Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

Bifurcation and positive solutions of a nonlinear fourth-order dynamic boundary value problem on time scales

Hua Luo*

*Correspondence: luohuanwnu@gmail.com
School of Mathematics and Quantitative Economics, Dongbei University of Finance and Economics, Dalian, 116025, P.R. China

Abstract

This paper discusses the spectrum properties of a linear fourth-order dynamic boundary value problem on time scales and obtains the existence result of positive solutions to a nonlinear fourth-order dynamic boundary value problem. The key condition which makes nonlinear problem have at least one positive solution is related to the first eigenvalue of the associated linear problem. The proof of the main result is based upon the Krein-Rutman theorem and the global bifurcation techniques on time scales.

MSC: 34N05; 34B18

Keywords: fourth-order dynamic equations; positive solutions; bifurcation; Krein-Rutman theorem; eigenvalue; time scales

1 Introduction

In 2006, Luo and Ma [1] considered the second-order dynamic boundary value problem on time scales

$$\begin{aligned} &u^{\Delta^2}(t) + \hat{f}\big(u^\sigma(t)\big) = 0, \quad t \in [0,1]_{\mathbb{T}}, \\ &u(0) = u(1) = 0, \end{aligned} \tag{1.1}$$

where $\hat{f} \in C([0,\infty),(0,\infty))$. They proved that (1.1) has at least one positive solution if either

$$\hat{f}_0 < \mu_1 < \hat{f}_\infty \tag{1.2}$$

or

$$\hat{f}_\infty < \mu_1 < \hat{f}_0, \tag{1.3}$$

where

$$\hat{f}_0 = \lim_{s\to 0^+} \frac{\hat{f}(s)}{s}, \qquad \hat{f}_\infty = \lim_{s\to\infty} \frac{\hat{f}(s)}{s}$$

and μ_1 is the first eigenvalue of the linear problem

$$u^{\Delta^2}(t) + \mu u^\sigma(t) = 0, \qquad u(0) = u(1) = 0.$$

Notice that conditions (1.2) and (1.3) are optimal! Inspired by this paper, we have a natural question if we could establish some optimal results for the fourth-order dynamic boundary value problem

$$\begin{aligned} &u^{\Delta^4}(t) = f\bigl(t, u(t), u^{\Delta^2}(t)\bigr), \quad t \in \bigl[0, \rho^2(T)\bigr]_{\mathbb{T}}, \\ &u(0) = u\bigl(\sigma^2(T)\bigr) = u^{\Delta^2}(0) = u^{\Delta^2}(T) = 0, \end{aligned} \tag{1.4}$$

where $\mathbb{T}$ is a time scale and $0, T \in \mathbb{T}$, $0 < \rho(T)$, $f : [0, \rho(T)]_{\mathbb{T}} \times [0, \infty) \times (-\infty, 0] \to [0, \infty)$ is continuous. To the best of our knowledge, few papers can be found to study such a problem in the literature. Wang and Sun [2] and Luo and Gao [3] applied the Schauder fixed point theorem to show the existence of positive solutions of a fourth-order dynamic boundary value problem under different boundary value conditions. However, the key conditions in these two papers are not directly related to the first eigenvalue of the associated linear eigenvalue problems.

For the continuous cases, Ma and Xu [4] discussed

$$\begin{aligned} &u^{(4)}(x) = g\bigl(x, u(x), u''(x)\bigr), \quad x \in (0,1), \\ &u(0) = u(1) = u''(0) = u''(1) = 0, \end{aligned}$$

where $g : [0,1] \times [0, \infty) \times (-\infty, 0] \to [0, \infty)$ is continuous. And for the discrete cases, there exists no work corresponding exactly to the continuous cases, but we can refer the reader to Xu, Gao and Ma [5] and Ma and Gao [6] and Gao and Xu [7] for the similar discussions. Moreover, nothing is known about the time scales extension analog (1.4) except for [8] for second-order dynamic equations. The likely reason is that few properties of the associated linear eigenvalue problem

$$\begin{aligned} &u^{\Delta^4}(t) + \lambda\bigl[B(t)u^{\Delta^2}(t) - A(t)u(t)\bigr] = 0, \quad t \in \bigl[0, \rho^2(T)\bigr]_{\mathbb{T}}, \\ &u(0) = u\bigl(\sigma^2(T)\bigr) = u^{\Delta^2}(0) = u^{\Delta^2}(T) = 0 \end{aligned} \tag{1.5}$$

are known.

In this paper, we establish spectrum properties of (1.5) in Section 2 and use the global bifurcation techniques on time scales (see Davidson and Rynne [9, Theorem 7.1] or Luo and Ma [1]) to discuss the existence of positive solutions for problem (1.4) in Section 3. Our existence result is related to the first eigenvalue of the associated linear eigenvalue problem (1.5) and thus should be optimal.

For convenience, here we will not introduce the concepts and notations about time scale. The reader is referred to [10, 11] or most of the time-scales-related papers for details.

