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# Periodic solutions for prescribed mean curvature Rayleigh equation with a deviating argument

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**Abstract**

By using Mawhin's continuation theorem in the coincidence degree theory, some criteria for guaranteeing the existence of periodic solutions for prescribed mean curvature Rayleigh equation with a deviating argument are provided.

**Keywords:** prescribed mean curvature Rayleigh equation; periodic solutions; Leray-Schauder degree

## 1 Introduction

Consider the prescribed mean curvature Rayleigh equation

$$\left(\frac{x'}{\sqrt{1+x'^2}}\right)' + f\bigl(t, x'(t)\bigr) + g\bigl(t, x\bigl(t-\tau(t)\bigr)\bigr) = e(t), \tag{1.1}$$

where $\tau, e \in C(\mathbb{R}, \mathbb{R})$ are $T$-periodic, and $f, g \in C(\mathbb{R}\times\mathbb{R}, \mathbb{R})$ are $T$-periodic in the first argument, $T > 0$ is a constant.

In recent years, there are many results on the existence of periodic solutions for various types of delay differential equation with deviating arguments, especially for the Liénard equation and Rayleigh equation (see [1–11]). Now as the prescribed mean curvature $(\frac{x'(t)}{\sqrt{1+x'^2(t)}})'$ of a function $x(t)$ frequently appears in different geometry and physics (see [12–14]), it is interesting to try to consider the existence of periodic solutions of prescribed mean curvature equations. However, to our best knowledge, the studies of delay equations with prescribed mean curvature is relatively infrequent. The main difficulty lies in the nonlinear term $(\frac{x'(t)}{\sqrt{1+x'2(t)}})'$, the existence of which obstructs the usual method of finding a priori bounds for delay Liénard or Rayleigh equations from working. In [15], Feng discussed a delay prescribed mean curvature Liénard equation of the form

$$\left(\frac{x'}{\sqrt{1+x'^2}}\right)' + f\bigl(x(t)\bigr)x'(t) + g\bigl(t, x\bigl(t-\tau(t)\bigr)\bigr) = e(t), \tag{1.2}$$

estimated a priori bounds by eliminating the nonlinear term $(\frac{x'(t)}{\sqrt{1+x'^2(t)}})'$, and established sufficient conditions on the existence of periodic solutions for (1.2) by using Mawhin's continuation theorem.

The conditions imposed on $f(x)$ and $g(t,x)$ in [15] were such as:

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($C_1$) There exists $\gamma > 0$ satisfies $|f(x)| \geq \gamma$.
($C_2$) There exists $l > 0$ such that $|g(t,x_1) - g(t,x_2)| \leq l|x_1 - x_2|$, $\forall t \in \mathbb{R}$, $x_1, x_2 \in \mathbb{R}$.

It is not difficult to see that the condition ($C_1$) is strong. It is natural to relax the conditions ($C_1$) and ($C_2$). Our purpose is studying the more general equation (1.1) under the more weaker conditions.

The rest of the paper is organized as follows. In Section 2, we shall state and prove some basic lemmas. In Section 3, we shall prove the main result. An example will be given to show the applications of our main result in the final section.

## 2 Preliminaries

In this section, we first recall Mawhin's continuation theorem, which our study is based upon.

Let $X$ and $Y$ be real Banach spaces and $L : X \supset \operatorname{Dom} L \to Y$ be a linear operator. $L$ is said to be a Fredholm operator with index zero provided that

(i) $\operatorname{Im} L$ is closed subset of $Y$,
(ii) $\dim \ker L = \operatorname{codim} \operatorname{Im} L < +\infty$.

Set $X = \ker L \oplus X_1$, $Y = \operatorname{Im} L \oplus Y_1$. Let $P : X \to \ker L$ and $Q : Y \to Y_1$ be the nature projections. It is easy to see that $\ker L \cap (\operatorname{Dom} L \cap X_1) = 0$. Thus, the restriction $L_p := L|_{\operatorname{Dom} L \cap X_1}$ is invertible. We denote by $k$ the inverse of $L_p$.

Let $\Omega$ be a open bounded subset of $X$ with $\operatorname{Dom} L \cap \Omega \neq \phi$. A map $N : \overline{\Omega} \to Y$ is said to be $L$-compact in $\overline{\Omega}$ if $QN : \overline{\Omega} \to Y$ and $k(I - Q)N : \overline{\Omega} \to X$ are compact.

