Park et al. *Advances in Difference Equations* 2014, 2014:1
http://www.advancesindifferenceequations.com/content/2014/1/1

Advances in Difference Equations
a SpringerOpen Journal

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The twisted Daehee numbers and polynomials

Jin-Woo Park[1], Seog-Hoon Rim[1] and Jongkyum Kwon[2*]

*Correspondence: mathkjk26@hanmail.net
[2]Department of Mathematics, Kyungpook National University, Taegu, 702-701, Republic of Korea
Full list of author information is available at the end of the article

Abstract
We consider the Witt-type formula for the *n*th twisted Daehee numbers and polynomials and investigate some properties of those numbers and polynomials. In particular, the *n*th twisted Daehee numbers are closely related to higher-order Bernoulli numbers and Bernoulli numbers of the second kind.

Keywords: the *n*th twisted Daehee numbers and polynomials; Bernoulli numbers of the second kind; higher-order Bernoulli numbers

1 Introduction

In this paper, we assume that $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will, respectively, denote the rings of p-adic integers, the fields of p-adic numbers and the completion of algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\cdot|_p$ is normalized by $|p|_p = 1/p$. Let $\mathrm{UD}[\mathbb{Z}_p]$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in \mathrm{UD}[\mathbb{Z}_p]$, the p-adic invariant integral on $\mathbb{Z}_p$ is defined by

$$I(f) \int_{\mathbb{Z}_p} f(x)\,d\mu_0(x) = \lim_{n\to\infty} \frac{1}{p^n} \sum_{x=0}^{p^n-1} f(x) \quad \text{(see [1, 2])}. \tag{1}$$

Let f_1 be the translation of f with $f_1(x) = f(x+1)$. Then, by (1), we get

$$I(f_1) = I(f) + f'(0), \quad \text{where } f'(0) = \left.\frac{df(x)}{dx}\right|_{x=0}. \tag{2}$$

As is known, the Stirling number of the first kind is defined by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l, \tag{3}$$

and the Stirling number of the second kind is given by the generating function to be

$$\left(e^t - 1\right)^m = m! \sum_{l=m}^{\infty} S_2(l,m)\frac{t^l}{l!} \quad \text{(see [3–5])}. \tag{4}$$

For $\alpha \in \mathbb{Z}$, the Bernoulli polynomials of order α are defined by the generating function to be

$$\left(\frac{t}{e^t - 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!} \quad \text{(see [3, 6, 7])}. \tag{5}$$

When $x = 0$, $B_n^{(\alpha)} = B_n^{(\alpha)}(0)$ are called the Bernoulli numbers of order α.

For $n \in \mathbb{N}$, let T_p be the p-adic locally constant space defined by

$$T_p = \bigcup_{n \geq 1} C_{p^n} = \lim_{n \to \infty} C_{p^n},$$

where $C_{p^n} = \{\omega | \omega^{p^n} = 1\}$ is the cyclic group of order p^n. It is well known that the twisted Bernoulli polynomials are defined as

$$\frac{t}{\xi e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_{n,\xi}(x) \frac{t^n}{n!}, \quad \xi \in T_p \text{ (see [8])},$$

and the twisted Bernoulli numbers $B_{n,\xi}$ are defined as $B_{n,\xi} = B_{n,\xi}(0)$.

Recently, Kim and Kim introduced the Daehee numbers and polynomials which are given by the generating function to be

$$\left(\frac{\log(1+t)}{t}\right)(1+t)^x = \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!} \quad \text{(see [9, 10])}. \tag{6}$$

In the special case, $x = 0$, $D_n = D_n(0)$ are called the nth Daehee numbers.

In the viewpoint of generalization of the Daehee numbers and polynomials, we consider the nth twisted Daehee polynomials defined by the generating function to be

$$\left(\frac{\log(1+\xi t)}{\xi t}\right)(1+\xi t)^x = \sum_{n=0}^{\infty} D_{n,\xi}(x) \frac{t^n}{n!}. \tag{7}$$

In the special case, $x = 0$, $D_{n,\xi} = D_{n,\xi}(0)$ are called the nth twisted Daehee numbers.