2 Generalized eigenvalues

For a compact interval $[a,b]_{\mathbb{T}}$, $C[a,b]_{\mathbb{T}}$ is a Banach space with the norm

$$\|y\|_{[a,b]_{\mathbb{T}}} = \sup\bigl\{\bigl|y(t)\bigr| \mid t \in [a,b]_{\mathbb{T}}\bigr\}.$$

For $h \in C[0,\rho(T)]_{\mathbb{T}}$, let us consider the linear problem

$$\begin{aligned} &u^{\Delta^4}(t) = h(t), \quad t \in \big[0,\rho^2(T)\big]_{\mathbb{T}}, \\ &u(0) = u\big(\sigma^2(T)\big) = u^{\Delta^2}(0) = u^{\Delta^2}(T) = 0. \end{aligned} \tag{2.1}$$

Let $u^{\Delta^2}(t) = y(t)$ for $t \in [0,T]_{\mathbb{T}}$. Then

$$\begin{aligned} &y^{\Delta^2}(t) = h(t), \quad t \in \big[0,\rho^2(T)\big]_{\mathbb{T}}, \\ &y(0) = y(T) = 0. \end{aligned}$$

Consequently,

$$y(t) = -\int_0^{\rho(T)} H_1(t,s)h(s)\Delta s, \quad t \in [0,T]_{\mathbb{T}}, \tag{2.2}$$

where

$$H_1(t,s) = \frac{1}{T}\begin{cases} t(T-\sigma(s)), & t \le s, \\ \sigma(s)(T-t), & \sigma(s) \le t. \end{cases} \tag{2.3}$$

Now, (2.1) is equivalent to

$$\begin{aligned} &u^{\Delta^2}(t) = y(t), \quad t \in [0,T]_{\mathbb{T}}, \\ &u(0) = u\big(\sigma^2(T)\big) = 0. \end{aligned}$$

Consequently again,

$$u(t) = -\int_0^{\sigma(T)} H(t,s)y(s)\Delta s, \quad t \in \big[0,\sigma^2(T)\big]_{\mathbb{T}}, \tag{2.4}$$

where

$$H(t,s) = \frac{1}{\sigma^2(T)}\begin{cases} t(\sigma^2(T)-\sigma(s)), & t \le s, \\ \sigma(s)(\sigma^2(T)-t), & \sigma(s) \le t. \end{cases} \tag{2.5}$$

And we can easily check that

$$H(t,s) \ge H_1(t,s), \quad t \in [0,T]_{\mathbb{T}}, s \in \big[0,\rho(T)\big]_{\mathbb{T}}. \tag{2.6}$$

To sum up, we have proved the following.

Lemma 2.1 *For each $h \in C[0,\rho^2(T)]_{\mathbb{T}}$, problem* (2.1) *has a unique solution*

$$u(t) = \int_0^{\sigma(T)} H(t,s)\int_0^{\rho(T)} H_1(s,\tau)h(\tau)\Delta\tau\,\Delta s, \quad t \in \big[0,\sigma^2(T)\big]_{\mathbb{T}}. \tag{2.7}$$

Let $0 < l < \tilde{l} < 1$: $\sin_{\sqrt{l}}(\sigma^2(T), 0) = 0$, $\sin_{\sqrt{\tilde{l}}}(T, 0) = 0$. Then $e(t) = \sin_{\sqrt{l}}(t, 0)$ solves the problem

$$\begin{aligned} &e^{\Delta^2}(t) + le(t) = 0, \quad t \in [0, T]_{\mathbb{T}}, \\ &e(0) = e\big(\sigma^2(T)\big) = 0 \end{aligned} \tag{2.8}$$

and $\tilde{e}(t) = \sin_{\sqrt{\tilde{l}}}(t, 0)$ solves the problem

$$\begin{aligned} &\tilde{e}^{\Delta^2}(t) + \tilde{l}\tilde{e}(t) = 0, \quad t \in \big[0, \rho^2(T)\big]_{\mathbb{T}}, \\ &\tilde{e}(0) = \tilde{e}(T) = 0 \end{aligned} \tag{2.9}$$

from [10, pp.92-93].

Remark 2.1

$$\begin{aligned} &\sin_{\sqrt{l}}(t, 0) = \frac{e_{i\sqrt{l}}(t, 0) - e_{-i\sqrt{l}}(t, 0)}{2i}; \\ &\sin_{\sqrt{\tilde{l}}}(t, 0) = \frac{e_{i\sqrt{\tilde{l}}}(t, 0) - e_{-i\sqrt{\tilde{l}}}(t, 0)}{2i}; \\ &e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}\big(p(\tau)\big)\Delta\tau\right); \\ &\xi_h(z) = \begin{cases} \frac{1}{h}\operatorname{Log}(1 + zh) = \frac{1}{h}[\log|1 + zh| + i\arg(1 + zh)], & h > 0, \\ z, & h = 0. \end{cases} \end{aligned}$$

It is clear that

$$e(t) > 0, \quad t \in \big(0, \sigma^2(T)\big)_{\mathbb{T}}, \qquad \tilde{e}(t) > 0, \quad t \in (0, T)_{\mathbb{T}} \tag{2.10}$$

and so there exist $\epsilon_1, \epsilon_2 > 0$ such that

$$\tilde{e}(t) \le \epsilon_1 e(t), \quad t \in \big[0, \rho(T)\big], \tag{2.11}$$

$$e(t) \le \epsilon_2 \tilde{e}(t), \quad t \in \big[0, \rho(T)\big]. \tag{2.12}$$

Remark 2.2 (2.11) holds for any time scale $\mathbb{T}$ with $0, T \in \mathbb{T}$, but (2.12) does not hold for the case of T: $\rho(T) = T < \sigma(T)$, *i.e.*, T is an left-dense and right-scattered point (abbreviated to ldrs point) of the time scale $\mathbb{T}$.