The following lemma due to Mawhin [16] is fundamental to prove our main result.

**Lemma 2.1** *Let $L$ be a Fredholm operator of index zero and Let $N$ be $L$-compact on $\overline{\Omega}$. If the following conditions hold*:

($h_1$) $Lx \neq \lambda Nx$, $\forall (x,\lambda) \in [(D(L) \setminus \operatorname{Ker} L) \cap \partial\Omega] \times (0,1)$;
($h_2$) $Nx \notin \operatorname{Im} L$, $\forall x \in \operatorname{Ker} L \cap \partial\Omega$;
($h_3$) $\deg(JQN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) \neq 0$.

*Then $Lx = Nx$ has at least one solution in $D(L) \cap \overline{\Omega}$.*

The following lemmas is useful in the proof of our main result.

**Lemma 2.2** ([17]) *Let $s \in C(\mathbb{R},\mathbb{R})$ with $s(t) \in [0,T]$, $\forall t \in \mathbb{R}$. Suppose $p \in (1,+\infty)$, $\alpha = \max_{t\in[0,T]} s(t)$ and $u \in C^1(\mathbb{R},\mathbb{R})$ with $u(t+\omega) = u(t)$. Then*

$$\int_0^T \bigl|u(t) - u\bigl(t - s(t)\bigr)\bigr|^p \, dt \leq \alpha^p \int_0^T \bigl|u'(t)\bigr|^p \, dt.$$

**Lemma 2.3** ([18]) *If $x \in C_T^1(\mathbb{R},\mathbb{R})$ and $\int_0^T x(t)\,dt = 0$, then*

$$\int_0^T \bigl|x(t)\bigr|^2 \, dt \leq \bigl(T^2/4\pi^2\bigr) \int_0^T \bigl|x'(t)\bigr|^2 \, dt$$

(*Wirtinger inequality*) *and*

$$|x|_\infty^2 \leq (T/12) \int_0^T \bigl|x'(t)\bigr|^2 \, dt$$

(*Sobolev inequality*).

**Lemma 2.4** ([18]) *Suppose* $x(t) \in C^1[0,T]$, *and* $x(0) = x(T) = 0$. *Then*

$$\int_0^T |x(t)|^2\,dt \le \frac{T^2}{\pi^2}\int_0^T |x'(t)|^2\,dt.$$

**Lemma 2.5** *Assume that* $x(t) \in C^1[0,T]$, *and* $x(0) = x(T) = 0$. *Then*

$$|x(t)| \le \frac{1}{2}\sqrt{T}\left(\int_0^T |x'(t)|^2\,dt\right)^2, \quad \forall t \in [0,T].$$

*Proof* It is easy to see that

$$|x(t)| = \left|x(0) + \int_0^t x'(s)\,ds\right| \le \int_0^t |x'(s)|\,ds, \quad \forall t \in [0,T],$$

and

$$|x(t)| = \left|x(T) - \int_t^T x'(s)\,ds\right| \le \int_t^T |x'(s)|\,ds, \quad \forall t \in [0,T].$$

Combining the above inequalities and using Hölder's inequality,

$$|x(t)| \le \frac{1}{2}\int_0^T |x'(s)|\,ds \le \frac{1}{2}\sqrt{T}\left(\int_0^T |x'(t)|^2\,dt\right)^2, \quad \forall t \in [0,T].$$

The proof is completed. □

In order to apply Mawhin's continuation theorem to study the existence of $T$-periodic solution of Equation (1.1), we rewrite (1.1) as

$$\begin{cases} x'(t) = \frac{y(t)}{\sqrt{1-y^2(t)}}, \\ y'(t) = -f(t, \frac{y(t)}{\sqrt{1-y^2(t)}}) - g(t, x(t-\tau(t))) + e(t). \end{cases} \tag{2.1}$$

Obviously, if $z(t) = (x(t), y(t))^\top$ is a $T$-periodic solution of (2.1), then $x(t)$ must be a $T$-periodic solution of (1.1). Hence, the problem of finding a $T$-periodic solution of (1.1) reduces to finding one of (2.1).

