big(\lambda E - A_2 - e^{-\lambda\tau} B_2\big) = 0,$$

where E is the unit matrix and

$$A_2 = \begin{bmatrix} -\frac{kI_*}{1+\alpha I_*^h} - (d+\mu) & -\frac{kI_*}{1+\alpha I_*^h} + \gamma \\ \mu & -(d+\gamma) \end{bmatrix}, \qquad B_2 = \begin{bmatrix} \frac{1+(1-h)\alpha I_*^h}{(1+\alpha I_*^h)^2} k(\frac{b}{d} - I_* - R_*) & 0 \\ 0 & 0 \end{bmatrix}.$$

Note that $E_* = (I_*, R_*)$ satisfies Eq. (6). Then the characteristic equation becomes

$$\lambda^2 + D_3\lambda + D_4 = e^{-\lambda\tau}(\lambda + D_5)D_6, \tag{12}$$

where

$$\begin{aligned}
D_3 &= \frac{kI_*}{1+\alpha I_*^h} + 2d + \mu + \gamma, \\
D_4 &= \frac{kI_*}{1+\alpha I_*^h}(\mu + d + \gamma) + d^2 + (\mu+\gamma)d, \\
D_5 &= d + \gamma, \\
D_6 &= k\left(\frac{b}{d} - \left(1 + \frac{\mu}{\gamma+d}\right) I_*\right) \frac{1+(1-h)\alpha I_*^h}{(1+\alpha I_*^h)^2}.
\end{aligned}$$

Suppose that there is a positive τ_0 such that Eq. (12) has a pair of purely imaginary roots $\pm i\omega$, $\omega > 0$. Then ω satisfies

$$-\omega^2 + iD_3\omega + D_4 = (i\omega + D_5)D_6(\cos\omega\tau_0 - i\sin\omega\tau_0) = 0.$$

Separating the real and imaginary parts, we have

$$\begin{cases} D_6\omega\cos\omega\tau_0 - D_5D_6\sin\omega\tau_0 = D_3\omega, \\ D_5D_6\cos\omega\tau_0 + D_6\omega\sin\omega\tau_0 = -\omega^2 + D_4. \end{cases}$$

By a simple computation, we have

$$\omega^4 + \left(D_3^2 - 2D_4 - D_6^2\right)\omega^2 + \left(D_4^2 - D_5^2 D_6^2\right) = 0. \tag{13}$$

From Theorem 3.1, the real parts of all roots for Eq. (12) with $\tau = 0$ are negative, which implies that

$$D_4 - D_5 D_6 > 0. \tag{14}$$

Furthermore, we have

$$\begin{aligned}
&D_3^2 - 2D_4 - D_6^2 \\
&\quad = \left(\frac{kI_*}{1+\alpha I_*^h} + 2d + \mu + \gamma\right)^2 \\
&\qquad - 2\left(\frac{kI_*}{1+\alpha I_*^h}(\mu + d + \gamma) + d^2 + (\mu + \gamma)d\right) \\
&\qquad - \left(k\left(\frac{b}{d} - \left(1 + \frac{\mu}{\gamma + d}\right)I_*\right)\frac{1+(1-h)\alpha I_*^h}{(1+\alpha I_*^h)^2}\right)^2 \\
&\quad = \frac{k^2 I_*^2}{(1+\alpha I_*^h)^2} + 2d^2 + (\mu+\gamma)^2 + 2d\frac{kI_*}{1+\alpha I_*^h} \\
&\qquad + 2d(\mu+\gamma) - \left(d + \mu - \frac{\mu\gamma}{d+\gamma}\right)^2\left(\frac{1+(1-h)\alpha I_*^h}{1+\alpha I_*^h}\right)^2 \\
&\quad \geq \frac{k^2 I_*^2}{(1+\alpha)^2} + d^2 + 2\mu\gamma + \gamma^2 + 2d\gamma + 2d\frac{kI_*}{1+\alpha I_*^h} \\
&\qquad + \left(d + \mu - \frac{\mu\gamma}{d+\gamma}\right)^2 \frac{\Lambda_2}{(1+\alpha I_*^h)^2}
\end{aligned}$$