Let

$$X = \left\{ u \in C^2\big[0, \sigma^2(T)\big]_{\mathbb{T}} \;\middle|\; \begin{aligned} &u(0) = u\big(\sigma^2(T)\big) = u^{\Delta^2}(0) = u^{\Delta^2}(T) = 0; \\ &\exists\gamma \in (0, \infty): -\gamma\tilde{e}(t) \le -u^{\Delta^2}(t) \le \gamma\tilde{e}(t) \text{ for } t \in [0, T]_{\mathbb{T}}; \\ &\exists\delta \in (0, \infty): -\delta e(t) \le u(t) \le \delta e(t) \text{ for } t \in \big[0, \sigma^2(T)\big]_{\mathbb{T}} \end{aligned} \right\}$$

and define the norm of $u \in X$ by

$$\|u\|_X := \inf\left\{ \left|(\gamma,\delta)\right| \;\middle|\; \begin{matrix} -\gamma\tilde{e}(t) \le -u^{\Delta^2}(t) \le \gamma\tilde{e}(t) \text{ for } t \in [0,T]_{\mathbb{T}}; \\ -\delta e(t) \le u(t) \le \delta e(t) \text{ for } t \in [0,\sigma^2(T)]_{\mathbb{T}} \end{matrix} \right\}, \tag{2.13}$$

where $|(\gamma,\delta)| = \sqrt{\gamma^2+\delta^2}$. It is easy to check that $(X, \|\cdot\|_X)$ is a Banach space. Let

$$P := \{u \in X \mid u^{\Delta^2}(t) \le 0 \text{ for } t \in [0,T]_{\mathbb{T}}; u(t) \ge 0 \text{ for } t \in [0,\sigma^2(T)]_{\mathbb{T}}\}.$$

Then the cone P is normal and has nonempty interiors $\operatorname{int} P$.

Lemma 2.2 *For* $u \in X$,

$$\|u\|_{C[0,\sigma^2(T)]_{\mathbb{T}}} \le q\|u\|_X, \qquad \|u^{\Delta^2}\|_{C[0,T]_{\mathbb{T}}} \le q\|u\|_X, \tag{2.14}$$

where $q := \max\{\|e\|_{C[0,\sigma^2(T)]_{\mathbb{T}}}, \|\tilde{e}\|_{C[0,T]_{\mathbb{T}}}\}$.

Proof Since $u \in X$, there exist $\delta \in (0,\infty)$: $-\delta e(t) \le u(t) \le \delta e(t)$, $t \in [0,\sigma^2(T)]_{\mathbb{T}}$ and $\gamma \in (0,\infty)$: $-\gamma\tilde{e}(t) \le -u^{\Delta^2}(t) \le \gamma\tilde{e}(t)$, $t \in [0,T]_{\mathbb{T}}$, we have

$$\|u\|_{C[0,\sigma^2(T)]_{\mathbb{T}}} \le \delta\|e\|_{C[0,\sigma^2(T)]_{\mathbb{T}}} \le \left|(\gamma,\delta)\right|\|e\|_{C[0,\sigma^2(T)]_{\mathbb{T}}}$$

and

$$\|u^{\Delta^2}\|_{C[0,T]_{\mathbb{T}}} \le \gamma\|\tilde{e}\|_{C[0,T]_{\mathbb{T}}} \le \left|(\gamma,\delta)\right|\|\tilde{e}\|_{C[0,T]_{\mathbb{T}}}.$$

Thus

$$\begin{aligned} &\|u\|_{C[0,\sigma^2(T)]_{\mathbb{T}}} \le \|e\|_{C[0,\sigma^2(T)]_{\mathbb{T}}}\|u\|_X \le q\|u\|_X, \\ &\|u^{\Delta^2}\|_{C[0,T]_{\mathbb{T}}} \le \|\tilde{e}\|_{C[0,T]_{\mathbb{T}}}\|u\|_X \le q\|u\|_X. \end{aligned}$$

□

Let us make the assumption

(H0) $A, B \in C([0,\rho(T)]_{\mathbb{T}}, [0,\infty))$ with $A(t) > 0$ or $B(t) > 0$ on $[0,\rho(T)]_{\mathbb{T}}$.

Define a linear operator $G : X \to C^4[0,\sigma^2(T)]_{\mathbb{T}}$ by

$$\begin{aligned} Gu(t) &:= \int_0^{\sigma(T)} H(t,s)\int_0^{\rho(T)} H_1(s,\tau)\left[A(\tau)u(\tau) - B(\tau)u^{\Delta^2}(\tau)\right]\Delta\tau\,\Delta s, \\ &\quad t \in [0,\sigma^2(T)]_{\mathbb{T}}. \end{aligned} \tag{2.15}$$

Then (1.5) is equivalent to

$$u(t) = \lambda Gu(t), \quad t \in [0,\sigma^2(T)]_{\mathbb{T}}.$$

Definition 2.1 We say λ is an *eigenvalue* of the linear problem

$$\begin{aligned} &u^{\Delta^4}(t)+\lambda\big[B(t)u^{\Delta^2}(t)-A(t)u(t)\big]=0, \quad t\in\big[0,\rho^2(T)\big]_{\mathbb{T}}, \\ &u(0)=u\big(\sigma^2(T)\big)=u^{\Delta^2}(0)=u^{\Delta^2}(T)=0 \end{aligned} \tag{2.16}$$

if (2.16) has nontrivial solutions.