Now, we set

$$X = Y = \{z : z(t) = (x(t), y(t))^\top \in C(\mathbb{R}^1, \mathbb{R}^2), z(t) = z(t+T)\},$$

with the norm $\|z\| = \max\{\|x\|_\infty, \|y\|_\infty\}$, where

$$\|x\|_\infty = \max_{t\in[0,T]}|x(t)|, \qquad \|y\|_\infty = \max_{t\in[0,T]}|y(t)|.$$

Clearly, $X$ and $Y$ are Banach spaces. Meanwhile, let

$$L : X \supset \operatorname{Dom} L \to Y, \qquad Lz = z' = (x'(t), y'(t))^\top,$$

where

$$\operatorname{Dom} L = \{z : z = (x(t), y(t))^\top \in C^1(\mathbb{R}, \mathbb{R}^2), z(t) = z(t+T)\}.$$

Define a nonlinear operator $N : X \to Y$ by

$$Nz = \begin{pmatrix} \frac{y(t)}{\sqrt{1-y^2(t)}} \\ -f(t, \frac{y(t)}{\sqrt{1-y^2(t)}}) - g(t, x_1(t-\tau(t))) + e(t) \end{pmatrix}.$$

Then the problem (2.1) can be written to $Lz = Nz$.

It is easy to see that $\ker L = \mathbb{R}^2$ and $\operatorname{Im} L = \{u \in Y : \int_0^T u(s)\,ds = 0\}$. So, $L$ is a Fredholm operator with index zero.

Let $P : X \to \ker L$ and $Q : Y \to \operatorname{Im} Q$ be defined by

$$Pz = \frac{1}{T}\int_0^T z(s)\,ds, \qquad Qu = \frac{1}{T}\int_0^T u(s)\,ds,$$

and denote by $k$ the inverse of $L|_{\ker P \cap \operatorname{Dom} L}$. Then $\ker L = \operatorname{Im} Q = \mathbb{R}^2$ and

$$ku(t) = \int_0^T G(t,s)u(s)\,ds, \tag{2.2}$$

where

$$G(t,s) = \begin{cases} \frac{s}{T}, & 0 \le s < t \le T, \\ \frac{s-T}{T}, & 0 \le t \le s \le T. \end{cases}$$

It follows from (2.2) that $N$ is $L$-compact on $\overline{\Omega}$, where $\Omega$ is an open, bounded subset of $X$.

## 3 Main results

In this section, we will state and prove our main results.

We first give the following assumptions:

(H1) $f(t,0) = 0$, and $xf(t,x) \ge 0$ (or $xf(t,x) \le 0$), $\forall t \in \mathbb{R}$.

(H2) $xg(t,x) \ge 0$ (or $xg(t,x) \le 0$), $\forall t \in \mathbb{R}$, and $|g(t,x)| > \|e\|_\infty$ for $|x| > d$.

(H3) There exists $r_1, r_2, r_3 > 0$ such that

$$r_1 \le \liminf_{|x|\to\infty} \frac{|f(t,x)|}{|x|} \le \limsup_{|x|\to\infty} \frac{|f(t,x)|}{|x|} \le r_2, \quad \text{uniformly in } t \in \mathbb{R},$$

and

$$\limsup_{|x|\to\infty} \frac{|g(t,x)|}{|x|} \le r_3, \quad \text{uniformly in } t \in \mathbb{R}.$$

(H4) There exists an integer $m$ such that $0 \le \tau(t) - mT \le T$, and $\alpha := \|\tau(t) - mT\|_\infty \le T$.

**Theorem 3.1** *Assume* (H1)-(H4) *hold. Then Equation* (1.1) *has at least one $T$-periodic solution provided*

$$\frac{r_3}{r_1} < \max\left\{\frac{2}{T}, \frac{1}{\alpha + \frac{T}{\pi}}\right\}. \tag{3.1}$$

*Proof* Consider the operator equation

$$Lz = \lambda Nz, \quad \forall \lambda \in (0,1). \tag{3.2}$$

Let $\Omega_1 = \{z \in X : Lz = \lambda Nz, \lambda \in (0,1)\}$. If $z(t) = (x(t), y(t))^\top \in \Omega_1$, we have

$$\begin{cases} x'(t) = \lambda \frac{y(t)}{\sqrt{1-y^2(t)}}, \\ y'(t) = -\lambda f(t, \frac{y(t)}{\sqrt{1-y^2(t)}}) - \lambda g(t, x(t-\tau(t))) + \lambda e(t), \end{cases} \tag{3.3}$$