and

$$\begin{aligned}
&D_4 + D_5 D_6 \\
&\quad = \frac{kI_*}{1+\alpha I_*^h}(\mu + d + \gamma) + d^2 + (\mu+\gamma)d \\
&\qquad + (d+\gamma)\left(d + \mu - \frac{\mu\gamma}{d+\gamma}\right)\frac{1+(1-h)\alpha I_*^h}{1+\alpha I_*^h} \\
&\quad = \frac{kI_*}{1+\alpha I_*^h}(\mu + d + \gamma) + \left(d^2 + (\mu+\gamma)d\right)\frac{\Lambda_3}{1+\alpha_*^h},
\end{aligned} \tag{15}$$

where

$$\begin{aligned}
\Lambda_2 &= \left(2 + (2-h)\alpha I_*^h\right)h\alpha I_*^h, \\
\Lambda_3 &= 2 + (2-h)\alpha I_*^h.
\end{aligned}$$

When $\mathscr{R}_0 > 1$ and $0 < h \leq 2$, we have Λ_2, Λ_3, $D_3^2 - 2D_4 - D_6^2$ and $D_4^2 - D_5^2 D_6^2$ are positive. In this case, Eq. (13) has no real root. That is, the real parts of all roots of Eq. (12) are negative. So, we obtain the following.

Theorem 4.2 *If $\mathscr{R}_0 > 1$ and $0 < h \leq 2$, Eq.* (3) *has a locally asymptotically stable EE $E_* = (I_*, R_*)$ for any $\tau > 0$.*

Now, we consider the case that $h > 2$ if $\mathscr{R}_0 > 1$ and $\tau > 0$. Firstly, we give the following lemma.

Lemma 4.1 *If $\mathscr{R}_0 > 1$, $h > 2$ and $H < 0$ where*

$$\begin{aligned} H &:= 2d(\mu + d + \gamma) - (h-2)(d+\mu)(d+\gamma)(\mathscr{R}_0 - 1) \\ &\quad + k(h-1)(\mu + d + \gamma) \times \sqrt[h]{\frac{(d+\gamma)(d+\mu)(\mathscr{R}_0 - 1)}{\alpha(d^2 + (\mu+\gamma)d)}}, \end{aligned} \tag{16}$$

then there is a positive τ_0 such that Eq. (12) *has a pair of purely imaginary eigenvalues $\pm\omega_0 i$ as $\tau = \tau_0$, and all eigenvalues with negative real parts as $0 < \tau < \tau_0$.*

Proof From Eq. (7), we have

$$I_* \leq \sqrt[h]{\frac{(d+\gamma)(d+\mu)(\mathscr{R}_0 - 1)}{\alpha(d^2 + (\mu+\gamma)d)}}. \tag{17}$$

Thus, we have

$$\begin{aligned} & D_4^2 - D_5^2 D_6^2 \\ &\quad = (D_4 - D_5 D_6)(D_4 + D_5 D_6) \\ &\quad = \Lambda_4 \big(kI_*(\mu + d + \gamma) + \big(d^2 + (\mu+\gamma)d\big)\big(2 + (2-h)\alpha I_*^h\big)\big) \\ &\quad = \Lambda_4 \bigg((h-1)k(\mu + d + \gamma)I_* + (h-2)\bigg(d^2 + (\mu+\gamma)d \\ &\qquad - \frac{kb}{d}(\gamma + d)\bigg) + 2\big(d^2 + (\mu+\gamma)d\big)\bigg) \\ &\quad \leq \Lambda_4 \bigg((h-1)k(\mu + d + \gamma) \cdot \sqrt[h]{\frac{(d+\gamma)(d+\mu)(\mathscr{R}_0 - 1)}{\alpha(d^2 + (\mu+\gamma)d)}} \\ &\qquad + (h-2)\bigg(d^2 + (\mu+\gamma)d - \frac{kb}{d}(\gamma + d)\bigg) + 2\big(d^2 + (\mu+\gamma)d\big)\bigg) \\ &\quad \leq \Lambda_4 \bigg((h-1)k(\mu + d + \gamma) \cdot \sqrt[h]{\frac{(d+\gamma)(d+\mu)(\mathscr{R}_0 - 1)}{\alpha(d^2 + (\mu+\gamma)d)}} \\ &\qquad + 2\big(d^2 + (\mu+\gamma)d\big) - (h-2)(d+\mu)(d+\gamma)(\mathscr{R}_0 - 1)\bigg), \end{aligned}$$

where

$$\Lambda_4 = \frac{D_4 + D_5 D_6}{1 + \alpha I_*^h} > 0.$$

Let $z = \omega^2$ of Eq. (13), we have

$$z^2 + \big(D_3^2 - 2D_4 - D_6^2\big)z + \big(D_4^2 - D_5^2 D_6^2\big) = 0. \tag{18}$$