Theorem 2.1 *Let* (H0) *hold and T be not an ldrs point of the time scale $\mathbb{T}$. Then*

1. *$G(P)\subseteq P$, and $G:P\to P$ is strongly positive*;
2. *Problem* (1.5) *has an algebraically simple eigenvalue, $\lambda_1(A,B)=(r(G))^{-1}$, with a nonnegative eigenfunction $\varphi_1(\cdot)\in \operatorname{int}P$*;
3. *There is no other eigenvalue with a nonnegative eigenfunction.*

Proof We prove $G:X\to X$ firstly.

For $u\in X$, we have

$$\begin{aligned} &-\delta e(t)\le u(t)\le \delta e(t), \quad t\in\big[0,\sigma^2(T)\big]_{\mathbb{T}} \quad \text{and} \\ &-\gamma\tilde{e}(t)\le -u^{\Delta^2}(t)\le \gamma\tilde{e}(t), \quad t\in[0,T]_{\mathbb{T}} \end{aligned}$$

for some positive δ and γ. From (2.15), the condition (H0), (2.11) and (2.12), we have

$$\begin{aligned} (Gu)(t) &= \int_0^{\sigma(T)} H(t,s)\int_0^{\rho(T)} H_1(s,\tau)\big[A(\tau)u(\tau)-B(\tau)u^{\Delta^2}(\tau)\big]\Delta\tau\,\Delta s \\ &\le \int_0^{\sigma(T)} H(t,s)\int_0^{\rho(T)} H_1(s,\tau)\big[\delta A(\tau)e(\tau)+\gamma B(\tau)\tilde{e}(\tau)\big]\Delta\tau\,\Delta s \\ &\le \int_0^{\sigma(T)} H(t,s)\int_0^{\rho(T)} H_1(s,\tau)\big[\epsilon_2\delta A(\tau)+\gamma B(\tau)\big]\tilde{e}(\tau)\Delta\tau\,\Delta s \\ &\le \big[\epsilon_2\delta\|A\|_{C[0,\rho(T)]_{\mathbb{T}}}+\gamma\|B\|_{C[0,\rho(T)]_{\mathbb{T}}}\big]\int_0^{\sigma(T)} H(t,s)\int_0^{\rho(T)} H_1(s,\tau)\tilde{e}(\tau)\Delta\tau\,\Delta s \\ &\le \frac{1}{\tilde{l}}\big[\epsilon_2\delta\|A\|_{C[0,\rho(T)]_{\mathbb{T}}}+\gamma\|B\|_{C[0,\rho(T)]_{\mathbb{T}}}\big]\int_0^{\sigma(T)} H(t,s)\epsilon_1 e(s)\Delta s \\ &= \frac{\epsilon_1}{l\tilde{l}}\big[\epsilon_2\delta\|A\|_{C[0,\rho(T)]_{\mathbb{T}}}+\gamma\|B\|_{C[0,\rho(T)]_{\mathbb{T}}}\big]e(t). \end{aligned} \tag{2.17}$$

By (2.15), the condition (H0) and (2.12), we have

$$\begin{aligned} -(Gu)^{\Delta^2}(t) &= \int_0^{\rho(T)} H_1(t,\tau)\big[A(\tau)u(\tau)-B(\tau)u^{\Delta^2}(\tau)\big]\Delta\tau \\ &\le \int_0^{\rho(T)} H_1(t,\tau)\big[\delta A(\tau)e(\tau)+\gamma B(\tau)\tilde{e}(\tau)\big]\Delta\tau \\ &\le \int_0^{\rho(T)} H_1(t,\tau)\big[\epsilon_2\delta A(\tau)+\gamma B(\tau)\big]\tilde{e}(\tau)\Delta\tau \\ &\le \big[\epsilon_2\delta\|A\|_{C[0,\rho(T)]_{\mathbb{T}}}+\gamma\|B\|_{C[0,\rho(T)]_{\mathbb{T}}}\big]\int_0^{\rho(T)} H_1(t,\tau)\tilde{e}(\tau)\Delta\tau \\ &= \frac{1}{\tilde{l}}\big[\epsilon_2\delta\|A\|_{C[0,\rho(T)]_{\mathbb{T}}}+\gamma\|B\|_{C[0,\rho(T)]_{\mathbb{T}}}\big]\tilde{e}(t). \end{aligned} \tag{2.18}$$

Similarly, we get

$$(Gu)(t) \geq -\frac{\epsilon_1}{l\tilde{l}}\left[\epsilon_2\delta\|A\|_{C[0,\rho(T)]_{\mathbb{T}}} + \gamma\|B\|_{C[0,\rho(T)]_{\mathbb{T}}}\right]e(t) \tag{2.19}$$

and

$$(Gu)^{\Delta^2}(t) \geq -\frac{1}{\tilde{l}}\left[\epsilon_2\delta\|A\|_{C[0,\rho(T)]_{\mathbb{T}}} + \gamma\|B\|_{C[0,\rho(T)]_{\mathbb{T}}}\right]\tilde{e}(t). \tag{2.20}$$

Thus $G: X \to X$. Furthermore, since $G(X) \subset C^4[0,\sigma^2(T)]_{\mathbb{T}} \cap X \hookrightarrow\hookrightarrow X$, we have that $G: X \to X$ is compact.