It follows from the first equation of (3.3) that

$$y(t) = \frac{\frac{1}{\lambda}x'(t)}{\sqrt{1+\frac{1}{\lambda^2}x'^2(t)}}.$$

Then (3.3) can be written to

$$\left(\frac{\frac{1}{\lambda}x'(t)}{\sqrt{1+\frac{1}{\lambda^2}x'^2(t)}}\right)' = -\lambda f\left(t, \frac{1}{\lambda}x'(t)\right) - \lambda g\bigl(t, x\bigl(t-\tau(t)\bigr)\bigr) + \lambda e(t). \tag{3.4}$$

Integrating the first equation of (3.3) from 0 to $T$, we have

$$\int_0^T \frac{y(t)}{\sqrt{1-y^2(t)}}\,dt = 0.$$

Then there exist $t_1, t_2 \in [0,T]$, such that

$$y(t_1) \geq 0, \qquad y(t_2) \leq 0.$$

Assume that $t_3, t_4 \in [0,T]$ are the maximum and minimum points, respectively. Then

$$y(t_3) \geq 0, \qquad y'(t_3) = 0, \tag{3.5}$$

and

$$y(t_4) \leq 0, \qquad y'(t_4) = 0. \tag{3.6}$$

It follows from the second equation of (3.3) that

$$0 = y'(t_3) = -\lambda f\left(t_3, \frac{y(t_3)}{\sqrt{1-y^2(t_3)}}\right) - \lambda g\bigl(t_3, x\bigl(t_3 - \tau(t_3)\bigr)\bigr) + \lambda e(t_3).$$

From (H1) and (H2), without loss of generality, we can assume that $xf(t,x) \geq 0$ and $xg(t,x) \geq 0$, $\forall x \in \mathbb{R}$. Then

$$g\big(t_3, x\big(t_3 - \tau(t_3)\big)\big) \leq e(t_3) \leq \|e\|_\infty.$$

If $x(t_3 - \tau(t_3)) > d$, then $g(t_3, x(t_3 - \tau(t_3))) > \|e\|_\infty$, which is a contradiction. It follows that

$$x\big(t_3 - \tau(t_3)\big) \leq d. \tag{3.7}$$

Similarly, from (3.6), we have

$$x\big(t_4 - \tau(t_4)\big) \geq -d. \tag{3.8}$$

Combining the above, we know that there exists a point $\xi \in [0, T]$ such that

$$\big|x\big(\xi - \tau(\xi)\big)\big| \leq d.$$

Note that there exist $k \in \mathbb{Z}$ and $t^* \in [0, T]$ such that $\xi - \tau(\xi) = kT + t^*$. Then we get

$$\big|x\big(t^*\big)\big| \leq d.$$

By Lemma 2.4, we obtain

$$\begin{aligned}
\big|x(t)\big| &\leq \big|x\big(t^*\big)\big| + \big|x(t) - x\big(t^*\big)\big| \\
&\leq \big|x\big(t^*\big)\big| + \frac{1}{2}\sqrt{T}\left(\int_{t^*}^{T^*+T} \big|x'(t)\big|^2\,dt\right)^{\frac{1}{2}} \\
&= d + \frac{1}{2}\sqrt{T}\big\|x'\big\|_2.
\end{aligned}$$

Hence,

$$\|x\|_\infty \leq d + \frac{1}{2}\sqrt{T}\big\|x'\big\|_2. \tag{3.9}$$

Meanwhile, by Lemma 2.3, we have

$$\begin{aligned}
\|x\|_2 &= \left(\int_0^T \big|x(t)\big|^2\,dt\right)^{\frac{1}{2}} \\
&= \left(\int_0^T \big|x\big(t + t^*\big)\big|^2\,dt\right)^{\frac{1}{2}} \\
&= \left(\int_0^T \big|x\big(t + t^*\big) - x\big(t^*\big) + x\big(t^*\big)\big|^2\,dt\right)^{\frac{1}{2}} \\
&\leq \left(\int_0^T \big(\big|x\big(t + t^*\big) - x\big(t^*\big)\big| + d\big)^2\,dt\right)^{\frac{1}{2}} \\
&= \left(\int_0^T \big(\big|x\big(t + t^*\big) - x\big(t^*\big)\big|^2 + 2d\big|x\big(t + t^*\big) - x\big(t^*\big)\big| + d^2\big)\,dt\right)^{\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
&\le \left(\frac{T^2}{\pi^2}\int_0^T |x'(t)|^2\,dt + 2d\sqrt{T}\frac{T}{\pi}\left(\int_0^T |x'(t)|^2\,dt\right)^{\frac{1}{2}} + d^2T\right)^{\frac{1}{2}} \\
&= \frac{T}{\pi}\|x'\|_2 + d\sqrt{T}.
\end{aligned} \tag{3.10}$$

