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Urysohn integral equations approach by common fixed points in complex-valued metric spaces

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Abstract

Recently, the complex-valued metric spaces which are more general than the metric spaces were first introduced by Azam *et al.* (Numer. Funct. Anal. Optim. 32:243-253, 2011). They also established the existence of fixed point theorems under the contraction condition in these spaces. The aim of this paper is to introduce the concepts of a C-Cauchy sequence and C-complete in complex-valued metric spaces and establish the existence of common fixed point theorems in C-complete complex-valued metric spaces. Furthermore, we apply our result to obtain the existence theorem for a common solution of the Urysohn integral equations

$$x(t) = \int_a^b K_1(t,s,x(s))\,ds + g(t),$$

$$x(t) = \int_a^b K_2(t,s,x(s))\,ds + h(t),$$

where $t \in [a,b] \subseteq \mathbb{R}$, $x, g, h \in C([a,b],\mathbb{R}^n)$ and $K_1, K_2 : [a,b] \times [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$.

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1 Introduction

The study of metric spaces has played a vital role in many branches of pure and applied sciences. We can find useful applications of metric spaces in mathematics, biology, medicine, physics and computer science (see [1–3]). Several mathematicians improved, generalized and extended the concept of metric spaces to vector-valued metric spaces of Perov [4], G-metric spaces of Mustafa and Sims [5], cone metric spaces of Huang and Zhang [6], modular metric spaces of Chistyakov [7], partial metric spaces of Matthews [8] and others. Since Banach [9] introduced his contraction principle in complete metric spaces in 1922, this field of fixed point theory has been rapidly growing. It has been very useful in many fields such as optimization problems, control theory, differential equations, economics and many others. A number of papers in this field have been dedicated to the improvement and generalization of Banach's contraction mapping principle in many spaces and ways (see [10–13]).

Recently, Azam *et al.* [14] introduced a new space, the so-called complex-valued metric space, and established a fixed point theorem for some type of contraction mappings as follows.

Theorem 1.1 (Azam *et al.* [14]) *Let (X,d) be a complete complex-valued metric space and $S,T:X\to X$ be two mappings. If S and T satisfy*

$$d(Sx,Ty)\precsim \lambda d(x,y)+\frac{\mu d(x,Sx)d(y,Ty)}{1+d(x,y)} \tag{1.1}$$

for all $x,y\in X$, where λ, μ are nonnegative reals with $\lambda+\mu<1$, then S and T have a unique common fixed point in X.

Theorem 1.1 of Azam *et al.* in [14] is an essential tool in the complex-valued metric space to claim the existence of a common fixed point for some mappings. However, it is most interesting to find another new auxiliary tool to claim the existence of a common fixed point. Some other works related to the results in a complex-valued metric space are [15, 16].

In this paper, we introduce the concept of a C-Cauchy sequence and C-complete in complex-valued metric spaces and also prove some common fixed point theorems for new generalized contraction mappings in C-complete complex-valued metric spaces.

On the other hand, integral equations arise naturally from many applications in describing numerous real world problems. These equations have been studied by many authors. Existence theorems for the Urysohn integral equations can be obtained applying various fixed point principles.

As applications, we show the existence of a common solution of the following system of Urysohn integral equations by using our common fixed point results:

$$x(t)=\int_a^b K_1\bigl(t,s,x(s)\bigr)\,ds+g(t), \tag{1.2}$$

$$x(t)=\int_a^b K_2\bigl(t,s,x(s)\bigr)ds+h(t), \tag{1.3}$$

where $t\in[a,b]\subseteq\mathbb{R}$, $x,g,h\in C([a,b],\mathbb{R}^n)$ and $K_1,K_2:[a,b]\times[a,b]\times\mathbb{R}^n\to\mathbb{R}^n$.

2 Preliminaries

In this section, we discuss some background of the complex-valued metric spaces of Azam *et al.* in [14] and give some notions for our results. Also, some essential lemmas which are useful for our results are given.