From $H < 0$, there exists a unique positive root z_0 of Eq. (18). Then Eq. (18) has a unique positive root $\omega_0 = \sqrt{z_0}$. Denote

$$\tau^j = \frac{1}{\omega_0} \arccos\left(\frac{(D_3 - D_5)\omega^2 + D_4 D_5}{D_6(\omega^2 + D_5^2)} + 2j\pi\right), \quad j = 0, 1, 2, \ldots.$$

Then $\pm j\omega_0$ is a pair of purely imaginary roots of Eq. (12) if $\tau = \tau^j$.

Now, we select $\tau^0 \in \{\tau^j, j = 0, 1, 2\}$ such that

$$\tau^0 = \min_{j \in \{0,1,\ldots\}} \{\tau^j\}. \tag{19}$$

Then Eq. (12) has a pair of purely imaginary roots $\pm i\omega^0$ if $\tau = \tau^0$. Note that any root of Eq. (12) has a negative real part for $\tau = 0$ and Eq. (12) has no zero root for any $\tau > 0$. Then all roots of Eq. (12) has a negative real part if $0 < \tau < \tau^0$ from the continuity of the roots on parameter τ. This implies the conclusion when we take

$$\tau_0 = \tau^0, \qquad \omega_0 = \omega^0, \qquad z_0 = \omega_0^2.$$

The proof is completed. □

Based on Lemma 4.1, we have the following.

Theorem 4.3 *Assume that* $\mathscr{R}_0 > 1$, $h > 2$, *and* $H < 0$, *then there is a positive real number* τ_0 *such that the following conclusions hold.*

(i) *If* $0 < \tau < \tau_0$, *Eq.* (3) *has an EE* E_* *which is locally asymptotically stable*;

(ii) *Equation* (3) *can undergo a Hopf bifurcation if* $\tau > \tau_0$, *and a periodic orbit exists in the small neighborhood of the EE* E_*.

Proof From Lemma 4.1, we have the conclusion in case (i). To obtain the Hopf bifurcation, we need to check the transversal condition in [18] for the complex eigenvalues of the EE E_* at $\tau = \tau_0$. Let $\lambda(\tau) = \xi(\tau) + i\omega(\tau)$ be a root of Eq. (12) as $\tau \geq \tau_0$, where ξ, ω are the real part and imaginary parts of λ, respectively. Then, from Eq. (12), we have

$$(2\lambda + D_3)\frac{d\lambda}{d\tau} = D_6 e^{-\lambda\tau}\frac{d\lambda}{d\tau} - D_6(\lambda + D_5)e^{\lambda\tau}\left(\lambda + \tau\frac{d\lambda}{d\tau}\right),$$

i.e.

$$\frac{d\lambda}{d\tau} = \frac{-D_6\lambda(\lambda + D_5)e^{-\lambda\tau}}{D_3 + 2\lambda + (D_6\tau(\lambda + D_5) - D_6)e^{-\lambda\tau}}.$$

Since $\xi(\tau_0) = 0$ and $\omega(\tau_0) = \omega_0$, we have

$$\begin{aligned}\left[\frac{d(\mathrm{Re})\lambda(\tau_0)}{d\tau}\right]^{-1} &= \mathrm{Re}\left[\frac{D_3 + 2\lambda + (D_6\tau(\lambda + D_5) - D_6)e^{-\lambda\tau}}{-D_6\lambda(\lambda + D_5)e^{-\lambda\tau}}\right]_{\tau=\tau_0} \\ &= \Lambda_5\big(D_3\cos\omega_0\tau_0 - 2\omega_0\sin\omega_0\tau_0 + D_5 D_6\tau_0 - D_6 \\ &\quad + i(D_3\sin\omega_0\tau_0 + 2\omega_0\cos\omega_0\tau_0 + D_6\omega_0\tau_0)\big)\end{aligned}$$