From (3.1), we have $\frac{T}{2}r_3 < r_1$, or $r_3(\alpha + \frac{T}{\pi}) < r_1$. Then there exists $\varepsilon > 0$ such that

$$\frac{T}{2}(r_3 + \varepsilon) < r_1 - \varepsilon, \tag{3.11}$$

or

$$(r_3 + \varepsilon)\left(\alpha + \frac{T}{\pi}\right) < (r_1 - \varepsilon). \tag{3.12}$$

For such a $\varepsilon > 0$, it follows from (H3), there exist $h_1, h_2 \ge 0$ such that

$$(r_1 - \varepsilon)|x| - h_1 \le |f(t,x)| \le (r_2 + \varepsilon)|x| + h_1, \quad \forall t, x \in \mathbb{R}, \tag{3.13}$$

and

$$|g(t,x)| \le (r_3 + \varepsilon)|x| + h_2, \quad \forall t, x \in \mathbb{R}. \tag{3.14}$$

Multiplying $x'(t)$ and (3.4) and integrating from 0 to $T$, we get

$$\lambda\int_0^T f\left(t, \frac{1}{\lambda}x'(t)\right)x'(t)\,dt + \lambda\int_0^T g(t, x(t - \tau(t)))x'(t)\,dt = \lambda\int_0^T e(t)x'(t)\,dt. \tag{3.15}$$

It follows from (H1) and (3.13) that

$$\begin{aligned}
\left|\lambda\int_0^T f\left(t, \frac{1}{\lambda}x'(t)\right)x'(t)\,dt\right| &= \lambda\int_0^T \left|f\left(t, \frac{1}{\lambda}x'(t)\right)x'(t)\right|\,dt \\
&\ge (r_1 - \varepsilon)\int_0^T |x'(t)|^2\,dt - \lambda\int_0^T h_1|x'(t)|\,dt.
\end{aligned} \tag{3.16}$$

Substituting (3.16) into (3.15) and from (3.14), we have

$$\begin{aligned}
&(r_1 - \varepsilon)\int_0^T |x'(t)|^2\,dt \\
&\quad \le \lambda\left|\int_0^T g(t, x(t - \tau(t)))x'(t)\,dt\right| + \lambda\left|\int_0^T e(t)x'(t)\,dt\right| \\
&\qquad + \lambda\left|\int_0^T h_1 x'(t)\,dt\right| \\
&\quad \le \int_0^T |g(t, x(t - \tau(t)))||x'(t)|\,dt + (\|e\|_\infty + h_1)\int_0^T |x'(t)|\,dt \\
&\quad \le (r_3 + \varepsilon)\int_0^T |x(t - \tau(t))||x'(t)|\,dt + (\|e\|_\infty + h_1 + h_2)\int_0^T |x'(t)|\,dt.
\end{aligned} \tag{3.17}$$

Case 1. (3.11) holds. It follows from (3.17) and Hölder inequality that

$$
\begin{aligned}
(r_1-\varepsilon)\|x'\|_2^2 &\le (r_3+\varepsilon)\|x\|_\infty\sqrt{T}\|x'\|_2 + \big(\|e\|_\infty + h_1 + h_2\big)\sqrt{T}\|x'\|_2 \\
&\le (r_3+\varepsilon)\left(\frac{1}{2}\sqrt{T}\|x'\|_2 + d\right)\sqrt{T}\|x'\|_2 + \big(\|e\|_\infty + h_1 + h_2\big)\sqrt{T}\|x'\|_2 \\
&= (r_3+\varepsilon)\frac{T}{2}\|x'\|_2^2 + \big((r_3+\varepsilon)d + \|e\|_\infty + h_1 + h_2\big)\sqrt{T}\|x'\|_2.
\end{aligned}
$$