Next, we show that $G: P \to P$ is strongly positive.

For $u \in P \setminus \{0\}$, if $A(t) > 0$ on $[0,\rho(T)]_{\mathbb{T}}$, then $A(t) \geq k$ on $[0,\rho(T)]_{\mathbb{T}}$ for some constant $k > 0$, and subsequently,

$$\begin{aligned} Gu(t) &= \int_0^{\sigma(T)} H(t,s)\int_0^{\rho(T)} H_1(s,\tau)\left[A(\tau)u(\tau) - B(\tau)u^{\Delta^2}(\tau)\right]\Delta\tau\,\Delta s \\ &\geq k\int_0^{\sigma(T)} H(t,s)\int_0^{\rho(T)} H_1(s,\tau)u(\tau)\Delta\tau\,\Delta s =: (G_1u)(t), \quad t \in \left[0,\sigma^2(T)\right]_{\mathbb{T}}. \end{aligned}$$

Since $(G_1u)(0) = (G_1u)(\sigma^2(T)) = 0$ and

$$-(G_1u)^{\Delta^2}(t) = k\int_0^{\rho(T)} H_1(t,\tau)u(\tau)\Delta\tau > 0, \quad t \in (0,T)_{\mathbb{T}},$$

it follows that there exists $r > 0$ such that $G_1u(t) \geq re(t)$ on $[0,\sigma^2(T)]_{\mathbb{T}}$. Thus

$$Gu(t) \geq re(t), \quad t \in \left[0,\sigma^2(T)\right]_{\mathbb{T}}. \tag{2.21}$$

If $B(t) > 0$ on $[0,\rho(T)]_{\mathbb{T}}$, then $B(t) \geq k_1$ on $[0,\rho(T)]_{\mathbb{T}}$ for some constant $k_1 > 0$, and subsequently,

$$\begin{aligned} Gu(t) &\geq k_1\int_0^{\sigma(T)} H(t,s)\int_0^{\rho(T)} H_1(s,\tau)\left[-u^{\Delta^2}(\tau)\right]\Delta\tau\,\Delta s \\ &= k_1\int_0^{\sigma(T)} H(t,s)u(s)\Delta s =: (G_2u)(t), \quad t \in \left[0,\sigma^2(T)\right]_{\mathbb{T}}. \end{aligned}$$

Then there exists $r_1 > 0$ such that $G_2u(t) \geq r_1e(t)$ on $[0,\sigma^2(T)]_{\mathbb{T}}$. So,

$$Gu(t) \geq r_1e(t), \quad t \in \left[0,\sigma^2(T)\right]_{\mathbb{T}}. \tag{2.22}$$

Similarly, for any $u \in P \setminus \{0\}$, if $A(t) > 0$ on $[0,\rho(T)]_{\mathbb{T}}$, then there exists $r_2 > 0$ such that for $t \in [0,T]_{\mathbb{T}}$,

$$\begin{aligned} -(Gu)^{\Delta^2}(t) &= \int_0^{\rho(T)} H_1(t,\tau)\left[A(\tau)u(\tau) - B(\tau)u^{\Delta^2}(\tau)\right]\Delta\tau \\ &\geq \int_0^{\rho(T)} H_1(t,\tau)A(\tau)u(\tau)\Delta\tau \end{aligned}$$

$$\begin{aligned} &\geq k\int_0^{\rho(T)} H_1(t,\tau)u(\tau)\Delta\tau \\ &\geq r_2\tilde{e}(t). \end{aligned} \tag{2.23}$$

If $B(t) > 0$ on $[0,\rho(T)]_{\mathbb{T}}$, then there exists $r_3 > 0$ such that for $t \in [0,T]_{\mathbb{T}}$,

$$\begin{aligned} -(Gu)^{\Delta^2}(t) &\geq \int_0^{\rho(T)} H_1(t,\tau)\big[-B(\tau)u^{\Delta^2}(\tau)\big]\Delta\tau \\ &\geq k_1\int_0^{\rho(T)} H_1(t,\tau)\big[-u^{\Delta^2}(\tau)\big]\Delta\tau \\ &\geq r_3\tilde{e}(t). \end{aligned} \tag{2.24}$$

It follows from (2.21)-(2.24) that $Gu \in \operatorname{int} P$.

Now, by the Krein-Rutman theorem ([12, Theorem 7.C] or [13, Theorem 19.3]), G has an algebraically simple eigenvalue $\lambda_1(A,B)$ with an eigenfunction $\varphi_1(\cdot) \in \operatorname{int} P$. Moreover, there is no other eigenvalue with a nonnegative eigenfunction. □

3 The main result

In this section, we will make the following assumptions:

(H1) T is not an ldrs point of the time scale $\mathbb{T}$.