In this paper, we give a p-adic integral representation of the nth twisted Daehee numbers and polynomials, which are called the Witt-type formula for the nth twisted Daehee numbers and polynomials. We can derive some interesting properties related to the nth twisted Daehee numbers and polynomials. For this idea, we are indebted to papers [9, 10].

2 Witt-type formula for the *n*th twisted Daehee numbers and polynomials

First, we consider the following integral representation associated with falling factorial sequences:

$$\int_{\mathbb{Z}_p} (x)_n \, d\mu_0(x), \quad \text{where } n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\} \text{ (see [10])}. \tag{8}$$

By (8), we get

$$\begin{aligned} \sum_{n=0}^{\infty} \xi^n \int_{\mathbb{Z}_p} (x)_n \, d\mu_0(x) \frac{t^n}{n!} &= \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} \xi^n \binom{x}{n} t^n \, d\mu_0(x) \\ &= \int_{\mathbb{Z}_p} (1+\xi t)^x \, d\mu_0(x), \end{aligned} \tag{9}$$

where $t \in \mathbb{C}_p$ with $|t|_p < -\frac{1}{p-1}$.

For $t \in C_p$ with $|t|_p < p^{-\frac{1}{p-1}}$, let us take $f(x) = (1 + \xi t)^x$. Then, from (2), we have

$$\int_{\mathbb{Z}_p} (1 + \xi t)^x \, d\mu_0(x) = \frac{\log(1 + \xi t)}{\xi t}. \tag{10}$$

By (9) and (10), we see that

$$\begin{aligned} \sum_{n=0}^{\infty} D_{n,\xi} \frac{t^n}{n!} &= \frac{\log(1 + \xi t)}{\xi t} \\ &= \int_{\mathbb{Z}_p} (1 + \xi t)^x \, d\mu_0(x) \\ &= \sum_{n=0}^{\infty} \xi^n \int_{\mathbb{Z}_p} (x)_n \, d\mu_0(x) \frac{t^n}{n!}. \end{aligned} \tag{11}$$

Therefore, by (11), we obtain the following theorem.

Theorem 1 *For $n \geq 0$, we have*

$$\xi^n \int_{\mathbb{Z}_p} (x)_n \, d\mu_0(x) = D_{n,\xi}.$$

For $n \in \mathbb{Z}$, it is known that

$$\left(\frac{\log(1 + t)}{t} \right)^n (1 + t)^{x-1} = \sum_{k=0}^{\infty} B_k^{(k-n+1)}(x) \frac{t^k}{k!} \quad \text{(see [3–5])}. \tag{12}$$

Thus, replacing t by $e^{\xi t} - 1$ in (12), we get

$$D_{k,\xi} = \xi^n \int_{\mathbb{Z}_p} (x)_k \, d\mu_0(x) = \xi^n B_k^{(k+2)}(1) \quad (k \geq 0), \tag{13}$$

where $B_k^{(n)}(x)$ are the Bernoulli polynomials of order n.

In the special case, $x = 0$, $B_k^{(n)} = B_k^{(n)}(0)$ are called the nth Bernoulli numbers of order n.

From (11), we note that

$$\begin{aligned} (1 + \xi t)^x \int_{\mathbb{Z}_p} (1 + \xi t)^y \, d\mu_0(y) &= \left(\frac{\log(1 + \xi t)}{\xi t} \right) (1 + \xi t)^x \\ &= \sum_{n=0}^{\infty} D_{n,\xi}(x) \frac{t^n}{n!}. \end{aligned} \tag{14}$$

Thus, by (14), we get

$$\xi^n \int_{\mathbb{Z}_p} (x + y)_n \, d\mu_0(y) = D_{n,\xi}(x) \quad (n \geq 0), \tag{15}$$

and, from (12), we have

$$D_{n,\xi}(x) = \xi^n B_n^{(n+2)}(x + 1). \tag{16}$$

Therefore, by (15) and (16), we obtain the following theorem.