$$
\begin{aligned}
&= \Lambda_5\bigl(2\omega_0^4 + \bigl(D_3^2 - 2D_4 - D_6^2\bigr)\omega_0^2\bigr) \\
&= \Lambda_5 z_0\bigl(2z_0 + \bigl(D_3^2 - 2D_4 - D_6^2\bigr)\bigr),
\end{aligned}
$$

where

$$\Lambda_5 = \frac{1}{D_6^2\omega^4 + D_5^2 D_6^2\omega_0^4}.$$

Noting $z^2 + (D_3^2 - 2D_4 - D_6^2)z + (D_4^2 - D_5^2D_6^2) = 0$ and $D_4^2 - D_5^2D_6^2 < 0$, we have

$$
\begin{aligned}
&2z_0^2 + \bigl(D_3^2 - 2D_4 - D_6^2\bigr)z_0 \\
&\quad = z_0^2 + \bigl(z_0^2 + \bigl(D_3^2 - 2D_4 - D_6^2\bigr)z_0 + \bigl(D_4^2 - D_5^2D_6^2\bigr)\bigr) - \bigl(D_4^2 - D_5^2D_6^2\bigr) \\
&\quad = z_0^2 - \bigl(D_4^2 - D_5^2D_6^2\bigr) > 0.
\end{aligned}
$$

Thus, we have

$$\frac{d\mathrm{Re}\lambda(\tau_0)}{d\tau} > 0.$$

Then we have the conclusion. □

5 Examples

According to the above theorems, we have the same results on the dynamics of the SIRI model (1) with one of its limit equations (3). In this section, we shall give some examples and their simulations to illustrate the effectiveness of these results. Furthermore, we show the influence of the nonlinear incidence rate and latent period on the dynamical behaviors of the SIRI model.

Consider the SIRI model (1) for the following cases of different parameters.

Example 5.1 Take $b = 0.3$; $d = 0.3$; $k = 0.3$; $h = 1$; $\gamma = 0.5$; $\mu = 0.2$; $\alpha = 10$.

Since $\mathscr{R}_0 = 0.85 < 1$, it follows from Theorem 3.1 and Theorem 4.1 that the disease-free equilibrium is globally asymptotically stable for any latent period. Figure 1 shows the global asymptotical stability of the equilibrium for the model (1) with $\tau = 0$, $\tau = 20$, respectively.

Example 5.2 Take $b = 0.3$; $d = 0.3$; $k = 0.375$; $h = 1$; $\gamma = 0.5$; $\mu = 0.2$; $\alpha = 10$.

Since $\mathscr{R}_0 = 1$, it follows from Theorem 3.1 and Theorem 4.1 that the disease-free equilibrium is globally asymptotically stable for any latent period. Figure 2 shows the globally asymptotical stability of the equilibrium for the model (1) with $\tau = 0$, $\tau = 20$, respectively.

Example 5.3 Take $b = 1.5$; $d = 0.3$; $k = 0.3$; $h = 1$; $\gamma = 0.5$; $\mu = 0.2$; $\alpha = 10$.

Since $\mathscr{R}_0 = 3.25 > 1$ and $0 < h < 2$, it follows from Theorem 3.1 (Theorem 4.2) that the endemic equilibrium is globally (locally) asymptotically stable for $\tau = 0$ ($\tau > 0$). Figure 3 shows the asymptotical stability of the equilibrium for the model (1) with $\tau = 0$, $\tau = 30$, respectively.

Example 5.4 Take $b = 1$; $d = 0.3$; $k = 0.5$; $h = 3$; $\gamma = 0.5$; $\mu = 0.2$; $\alpha = 1{,}000$.