(H2) $f : [0,\rho(T)]_{\mathbb{T}} \times [0,\infty) \times (-\infty,0] \to [0,\infty)$ is continuous and there exist functions $a,b,c,d \in C([0,\rho(T)]_{\mathbb{T}},[0,\infty))$ with $a(t)+b(t) > 0$ and $c(t)+d(t) > 0$ on $[0,\rho(T)]_{\mathbb{T}}$ such that

$$f(t,u,p) = a(t)u - b(t)p + o\big(\big|(u,p)\big|\big) \quad \text{as } \big|(u,p)\big| \to (0,0)$$

uniformly for $t \in [0,\rho(T)]_{\mathbb{T}}$, and

$$f(t,u,p) = c(t)u - d(t)p + o\big(\big|(u,p)\big|\big) \quad \text{as } \big|(u,p)\big| \to \infty$$

uniformly for $t \in [0,\rho(T)]_{\mathbb{T}}$. Here

$$\big|(u,p)\big| := \sqrt{u^2+p^2}.$$

(H3) $f(t,u,p) > 0$ for $t \in [0,\rho(T)]_{\mathbb{T}}$ and $(u,p) \in ([0,\infty) \times (-\infty,0]) \setminus \{(0,0)\}$.

(H4) There exist functions $a_0 : [0,\rho(T)]_{\mathbb{T}} \to [0,\infty)$ and $b_0 \in C([0,\rho(T)]_{\mathbb{T}},(0,\infty))$ such that

$$f(t,u,p) \geq a_0(t)u - b_0(t)p, \quad (t,u,p) \in \big[0,\rho(T)\big]_{\mathbb{T}} \times [0,\infty) \times (-\infty,0].$$

Theorem 3.1 *Let* (H1)-(H4) *hold. Assume that either*

$$\lambda_1(c,d) < 1 < \lambda_1(a,b) \tag{3.1}$$

or

$$\lambda_1(a,b) < 1 < \lambda_1(c,d). \tag{3.2}$$

Then problem (1.4) *has at least one positive solution.*

Denote $L : D(L) \to C[0,\sigma^2(T)]_{\mathbb{T}}$ by setting

$$(Lu)(t) := u^{\Delta^4}(t), \quad u \in D(L), t \in \left[0,\rho^2(T)\right]_{\mathbb{T}}, \tag{3.3}$$

where

$$D(L) = \left\{u \in C^4\left[0,\sigma^2(T)\right]_{\mathbb{T}} \mid u(0) = u\left(\sigma^2(T)\right) = u^{\Delta^2}(0) = u^{\Delta^2}(T) = 0\right\}. \tag{3.4}$$

From [9, Lemma 3.7] and standard properties of compact linear operators, we can get that $L^{-1} : C[0,\sigma^2(T)]_{\mathbb{T}} \to C^1[0,\sigma(T)]_{\mathbb{T}}$ is compact.

Let $\zeta, \xi : [0,\rho(T)]_{\mathbb{T}} \times [0,\infty) \times (-\infty,0] \to \mathbb{R}$ be continuous and satisfy

$$f(t,u,p) = a(t)u - b(t)p + \zeta(t,u,p), \tag{3.5}$$

$$f(t,u,p) = c(t)u - d(t)p + \xi(t,u,p). \tag{3.6}$$

Obviously, (H2) implies that

$$\lim_{|(u,p)|\to 0} \frac{\zeta(t,u,p)}{|(u,p)|} = 0 \quad \text{uniformly for } t \in \left[0,\rho(T)\right]_{\mathbb{T}}, \tag{3.7}$$

$$\lim_{|(u,p)|\to \infty} \frac{\xi(t,u,p)}{|(u,p)|} = 0 \quad \text{uniformly for } t \in \left[0,\rho(T)\right]_{\mathbb{T}}. \tag{3.8}$$

Let

$$\tilde{\xi}(r) = \max\left\{\left|\xi(t,u,p)\right| \mid 0 \le \left|(u,p)\right| \le r, t \in \left[0,\rho(T)\right]_{\mathbb{T}}\right\}. \tag{3.9}$$

Then $\tilde{\xi}$ is nondecreasing and

$$\lim_{r\to\infty} \frac{\tilde{\xi}(r)}{r} = 0. \tag{3.10}$$

Next we consider

$$(Lu)(t) = \lambda\left[a(t)u(t) - b(t)u^{\Delta^2}(t)\right] + \lambda\zeta\left(t,u(t),u^{\Delta^2}(t)\right), \quad \lambda > 0, \tag{3.11}$$

as a bifurcation problem from the trivial solution $u \equiv 0$. It is easy to check that (3.11) can be converted to the equivalent equation

$$\begin{aligned} u(t) &= \lambda\left\{\int_0^{\sigma(T)} H(t,s)\int_0^{\rho(T)} H_1(s,\tau)\left[a(\tau)u(\tau) - b(\tau)u^{\Delta^2}(\tau)\right]\Delta\tau\,\Delta s\right\} \\ &\quad + \lambda\left\{\int_0^{\sigma(T)} H(t,s)\int_0^{\rho(T)} H_1(s,\tau)\zeta\left(\tau,u(\tau),u^{\Delta^2}(\tau)\right)\Delta\tau\,\Delta s\right\} \\ &=: R(\lambda,u)(t). \end{aligned} \tag{3.12}$$