Theorem 2 *For $n \geq 0$, we have*

$$D_{n,\xi}(x) = \xi^n \int_{\mathbb{Z}_p} (x+y)_n \, d\mu_0(y)$$

and

$$D_{n,\xi}(x) = \xi^n B_n^{(n+2)}(x+1).$$

By Theorem 1, we easily see that

$$D_{n,\xi} = \xi^n \sum_{l=0}^{n} S_1(n,l) B_l, \tag{17}$$

where B_l are the ordinary Bernoulli numbers.

From Theorem 2, we have

$$\begin{aligned} D_{n,\xi}(x) &= \xi^n \int_{\mathbb{Z}_p} (x+y)_n \, d\mu_0(y) \\ &= \xi^n \sum_{l=0}^{n} S_1(n,l) B_l(x), \end{aligned} \tag{18}$$

where $B_l(x)$ are the Bernoulli polynomials defined by a generating function to be

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

Therefore, by (17) and (18), we obtain the following corollary.

Corollary 3 *For $n \geq 0$, we have*

$$D_{n,\xi}(x) = \xi^n \sum_{l=0}^{n} S_1(n,l) B_l(x).$$

In (11), we have

$$\frac{\log(1+\xi t)}{\xi t} (1+\xi t)^x = \sum_{n=0}^{\infty} D_{n,\xi}(x) \frac{t^n}{n!}. \tag{19}$$

Replacing t by $e^t - \frac{1}{\xi}$, we put

$$\begin{aligned} &\sum_{n=0}^{\infty} D_{n,\xi}(x) \frac{1}{n!} \left(e^t - \frac{1}{\xi} \right)^n \\ &\quad = \frac{\log(1 + \xi(e^t - \frac{1}{\xi}))}{\xi(e^t - \frac{1}{\xi})} \left(1 + \xi \left(e^t - \frac{1}{\xi} \right) \right)^x \\ &\quad = \frac{t}{\xi e^t - 1} (\xi e^t)^x \end{aligned}$$

$$
\begin{aligned}
&= \xi^x \frac{t}{\xi e^t - 1} e^{tx} \\
&= \xi^x \sum_{n=0}^{\infty} B_{n,\xi}(x) \frac{t^n}{n!}.
\end{aligned} \tag{20}
$$

Therefore, we have

$$
\begin{aligned}
\sum_{n=0}^{\infty} B_{n,\xi}(x) \frac{t^n}{n!} &= \sum_{n=0}^{\infty} D_{n,\xi}(x) \frac{1}{n!} \left(e^t - \frac{1}{\xi} \right)^n \\
&= \sum_{n=0}^{\infty} D_{n,\xi}(x) \frac{1}{n!} \xi^{-n} n! \sum_{m=n}^{\infty} S_2(m,n) \frac{t^m}{m!} \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{m} D_{n,\xi}(x) \xi^{-n} S_2(m,n),
\end{aligned} \tag{21}
$$

where $S_2(m,n)$ is the Stirling number of the second kind.

Hence,

$$
\xi^x B_{n,\xi}(x) = \sum_{n=0}^{m} D_{n,\xi}(x) \xi^{-n} S_2(m,n). \tag{22}
$$

Therefore, we have

$$
B_{m,\xi}(x) = \sum_{n=0}^{m} D_{n,\xi}(x) \xi^{-n-x} S_2(m,n). \tag{23}
$$

In particular,

$$
B_{m,\xi} = \sum_{n=0}^{m} D_{n,\xi} \xi^{-n} S_2(m,n). \tag{24}
$$

Therefore, by (20) and (23), we obtain the following theorem.