From (3.9) and (3.11), we obtain that there exists a positive constant $M_1$ such that

$$\|x'\|_2 \le M_1, \quad \text{and} \quad \|x\|_\infty \le M_1.$$

Case 2. (3.12) holds. It follows from (3.17), Lemma 2.2 and Hölder inequality that

$$
\begin{aligned}
(r_1-\varepsilon)\|x'\|_2^2 &\le (r_3+\varepsilon)\int_0^T \big|x(t)-x\big(t-\tau(t)\big)\big|\big|x'(t)\big|\,dt + (r_3+\varepsilon)\int_0^T \big|x(t)\big|\big|x'(t)\big|\,dt \\
&\quad + \big(\|e\|_\infty + h_1 + h_2\big)\int_0^T \big|x'(t)\big|\,dt \\
&\le (r_3+\varepsilon)\left(\int_0^T \big|x(t)-x\big(t-\tau(t)\big)\big|^2\,dt\right)^{\frac{1}{2}}\left(\int_0^T \big|x'(t)\big|^2\,dt\right)^{\frac{1}{2}} \\
&\quad + (r_3+\varepsilon)\left(\int_0^T \big|x(t)\big|^2\,dt\right)^{\frac{1}{2}}\left(\int_0^T \big|x'(t)\big|^2\,dt\right)^{\frac{1}{2}} \\
&\quad + \big(\|e\|_\infty + h_1 + h_2\big)\int_0^T \big|x'(t)\big|\,dt \\
&\le (r_3+\varepsilon)\left(\alpha^2\int_0^T \big|x'(t)\big|^2\,dt\right)^{\frac{1}{2}}\left(\int_0^T \big|x'(t)\big|^2\,dt\right)^{\frac{1}{2}} \\
&\quad + (r_3+\varepsilon)\left(\frac{T}{\pi}\|x'\|_2 + d\sqrt{T}\right)\|x'\|_2 + \big(\|e\|_\infty + h_1 + h_2\big)\sqrt{T}\|x'\|_2 \\
&= (r_3+\varepsilon)\left(\alpha + \frac{T}{\pi}\right)\|x'\|_2^2 + \big((r_3+\varepsilon)d + \|e\|_\infty + h_1 + h_2\big)\sqrt{T}\|x'\|_2.
\end{aligned}
$$

From (3.9) and (3.12), we know there exists a positive constant $M_2$ such that

$$\|x'\|_2 \le M_2, \quad \text{and} \quad \|x\|_\infty \le M_2.$$

Take $R_1 = \max\{M_1, M_2\}$. Then, if (3.1) holds, we have

$$\|x'\|_2 \le R_1, \quad \text{and} \quad \|x\|_\infty \le R_1.$$

By the first equation of (3.3), we have

$$\int_0^T \frac{y(t)}{\sqrt{1-y^2(t)}}\,dt = 0.$$

Then there exists $\eta \in [0,T]$ such that $y(\eta) = 0$. It implies that

$$y(t) = \int_\eta^t y'(s)\,ds + y(\eta) = \int_\eta^t y'(s)\,ds,$$

and

$$\|y\|_\infty \le \int_0^T |y'(s)|\,ds.$$

From the second equation of (3.3), we get

$$\int_0^T |y'(s)|\,ds \le \lambda \int_0^T \left| f\left(t, \frac{1}{\lambda}x'(t)\right)\right| dt + \lambda \int_0^T \left|g\big(t, x\big(t-\tau(t)\big)\big)\right| dt + \lambda \int_0^T |e(t)|\,dt.$$

Noticing that $\|x\|_\infty \le R_1$, we have there exists $k > 0$, such that

$$\left|g\big(t, x\big(t-\tau(t)\big)\big)\right| \le k, \quad \forall t \in [0,T].$$

Then, from (3.13), we have

$$\begin{aligned}\int_0^T |y'(t)|\,dt &\le (r_2+\varepsilon)\int_0^T |x'(t)|\,dt + \lambda\int_0^T h_1\,dt + \lambda\int_0^T k\,dt + \lambda\int_0^T |e(t)|\,dt \\ &\le (r_2+\varepsilon)\sqrt{T}\|x'\|_2 + \big(h_1 + k + \|e\|_\infty\big)T \\ &\le (r_2+\varepsilon)\sqrt{T}R_1 + \big(h_1 + k + \|e\|_\infty\big)T := R_2.\end{aligned}$$

Hence, $\|y\|_\infty \le R_2$.