From Theorem 2.1, we have that for each fixed $\lambda > 0$, the operator $\hat{G} : P \to P$,

$$\hat{G}u(t) := \int_0^{\sigma(T)} H(t,s)\int_0^{\rho(T)} H_1(s,\tau)\left[a(\tau)u(\tau) - b(\tau)u^{\Delta^2}(\tau)\right]\Delta\tau\,\Delta s, \tag{3.13}$$

is compact and strongly positive. Define $F : [0,\infty) \times X \to X$ by

$$F(\lambda,u) := \lambda \int_0^{\sigma(T)} H(t,s) \int_0^{\rho(T)} H_1(s,\tau)\zeta\big(\tau,u(\tau),u^{\Delta^2}(\tau)\big)\Delta\tau\,\Delta s. \tag{3.14}$$

Then we have from (3.7) and Lemma 2.2 that

$$\big\|F(\lambda,u)\big\|_X = o\big(\|u\|_X\big) \quad \text{as } \|u\|_X \to 0 \tag{3.15}$$

locally uniformly in λ. Now, we have from Davidson and Rynne [9, Theorem 7.1], or Luo and Ma [1], to conclude that there exists an unbounded connected subset $\mathcal{C}$ in the set

$$\big\{(\lambda,u) \in [0,\infty) \times P \colon u = \lambda\hat{G}u + F(\lambda,u), u \in \operatorname{int} P\big\} \cup \big\{\big(\lambda_1(a,b),0\big)\big\}$$

such that $(\lambda_1(a,b),0) \in \mathcal{C}$.

Proof of Theorem 3.1 It is clear that any solution of (3.11) of the form $(1,u)$ yields a solution u of (1.4). We will show that $\mathcal{C}$ crosses the hyperplane $\{1\} \times X$ in $\mathbb{R} \times X$. To do this, it is enough to show that $\mathcal{C}$ joins $(\lambda_1(a,b),0)$ to $(\lambda_1(c,d),\infty)$. Let $(\eta_n,y_n) \in \mathcal{C}$ satisfy

$$\eta_n + \|y_n\|_X \to \infty. \tag{3.16}$$

We note that $\eta_n > 0$ for all $n \in \mathbb{N}$ since $(0,0)$ is the only solution of (3.11) for $\lambda = 0$ and $\mathcal{C} \cap (\{0\} \times X) = \emptyset$.

Case 1. $\lambda_1(c,d) < 1 < \lambda_1(a,b)$.

In this case, we show that

$$\big(\lambda_1(c,d),\lambda_1(a,b)\big) \subseteq \big\{\lambda \in \mathbb{R} \mid (\lambda,u) \in \mathcal{C}\big\}.$$

We divide the proof into two steps.

Step 1. We show that if there exists a constant number $M > 0$ such that

$$\eta_n \subset (0,M], \tag{3.17}$$

then C joins $(\lambda_1(a,b),0)$ to $(\lambda_1(c,d),\infty)$.

If (3.17) holds, we have that $\|y_n\|_X \to \infty$. We divide the equation

$$Ly_n(t) = \eta_n\big[c(t)y_n(t) - d(t)y_n^{\Delta^2}(t)\big] + \eta_n\xi\big(t,y_n(t),y_n^{\Delta^2}(t)\big), \quad t \in \big[0,\rho^2(T)\big]_{\mathbb{T}} \tag{3.18}$$

by $\|y_n\|_X$ and set $\bar{y}_n = \frac{y_n}{\|y_n\|_X}$. Since $\bar{y}_n$ is bounded in X, choosing a subsequence and relabeling if necessary, we see that $\bar{y}_n \to \bar{y}$ for some $\bar{y} \in X$ with $\|\bar{y}\|_X = 1$. Moreover, from Lemma 2.2, we have

$$\frac{|\xi(t,y_n(t),y_n^{\Delta^2}(t))|}{\|y_n\|_X} \le \frac{\tilde{\xi}(\sqrt{2}q\|y_n\|_X)}{\|y_n\|_X}$$

and from (3.10),

$$\lim_{n\to\infty}\frac{|\xi(t,y_n(t),y_n^{\Delta^2}(t))|}{\|y_n\|_X}=0.$$

Thus

$$\bar{y}(t):=\int_0^{\sigma(T)}H(t,s)\int_0^{\rho(T)}H_1(s,\tau)\bar{\eta}\big[c(\tau)\bar{y}(\tau)-d(\tau)\bar{y}^{\Delta^2}(\tau)\big]\Delta\tau\,\Delta s,$$

where $\bar{\eta}:=\lim_{n\to\infty}\eta_n$, again choosing a subsequence and relabeling if necessary. Therefore,

$$L\bar{y}(t)=\bar{\eta}\big[c(t)\bar{y}(t)-d(t)\bar{y}^{\Delta^2}(t)\big],\quad t\in\big[0,\rho^2(T)\big]_{\mathbb{T}}.$$

This together with Theorem 2.1 yields $\bar{\eta}=\lambda_1(c,d)$. Thus $\mathcal{C}$ joins $(\lambda_1(a,b),0)$ to $(\lambda_1(c,d),\infty)$.

Step 2. We show that there exists a constant M such that $\eta_n\in(0,M]$ for all n.