Theorem 4 *For $m \geq 0$, we have*

$$
B_{m,\xi}(x) = \sum_{n=0}^{m} \xi^{-n-x} D_{n,\xi}(x) S_2(m,n).
$$

In particular,

$$
B_{m,\xi} = \sum_{n=0}^{m} \xi^{-n} D_{n,\xi} S_2(m,n).
$$

Remark For $m \geq 0$, by (18), we have

$$
\xi^n \int_{\mathbb{Z}_p} (x+y)^m \, d\mu_0(y) = \xi^n \sum_{n=0}^{m} D_n(x) S_2(m,n).
$$

For $n \in \mathbb{Z}_{n\geq 0}$, the rising factorial sequence is defined by

$$x^{(n)} = x(x+1)\cdots(x+n-1). \tag{25}$$

Let us define the nth twisted Daehee numbers of the second kind as follows:

$$\widehat{D}_{n,\xi} = \xi^n \int_{\mathbb{Z}_p} (-x)_n \, d\mu_0(x) \quad (n \in \mathbb{Z}_{n\geq 0}). \tag{26}$$

By (26), we get

$$x^{(n)} = (-1)^n(-x)_n = \sum_{l=0}^{n} S_1(n,l)(-1)^{n-l}x^l. \tag{27}$$

From (26) and (27), we have

$$\begin{aligned} \widehat{D}_{n,\xi} &= \xi^n \int_{\mathbb{Z}_p} (-x)_n \, d\mu_0(x) \\ &= \xi^n \int_{\mathbb{Z}_p} x^{(n)}(-1)^n \, d\mu_0(x) \\ &= \xi^n \sum_{l=0}^{n} S_1(n,l)(-1)^l B_l. \end{aligned} \tag{28}$$

Therefore, by (28), we obtain the following theorem.

Theorem 5 *For $n \geq 0$, we have*

$$\widehat{D}_{n,\xi} = \xi^n \sum_{l=0}^{n} S_1(n,l)(-1)^l B_l.$$

Let us consider the generating function of the nth twisted Daehee numbers of the second kind as follows:

$$\begin{aligned} \sum_{n=0}^{\infty} \widehat{D}_{n,\xi} \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \xi^n \int_{\mathbb{Z}_p} (-x)_n \, d\mu_0(x) \frac{t^n}{n!} \\ &= \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} \xi^n \binom{-x}{n} t^n \, d\mu_0(x) \\ &= \int_{\mathbb{Z}_p} (1+\xi t)^{-x} \, d\mu_0(x). \end{aligned} \tag{29}$$

From (2), we can derive the following equation:

$$\int_{\mathbb{Z}_p} (1+\xi t)^{-x} \, d\mu_0(x) = \frac{(1+\xi t)\log(1+\xi t)}{\xi t}, \tag{30}$$

where $|t|_p < p^{-\frac{1}{p}}$.

By (29) and (30), we get

$$\begin{aligned}\frac{1}{\xi t}(1+\xi t)\log(1+\xi t) &= \int_{\mathbb{Z}_p}(1+\xi t)^{-x}\,d\mu_0(x)\\ &= \sum_{n=0}^{\infty}\widehat{D}_{n,\xi}\frac{t^n}{n!}.\end{aligned} \tag{31}$$

Let us consider the nth twisted Daehee polynomials of the second kind as follows:

$$\frac{(1+\xi t)\log(1+\xi t)}{\xi t}\frac{1}{(1+\xi t)^x} = \sum_{n=0}^{\infty}\widehat{D}_{n,\xi}(x)\frac{t^n}{n!}. \tag{32}$$

Then, by (32), we get

$$\int_{\mathbb{Z}_p}(1+\xi t)^{-x-y}\,d\mu_0(y) = \sum_{n=0}^{\infty}\widehat{D}_{n,\xi}(x)\frac{t^n}{n!}. \tag{33}$$

From (33), we get

$$\begin{aligned}\widehat{D}_{n,\xi}(x) &= \xi^n\int_{\mathbb{Z}_p}(-x-y)_n\,d\mu_0(y) \quad (n\geq 0)\\ &= \xi^n\sum_{l=0}^{n}(-1)^l S_1(n,l)B_l(x).\end{aligned} \tag{34}$$

Therefore, by (34), we obtain the following theorem.