Let $\mathbb{C}$ be the set of complex numbers. For $z_1,z_2\in\mathbb{C}$, we will define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1\precsim z_2 \quad\Longleftrightarrow\quad \operatorname{Re}(z_1)\le\operatorname{Re}(z_2) \quad\text{and}\quad \operatorname{Im}(z_1)\le\operatorname{Im}(z_2).$$

We note that $z_1\precsim z_2$ if one of the following holds:

(C1) $\operatorname{Re}(z_1)=\operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1)=\operatorname{Im}(z_2)$;

(C2) $\operatorname{Re}(z_1)<\operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1)=\operatorname{Im}(z_2)$;

(C3) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$;
(C4) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$.

It obvious that if $a, b \in \mathbb{R}$ such that $a \leq b$, then $az \precsim bz$ for all $z \in \mathbb{C}$.

In particular, we write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (C2), (C3) and (C4) is satisfied, and we write $z_1 \prec z_2$ if only (C4) is satisfied. The following are well known:

$$0 \precsim z_1 \precnsim z_2 \quad \Longrightarrow \quad |z_1| < |z_2|,$$

$$z_1 \precsim z_2, \qquad z_2 \prec z_3 \quad \Longrightarrow \quad z_1 \prec z_3.$$

Definition 2.1 [14] Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$ satisfies the following conditions:
(d1) $0 \precsim d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d3) $d(x, y) \precsim d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a *complex-valued metric* on X and (X, d) is called a *complex-valued metric space*.

Example 2.2 Let $X = \mathbb{C}$. Define the mapping $d : X \times X \to \mathbb{C}$ by

$$d(z_1, z_2) = |x_1 - x_2| + |y_1 - y_2|i,$$

where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then (X, d) is a complex-valued metric space.

Example 2.3 Let $X = X_1 \cup X_2$, where

$$X_1 = \{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0, \mathrm{Im}(z) = 0\}$$

and

$$X_2 = \{z \in \mathbb{C} : \mathrm{Re}(z) = 0, \mathrm{Im}(z) \geq 0\}.$$

Define the mapping $d : X \times X \to \mathbb{C}$ by

$$d(z_1, z_2) = \begin{cases} \frac{4}{3}|x_1 - x_2| + i|x_1 - x_2|, & z_1, z_2 \in X_1, \\ |y_1 - y_2| + \frac{2i}{3}|y_1 - y_2|, & z_1, z_2 \in X_2, \\ \frac{4}{3}x_1 + y_2 + i(x_1 + \frac{2}{3}y_2), & z_1 \in X_1, z_2 \in X_2, \\ \frac{4}{3}x_2 + y_1 + i(x_2 + \frac{2}{3}y_1), & z_1 \in X_2, z_2 \in X_1, \end{cases}$$

where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then (X, d) is a complex-valued metric space.

Example 2.4 Let $X = X_1 \cup X_2$, where

$$X_1 = \{z \in \mathbb{C} : 0 \leq \mathrm{Re}(z) \leq 1, \mathrm{Im}(z) = 0\}$$

and

$$X_2 = \{z \in \mathbb{C} : \mathrm{Re}(z) = 0, 0 \leq \mathrm{Im}(z) \leq 1\}.$$

Define the mapping $d : X \times X \to \mathbb{C}$ by

$$d(z_1, z_2) = \begin{cases} \frac{2}{3}|x_1 - x_2| + \frac{i}{2}|x_1 - x_2|, & z_1, z_2 \in X_1, \\ \frac{1}{2}|y_1 - y_2| + \frac{i}{3}|y_1 - y_2|, & z_1, z_2 \in X_2, \\ \frac{2}{3}x_1 + \frac{1}{2}y_2 + i(\frac{1}{2}x_1 + \frac{1}{3}y_2), & z_1 \in X_1, z_2 \in X_2, \\ \frac{1}{2}y_1 + \frac{2}{3}y_2 + i(\frac{1}{3}y_1 + \frac{1}{2}x_2), & z_1 \in X_2, z_2 \in X_1, \end{cases}$$

where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then (X, d) is a complex-valued metric space.

Definition 2.5 [14] Let (X, d) be a complex-valued metric space.