Since $(y_n,\eta_n)\in\mathcal{C}$, we have

$$Ly_n=\eta_n f\big(t,y_n,y_n^{\Delta^2}\big).$$

Therefore from (H4), we get

$$\begin{aligned}y_n(t)&=\eta_n\int_0^{\sigma(T)}H(t,s)\int_0^{\rho(T)}H_1(s,\tau)f\big(\tau,y_n(\tau),y_n^{\Delta^2}(\tau)\big)\Delta\tau\,\Delta s\\&\ge\eta_n\int_0^{\sigma(T)}H(t,s)\int_0^{\rho(T)}H_1(s,\tau)\big[a_0(t)y_n(\tau)-b_0(t)y_n^{\Delta^2}(\tau)\big]\Delta\tau\,\Delta s\\&\ge\eta_n b^*\int_0^{\sigma(T)}H(t,s)\int_0^{\rho(T)}H_1(s,\tau)\big[-y_n^{\Delta^2}(\tau)\big]\Delta\tau\,\Delta s\\&=\eta_n b^*\int_0^{\sigma(T)}H(t,s)y_n(s)\Delta s,\end{aligned}\tag{3.19}$$

where $b^*=\min_{t\in[0,\rho(T)]_{\mathbb{T}}}b_0(t)>0$. Set $v_n(t)=\int_0^{\sigma(T)}H(t,s)y_n(s)\Delta s$, we have $-v_n^{\Delta^2}(t)=y_n(t)$, $t\in[0,\sigma^2(T)]_{\mathbb{T}}$, $v_n(0)=v_n(\sigma^2(T))=0$ and

$$-v_n^{\Delta^2}(t)\ge\eta_n b^* v_n(t).$$

Furthermore,

$$v_n(t)>0,\quad t\in[\alpha,\beta]_{\mathbb{T}}.$$

From $y_n(t)\ge 0$, $t\in[0,\sigma^2(T)]_{\mathbb{T}}$. Now we can obtain $\eta_n\in(0,M]$, $\forall n$ for some positive constant M according to [1, Lemma 2.2].

Case 2. $\lambda_1(a,b)<1<\lambda_1(c,d)$.

According to Step 2 of Case 1, we have

$$\eta_n\in(0,M],\quad n\in\mathbb{N}$$

for some $M > 0$. Then if $(\eta_n, y_n) \in \mathcal{C}$ is such that

$$\lim_{n\to\infty}\left(\eta_n + \|y_n\|_X\right) = \infty,$$

applying a similar argument to that used in Step 1 of Case 1, after taking a subsequence and relabeling if necessary, it follows that

$$(\eta_n, y_n) \to \left(\lambda_1(c,d), \infty\right), \quad n \to \infty.$$

Again $\mathcal{C}$ joins $(\lambda_1(a,b), 0)$ to $(\lambda_1(c,d), \infty)$ and the result follows. □

Competing interests
The author declares that they have no competing interests.

Acknowledgements
We would like to thank the referees for carefully reading this paper and suggesting many valuable comments. This work was supported by China Postdoctoral Science Foundation Funded Project (Nos. 201104602, 20100481239), General Project for Scientific Research of Liaoning Educational Committee (Nos. L2011200, L2012409), Teaching and Research Project of DUFE (No. YY12012) and the NSFC (Nos. 71171035, 71201019).

Received: 29 September 2012 Accepted: 2 March 2013 Published: 20 March 2013

References
1. Luo, H, Ma, R: Nodal solutions to nonlinear eigenvalue problems on time scales. Nonlinear Anal. **65**, 773-784 (2006)
2. Wang, D, Sun, J: Existence of a solution and a positive solution of a boundary value problem for a nonlinear fourth-order dynamic equation. Nonlinear Anal. **69**, 1817-1823 (2008)
3. Luo, H, Gao, C: Positive solutions of a nonlinear fourth-order dynamic eigenvalue problem on time scales. Abstr. Appl. Anal. **2012**, Article ID 798796 (2012)
4. Ma, R, Xu, L: Existence of positive solutions of a nonlinear fourth-order boundary value problem. Appl. Math. Lett. **23**, 537-543 (2010)
5. Xu, Y, Gao, C, Ma, R: Solvability of a nonlinear fourth-order discrete problem at resonance. Appl. Math. Comput. **216**, 662-670 (2010)
6. Ma, R, Gao, C: Bifurcation of positive solutions of a nonlinear discrete fourth-order boundary value problem. Z. Angew. Math. Phys. (2012). doi:10.1007/s00033-012-0243-7
7. Gao, C, Xu, J: Bifurcation techniques and positive solutions of discrete Lidstone boundary value problems. Appl. Math. Comput. **218**, 434-444 (2011)
8. Luo, H: Bifurcation from interval and positive solutions of a nonlinear second-order dynamic boundary value problem on time scales. Abstr. Appl. Anal. **2012**, Article ID 316080 (2012)
9. Davidson, FA, Rynne, BP: Global bifurcation on time scales. J. Math. Anal. Appl. **267**, 345-360 (2002)
10. Bohner, M, Peterson, A: Dynamic Equations on Time Scales: An Introduction with Applications. Birkhäuser, Boston (2001)
11. Bohner, M, Peterson, A (eds.): Advances in Dynamic Equations on Time Scales. Birkhäuser, Boston (2003)
12. Zeidler, E: Nonlinear Functional Analysis and Its Applications. I. Fixed-Point Theorems. Springer, New York (1985)
13. Deimling, K: Nonlinear Functional Analysis. Springer, Berlin (1985)

doi:10.1186/1687-1847-2013-64
Cite this article as: Luo: **Bifurcation and positive solutions of a nonlinear fourth-order dynamic boundary value problem on time scales.** *Advances in Difference Equations* 2013 **2013**:64.