Theorem 6 *For $n \geq 0$, we have*

$$\widehat{D}_{n,\xi}(x) = \xi^n\int_{\mathbb{Z}_p}(-x-y)_n\,d\mu_0(y) = \xi^n\sum_{l=0}^{n}(-1)^l S_1(n,l)B_l(x).$$

From (32) and (33), we have

$$\frac{\log(1+\xi t)}{\xi t}(1+\xi t)^{1-x} = \sum_{n=0}^{\infty}\widehat{D}_{n,\xi}(x)\frac{t^n}{n!}. \tag{35}$$

Replacing t by $e^t - \frac{1}{\xi}$, we get

$$\begin{aligned}\sum_{n=0}^{\infty}\widehat{D}_{n,\xi}(x)\frac{1}{n!}\left(e^t-\frac{1}{\xi}\right)^n &= \frac{\log(1+\xi(e^t-\frac{1}{\xi}))}{\xi(e^t-\frac{1}{\xi})}\left(1+\xi\left(e^t-\frac{1}{\xi}\right)\right)^{1-x}\\ &= \frac{t}{\xi e^t-1}\left(\xi e^t\right)^{1-x}\\ &= \xi^{1-x}\frac{t}{\xi e^t-1}e^{t(1-x)}\\ &= \xi^{1-x}\sum_{n=0}^{\infty}B_{n,\xi}(1-x)\frac{t^n}{n!}.\end{aligned} \tag{36}$$

Therefore, we have

$$\begin{aligned}\xi^{1-x}\sum_{m=0}^{\infty}B_{m,\xi}(1-x)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\widehat{D}_{n,\xi}(x)\frac{(e^t-\frac{1}{\xi})^n}{n!}\\ &= \sum_{n=0}^{\infty}\widehat{D}_{n,\xi}(x)\frac{1}{n!}\xi^{-n}n!\sum_{m=n}^{\infty}S_2(m,n)\frac{t^m}{m!}\\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}\widehat{D}_{n,\xi}(x)\xi^{-n}S_2(m,n)\right)\frac{t^m}{m!}. \qquad (37)\end{aligned}$$

Hence,

$$\xi^{1-x}B_{n,\xi}(1-x) = \sum_{n=0}^{m}\widehat{D}_{n,\xi}(x)\xi^{-n}S_2(m,n). \qquad (38)$$

Therefore, we have

$$B_{m,\xi}(1-x) = \sum_{n=0}^{m}\widehat{D}_{n,\xi}(x)\xi^{-n+x-1}S_2(m,n). \qquad (39)$$

Therefore, by (37) and (38), we obtain the following theorem.

Theorem 7 *For $m \geq 0$, we have*

$$B_{m,\xi}(1-x) = \sum_{n=0}^{m}\xi^{-m+x-1}\widehat{D}_{n,\xi}(x)S_2(m,n).$$

From Theorem 1 and (26), we have

$$\begin{aligned}(-1)^n\frac{D_{n,\xi}}{n!} &= (-1)^n\xi^n\int_{\mathbb{Z}_p}\binom{x}{n}d\mu_0(x)\\ &= \xi^n\int_{\mathbb{Z}_p}\binom{-x+n-1}{n}d\mu_0(x)\\ &= \xi^n\sum_{m=0}^{n}\binom{n-1}{n-m}\int_{\mathbb{Z}_p}\binom{-x}{m}d\mu_0(x)\\ &= \sum_{m=0}^{n}\binom{n-1}{n-m}\xi^{n-m}\frac{\widehat{D}_{m,\xi}}{m!}\\ &= \sum_{m=1}^{n}\binom{n-1}{m-1}\xi^{n-m}\frac{\widehat{D}_{m,\xi}}{m!} \qquad (40)\end{aligned}$$