(1) A point $x \in X$ is called an *interior point* of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that

$$B(x, r) = \{y \in X : d(x, y) \prec r\} \subseteq A.$$

(2) A point $x \in X$ is called a *limit point* of A whenever, for all $0 \prec r \in \mathbb{C}$,

$$B(x, r) \cap (A - X) \neq \emptyset.$$

(3) A set $A \subseteq X$ is called *open* whenever each element of A is an interior point of A.
(4) A set $A \subseteq X$ is called *closed* whenever each limit point of A belongs to A.
(5) A *sub-basis* for a Hausdorff topology τ on X is the family

$$F = \{B(x, r) : x \in X \text{ and } 0 \prec r\}.$$

Definition 2.6 [14] Let (X, d) be a complex-valued metric space, $\{x_n\}$ be a sequence in X and let $x \in X$.

(1) If, for any $c \in \mathbb{C}$ with $0 \prec c$, there exists $N \in \mathbb{N}$ such that, for all $n > N$, $d(x_n, x) \prec c$, then $\{x_n\}$ is said to be *convergent* to a point $x \in X$ or $\{x_n\}$ converges to a point $x \in X$ and x is the limit point of $\{x_n\}$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.
(2) If, for any $c \in \mathbb{C}$ with $0 \prec c$, there exists $N \in \mathbb{N}$ such that, for all $n > N$, $d(x_n, x_{n+m}) \prec c$, where $m \in \mathbb{N}$, then $\{x_n\}$ is called a *Cauchy sequence* in X.
(3) If every Cauchy sequence in X is convergent, then (X, d) is said to be a *complete complex-valued metric space*.

Next, we give some lemmas which are an essential tool in the proof of main results.

Lemma 2.7 [14, see Definition 2.5] *Let (X, d) be a complex-valued metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to a point $x \in X$ if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 2.8 [14] *Let (X, d) be a complex-valued metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.*

Definition 2.9 Let S and T be self-mappings of a nonempty set X.

(1) A point $x \in X$ is called a *fixed point* of T if $Tx = x$.

(2) A point $x \in X$ is called a *coincidence point* of S and T if $Sx = Tx$ and the point $w \in X$ such that $w = Sx = Tx$ is called a *point of coincidence* of S and T.
(3) A point $x \in X$ is called a *common fixed point* of S and T if $x = Sx = Tx$.

Lemma 2.10 [17] *Let X be a nonempty set and $T : X \to X$ be a function. Then there exists a subset $E \subseteq X$ such that $T(E) = T(X)$ and $T : E \to X$ is one-to-one.*

3 Common fixed points (I)

Throughout this paper, $\mathbb{R}$ denotes a set of real numbers, $\mathbb{C}_+$ denotes a set $\{c \in \mathbb{C} : 0 \precsim c\}$ and Γ denotes the class of all functions $\gamma : \mathbb{C}_+ \to [0,1)$ which satisfies the condition: for any sequences $\{x_n\}$ in $\mathbb{C}_+$,

$$\gamma(x_n) \to 1 \quad \Longrightarrow \quad |x_n| \to 0.$$

The following are examples of the function in Γ:
(1) $\gamma_1(x) = k$, where $k \in [0,1)$;
(2) $\gamma_2(x) = \frac{1}{1+k|x|}$, where $k \in (0,\infty)$.

Now, we introduce the concepts of a C-Cauchy sequence and C-complete in complex-valued metric spaces.

Definition 3.1 Let (X,d) be a complex-valued metric space and $\{x_n\}$ be a sequence in X.
(1) If, for any $c \in \mathbb{C}$ with $0 \prec c$, there exists $N \in \mathbb{N}$ such that, for all $m,n > N$, $d(x_n,x_m) \prec c$, then $\{x_n\}$ is called a *C-Cauchy sequence* in X.
(2) If every C-Cauchy sequence in X is convergent, then (X,d) is said to be a *C-complete complex-valued metric space.*

Next, we prove our main results.