Let $\Omega_2 = \{z \in \ker L : Nz \in \operatorname{Im} L\}$. If $z \in \Omega_2$, then $z \in \ker L$ and $QNz = 0$. Obviously,

$$|x(t)| \le R_1, \qquad y(t) = 0 \le R_2.$$

Set

$$\Omega = \left\{z = (x,y)^\top \in X : \|x\|_\infty < R_1 + 1, \|y\|_\infty < R_2 + 1\right\},$$

then (1) and (2) of Lemma 2.1 are satisfied.

Next, we claim that (3) of Lemma 2.1 is also satisfied. For this, we define the isomorphism $J : \operatorname{Im} Q \to \ker L$ by

$$J(x,y) = (-y,x),$$

and let $H(\nu,\mu) = \mu\nu + (1-\mu)JQN\nu$, $\forall(\nu,\mu) \in \Omega \times [0,T]$.

By simple calculations, we obtain, for $(z,\mu) \in \partial(\Omega \cap \ker L) \times [0,1]$,

$$z^\top H(z,\mu) = \mu\big(x^2 + y^2\big) + \frac{1-\mu}{T}x\int_0^T \Big(g\big(t, x\big(t-\tau(t)\big)\big) - e(t)\Big)\,dt.$$

Obviously, it follows from (H2) that $z^\top H(z,\mu) > 0$.

Then

$$\begin{aligned}\deg(JQN, \Omega \cap \ker L, 0) &= \deg\bigl(H(z,0), \Omega \cap \ker L, 0\bigr)\\ &= \deg\bigl(H(z,1), \Omega \cap \ker L, 0\bigr)\\ &= \deg(I, \Omega \cap \ker L, 0) \neq 0,\end{aligned}$$

which implies condition (3) of Lemma 2.1 is also satisfied.

Thus $Lz = Nz$ has a solution $z = (x(t), y(t))^\top$, *i.e.*, Equation (1.1) has a $T$-periodic solution $x(t)$ with $\|x\|_\infty \leq R_1$. This completes the proof. □

**Remark 3.1** If $\frac{r_3}{r_1} < \frac{2}{T}$, the condition (H4) can be not assumed, *i.e.*, it follows only from (H1)-(H3) that Equation (1.1) has a $T$-periodic solution.

## 4 An example

In this section, as applications for Theorem 3.1, we list the following example.

**Example 4.1** Consider prescribed mean curvature Rayleigh equation with a deviating argument

$$\left(\frac{x'}{\sqrt{1+x'^2}}\right)' + f\bigl(t, x'(t)\bigr) + g\left(t, x\left(t - \frac{1}{2}\cos^2 t\right)\right) = \cos t, \tag{4.1}$$

where $f(t,x) = (2 + \frac{1}{2}\sin^2 t)\frac{x^3}{\sqrt{1+x^4}}$, $g(t,x) = \frac{1}{3}(1 + \sin^4 t)\frac{x^5}{\sqrt{1+x^8}}$.

Let $T = 2\pi$. Clearly, $r_1 = 2$, $r_2 = \frac{3}{2}$, $r_3 = \frac{2}{3}$, $\alpha = \frac{1}{2}$, and

$$\frac{r_3}{r_1} = \frac{1}{3} < \max\left\{\frac{2}{2\pi}, \frac{1}{\frac{1}{2} + \frac{2\pi}{\pi}}\right\} = \frac{2}{5}.$$

From Theorem 3.1, Equation (4.1) has at least one $T$-periodic solution.

**Remark 4.1** If $f(t,x) = (2 + \frac{1}{2}\sin^2 t)x$, $g(t,x) = \frac{1}{3}(1 + \sin^4 t)x$, Equation (4.1) is a prescribed mean curvature Liénard equation. By using Theorem 3.1, it has at least one $2\pi$-periodic solution, which cannot be obtained by [15]. This implies that the results of this paper are essentially new.

### Competing interests

The authors declare that they have no competing interests.

### Authors' contributions

All authors have equally contributed in obtaining new results in this article and also read and approved the final manuscript.

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