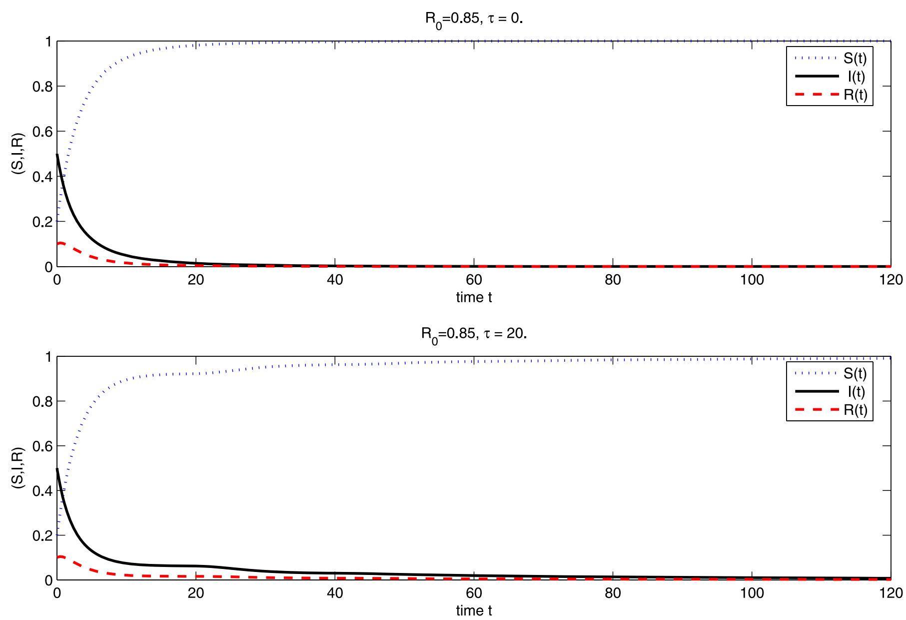

Figure 1 The disease-free equilibrium is globally asymptotically stable if $\mathscr{R}_0 < 1$ as $\tau = 0$ and $\tau = 20$.

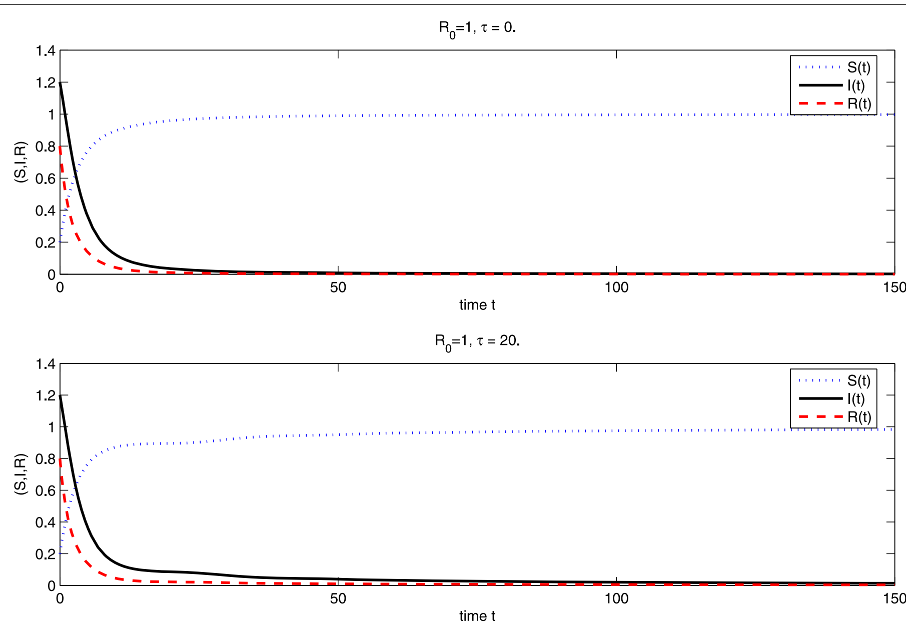

Figure 2 The disease-free equilibrium is globally asymptotically stable if $\mathscr{R}_0 = 1$ as $\tau = 0$ and $\tau = 20$.

We compute $\mathscr{R}_0 = 3.5833 > 1$, $h > 2$, $H = -0.2823 < 0$. From Theorem 4.3, the endemic equilibrium is locally asymptotically stable for a small latent period, while the SIRI model can undergo a Hopf bifurcation and produce a periodic orbit for a large latent period. Figure 4 shows the asymptotical stability of the equilibrium as $\tau = 8$, and Figure 5 shows the periodic orbit as $\tau = 30$.