and

$$\begin{aligned}(-1)^n\frac{\widehat{D}_{n,\xi}}{n!} &= (-1)^n\xi^n\int_{\mathbb{Z}_p}\binom{-x}{n}d\mu_0(x)\\ &= \xi^n\int_{\mathbb{Z}_p}\binom{x+n-1}{n}d\mu_0(x)\end{aligned}$$

$$= \xi^n \sum_{m=0}^{n} \binom{n-1}{n-m} \int_0^1 \binom{x}{m} d\mu_0(x)$$
$$= \sum_{m=0}^{n} \binom{n-1}{m-1} \xi^{n-m} \frac{D_{m,\xi}}{m!}$$
$$= \sum_{m=1}^{n} \binom{n-1}{m-1} \xi^{n-m} \frac{D_{m,\xi}}{m!}. \tag{41}$$

Therefore, by (40) and (41), we obtain the following theorem.

Theorem 8 *For $n \in \mathbb{N}$, we have*

$$(-1)^n \frac{D_{n,\xi}}{n!} = \sum_{m=1}^{n} \binom{n-1}{m-1} \xi^{n-m} \frac{\widehat{D}_{m,\xi}}{m!}$$

and

$$(-1)^n \frac{\widehat{D}_{n,\xi}}{n!} = \sum_{m=1}^{n} \binom{n-1}{m-1} \xi^{n-m} \frac{D_{m,\xi}}{m!}.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

Author details
[1]Department of Mathematics Education, Kyungpook National University, Taegu, 702-701, Republic of Korea.
[2]Department of Mathematics, Kyungpook National University, Taegu, 702-701, Republic of Korea.

Acknowledgements
The authors are grateful for the valuable comments and suggestions of the referees.

Received: 30 October 2013 Accepted: 6 December 2013 Published: 02 Jan 2014

References
1. Kim, T, Kim, DS, Mansour, T, Rim, SH, Schork, M: Umbral calculus and Sheffer sequence of polynomials. J. Math. Phys. **54**, 083504 (2013). doi:10.1063/1.4817853
2. Kim, T: An invariant *p*-adic integral associated with Daehee numbers. Integral Transforms Spec. Funct. **13**(1), 65-69 (2002)
3. Bayad, A: Special values of Lerch zeta function and their Fourier expansions. Adv. Stud. Contemp. Math. **21**(1), 1-4 (2011)
4. Carlitz, L: A note on Bernoulli and Euler polynomials of the second kind. Scr. Math. **25**, 323-330 (1961)
5. Comtet, L: Advanced Combinatorics. Reidel, Dordrecht (1974)
6. Araci, S, Acikgoz, M: A note on the Frobenius-Euler numbers and polynomials associated with Bernstein polynomials. Adv. Stud. Contemp. Math. **22**(3), 399-406 (2012)
7. Kurt, V: Some relation between the Bernstein polynomials and second kind Bernoulli polynomials. Adv. Stud. Contemp. Math. **23**(1), 43-48 (2013)
8. Kim, T: An analogue of Bernoulli numbers and their congruences. Rep. Fac. Sci. Eng. Saga Univ., Math. **22**(2), 21-26 (1994)
9. Kim, DS, Kim, T, Lee, SH, Seo, JJ: A note on the twisted λ-Daehee polynomials. Appl. Math. Sci. **7** (2013)
10. Kim, DS, Kim, T: Daehee numbers and polynomials. Appl. Math. Sci. **7**(120), 5969-5976 (2013)

10.1186/1687-1847-2014-1
Cite this article as: Park et al.: **The twisted Daehee numbers and polynomials.** *Advances in Difference Equations* 2014, 2014:1