From the above results and numerical examples, we can draw the following conclusions.

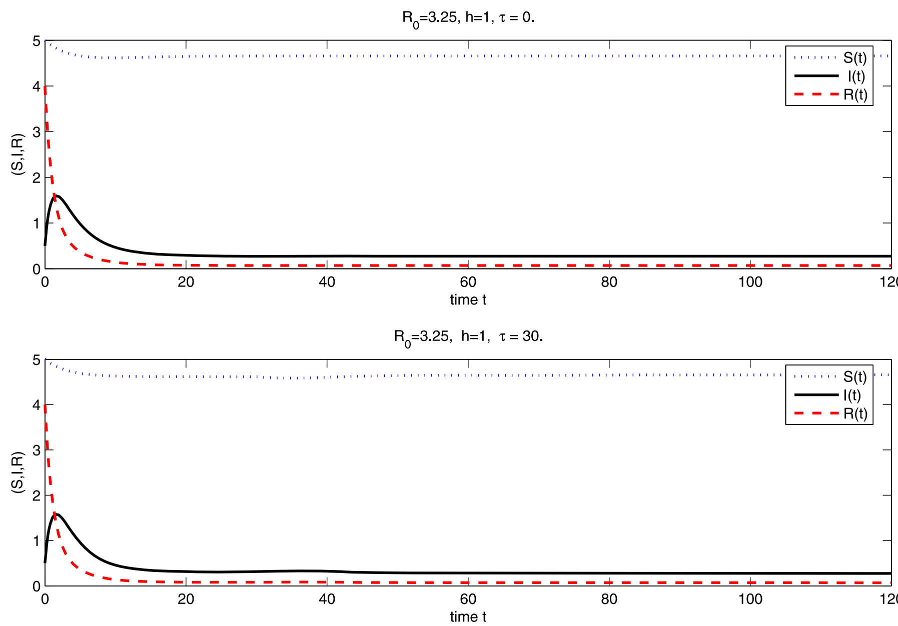

Figure 3 The endemic equilibrium is locally asymptotically stable if $\mathscr{R}_0 > 1$ and $0 < h < 2$ as $\tau = 0$ and $\tau = 30$.

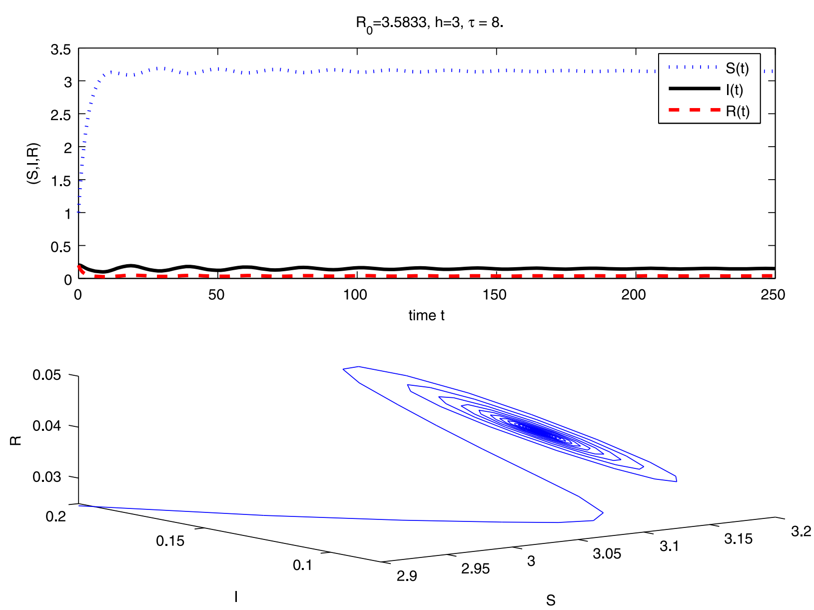

Figure 4 The endemic equilibrium is locally asymptotically stable if $\mathscr{R}_0 > 1$ and $h > 2$ as $\tau = 8$.

(1) The basic reproductive number $\mathscr{R}_0$ determines the existence of the equilibrium. When $\mathscr{R}_0 \leq 1$, model (1) has one unique disease-free equilibrium. When $\mathscr{R}_0 > 1$, model (1) has a disease-free equilibrium and one unique endemic equilibrium.

(2) For model (1) without latent period, its threshold dynamics is determined by $\mathscr{R}_0$. That is, the disease-free equilibrium is globally asymptotically stable if $\mathscr{R}_0 \leq 1$, while the unique endemic equilibrium is globally asymptotically stable if $\mathscr{R}_0 > 1$.

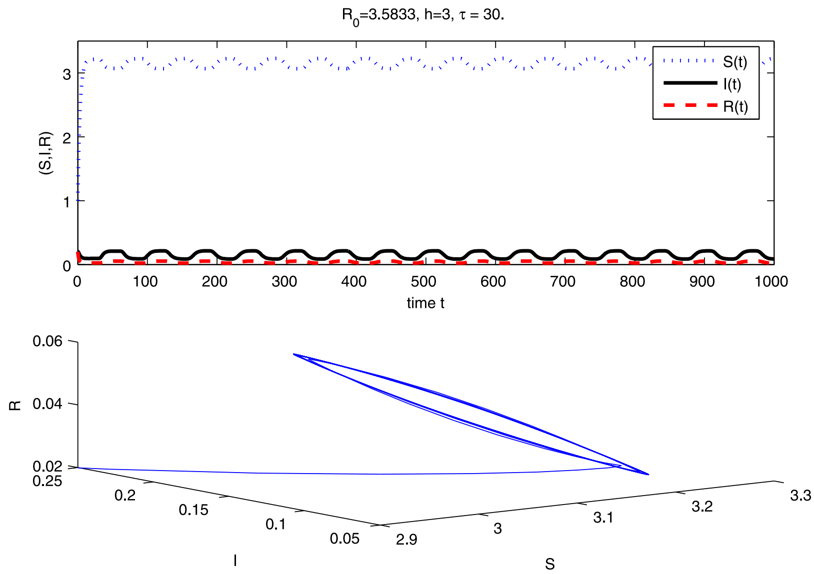

Figure 5 A periodic orbit appears if $\mathscr{R}_0 > 1$ and $h > 2$ as $\tau = 30$.

(3) When the SIRI model has a latent period, the dynamical behaviors of the SIRI model (1) become complex, which are dependent on the values of $\mathscr{R}_0$, τ, h. If $\mathscr{R}_0 \leq 1$, the latent period does not impact on the stability of the disease-free equilibrium, which is globally asymptotically stable for any latent period. However, the parameter h in the nonlinear incidence rate and the latent period τ are essential for the stability of the endemic equilibrium if $\mathscr{R}_0 > 1$. When $h > 2$ and $\mathscr{R}_0 > 1$, the SIRI model may undergo a Hopf bifurcation and produce a periodic orbit for a large latent period. When $0 < h < 2$ and $\mathscr{R}_0 > 1$, the endemic equilibrium is locally asymptotically stable.

In addition, we leave two unsolved problems in this paper for future work.

(i) Is the endemic equilibrium globally asymptotically stable if $0 < h < 2$ and $\mathscr{R}_0 > 1$? We conjecture it is positive by some numerical simulations.

(ii) Does the model undergo a Hopf bifurcation and produce a periodic orbit if $\mathscr{R}_0 > 1$, $h > 2$, but $H < 0$ is not satisfied?

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Chongqing Normal University, No. 12 Tianchen Road, Shapingba District, Chongqing, 400047, China. [2]Department of Mathematics, Sichuan University, No. 24 South Section 1, Yihuan Road, Chengdu, 610065, China.

Acknowledgements
The work is supported partially by National Natural Science Foundation of China under Grant No. 10971240, No. 61263020, the Program of Chongqing Innovation Team Project in University under Grant No. KJTD201308, Natural Science Foundation of Chongqing under Grant CQ CSTC 2012jjA40052, Foundation of Science and Technology project of Chongqing Education Commission under Grant KJ120615, KJ130613.

Received: 27 January 2014 Accepted: 14 May 2014 Published: 04 Jun 2014

References

1. Tudor, D: A deterministic model for herpes infections in human and animal populations. SIAM Rev. **32**, 136-139 (1990)
2. Chin, J: Control of Communicable Diseases Manual. American Public Health Association, Washington (1999)
3. Martin, SW: Livestock Disease Eradication: Evaluation of the Cooperative State Federal Bovine Tuberculosis Eradication Program. National Academy Press, Washington (1994)
4. VanLandingham, KE, Marsteller, HB, Ross, GW, Hayden, FG: Relapse of herpes simplex encephalitis after conventional acyclovir therapy. JAMA **259**, 1051-1053 (1988)
5. Moreira, HN, Wang, Y: Global stability in an $\mathrm{S} \to \mathrm{I} \to \mathrm{R} \to \mathrm{I}$ model. SIAM Rev. **39**, 496-502 (1997)
6. van den Driessche, P, Wang, L, Zou, X: Modeling diseases with latency and relapse. Math. Biosci. Eng. **4**, 205-219 (2007)
7. Capasso, V, Serio, G: A generalization of the Kermack-McKendrick deterministic epidemic model. Math. Biosci. **42**, 43-61 (1978)
8. Ruan, S, Wang, W: Dynamical behavior of an epidemic model with a nonlinear incidence rate. J. Differ. Equ. **188**, 135-163 (2003)
9. Xiao, D, Ruan, S: Global analysis of an epidemic model with nonmonotone incidence rate. Math. Biosci. **208**, 419-429 (2007)
10. Yang, Y, Xiao, D: Influence of latent period and nonlinear incidence rate on the dynamics of SIRS epidemiological models. Discrete Contin. Dyn. Syst., Ser. B **13**, 195-211 (2010)
11. Liu, WM, Levin, SA, Iwasa, Y: Influence of nonlinear incidence rates upon the behavior of SIRS epidemiological models. J. Math. Biol. **23**, 187-204 (1986)
12. Derrick, WR, van der Driessche, P: A disease transmission model in a nonconstant population. J. Math. Biol. **31**, 495-512 (1993)
13. Hethcote, HW, Lewis, MA, van der Driessche, P: An epidemiological model with a delay and a nonlinear incidence rate. J. Math. Biol. **27**, 49-64 (1989)
14. Beretta, E, Takeuchi, Y: Global stability of an SIR epidemic model with time delays. J. Math. Biol. **33**, 250-260 (1995)
15. Lizana, M, Rivero, J: Multiparametric bifurcations for a model in epidemiology. J. Math. Biol. **35**, 21-36 (1996)
16. Jin, Y, Wang, W, Xiao, S: An SIRS model with a nonlinear incidence rate. Chaos Solitons Fractals **34**, 1482-1497 (2007)
17. Li, G, Wang, W: Bifurcation analysis of an epidemic model with nonlinear incidence. Appl. Math. Comput. **214**, 411-423 (2009)
18. Hale, JK: Theory of Functional Differential Equations, 2nd edn. Applied Mathematical Sciences, vol. 3. Springer, New York (1977)
19. Kuang, Y: Delay Differential Equations with Applications in Population Dynamics. Academic Press, San Diego (1993)
20. van der Driessche, P, Watmough, J: Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission. Math. Biosci. **180**, 29-48 (2002)
21. Brauer, F, van der Driessche, P, Wu, J: Mathematical Epidemiology. Lecture Notes in Mathematics, vol. 1945. Springer, New York (2008)
22. Zhang, ZF, Ding, TR, Huang, WZ, Dong, ZX: Qualitative Theory of Differential Equations. Translations of Mathematical Monographs, vol. 101. Am. Math. Soc., Providence (1992)
23. Lasalle, JP: An invariance principle in the theory of stability. In: International Symposium on Differential Equations and Dynamical Systems, pp. 277-286. Academic Press, New York (1967)

10.1186/1687-1847-2014-164
Cite this article as: Guo et al.: **Dynamical behaviors of an SIRI epidemic model with nonlinear incidence and latent period.** *Advances in Difference Equations* 2014, 2014:164

Submit your manuscript to a SpringerOpen journal and benefit from:

- ► Convenient online submission
- ► Rigorous peer review
- ► Immediate publication on acceptance
- ► Open access: articles freely available online
- ► High visibility within the field
- ► Retaining the copyright to your article

Submit your next manuscript at ► springeropen.com