Theorem 3.2 *Let (X,d) be a C-complete complex-valued metric space and $S,T : X \to X$ be mappings. If there exist two mappings $\alpha,\beta : \mathbb{C}_+ \to [0,1)$ such that, for all $x,y \in X$,*
(a) $\alpha(x) + \beta(x) < 1$;
(b) *the mapping $\gamma : \mathbb{C}_+ \to [0,1)$ defined by $\gamma(x) := \frac{\alpha(x)}{1-\beta(x)}$ belongs to Γ;*
(c) $d(Sx,Ty) \precsim \alpha(d(x,y))d(x,y) + \frac{\beta(d(x,y))d(x,Sx)d(y,Ty)}{1+d(x,y)}$.
Then S and T have a unique common fixed point in X.

Proof Let x_0 be an arbitrary point in X. We construct the sequence $\{x_n\}$ in X such that

$$x_{2n+1} = Sx_{2n}, \qquad x_{2n+2} = Tx_{2n+1} \tag{3.1}$$

for all $n \geq 0$. For all $n \geq 0$, we get

$$\begin{aligned} &d(x_{2n+1},x_{2n+2}) \\ &\quad = d(Sx_{2n}, Tx_{2n+1}) \\ &\quad \precsim \alpha\bigl(d(x_{2n},x_{2n+1})\bigr)d(x_{2n},x_{2n+1}) \\ &\qquad + \frac{\beta(d(x_{2n},x_{2n+1}))d(x_{2n},Sx_{2n})d(x_{2n+1},Tx_{2n+1})}{1+d(x_{2n},x_{2n+1})} \end{aligned}$$

$$\begin{aligned}
&= \alpha\big(d(x_{2n},x_{2n+1})\big)d(x_{2n},x_{2n+1}) \\
&\quad + \frac{\beta(d(x_{2n},x_{2n+1}))d(x_{2n},x_{2n+1})d(x_{2n+1},x_{2n+2})}{1+d(x_{2n},x_{2n+1})} \\
&= \alpha\big(d(x_{2n},x_{2n+1})\big)d(x_{2n},x_{2n+1}) \\
&\quad + \beta\big(d(x_{2n},x_{2n+1})\big)d(x_{2n+1},x_{2n+2})\left(\frac{d(x_{2n},x_{2n+1})}{1+d(x_{2n},x_{2n+1})}\right) \\
&\precsim \alpha\big(d(x_{2n},x_{2n+1})\big)d(x_{2n},x_{2n+1}) + \beta\big(d(x_{2n},x_{2n+1})\big)d(x_{2n+1},x_{2n+2}),
\end{aligned}$$

which implies that

$$\begin{aligned}
d(x_{2n+1},x_{2n+2}) &\precsim \left(\frac{\alpha(d(x_{2n},x_{2n+1}))}{1-\beta(d(x_{2n},x_{2n+1}))}\right)d(x_{2n},x_{2n+1}) \\
&= \gamma\big(d(x_{2n},x_{2n+1})\big)d(x_{2n},x_{2n+1}).
\end{aligned} \tag{3.2}$$

Similarly, for all $n \geq 0$, we get

$$\begin{aligned}
&d(x_{2n+2},x_{2n+3}) \\
&\quad = d(x_{2n+3},x_{2n+2}) \\
&\quad = d(Sx_{2n+2},Tx_{2n+1}) \\
&\quad \precsim \alpha\big(d(x_{2n+2},x_{2n+1})\big)d(x_{2n+2},x_{2n+1}) \\
&\qquad + \frac{\beta(d(x_{2n+2},x_{2n+1}))d(x_{2n+2},Sx_{2n+2})d(x_{2n+1},Tx_{2n+1})}{1+d(x_{2n+2},x_{2n+1})} \\
&\quad = \alpha\big(d(x_{2n+2},x_{2n+1})\big)d(x_{2n+2},x_{2n+1}) \\
&\qquad + \frac{\beta(d(x_{2n+2},x_{2n+1}))d(x_{2n+2},x_{2n+3})d(x_{2n+1},x_{2n+2})}{1+d(x_{2n+1},x_{2n+2})} \\
&\quad = \alpha\big(d(x_{2n+2},x_{2n+1})\big)d(x_{2n+2},x_{2n+1}) \\
&\qquad + \beta\big(d(x_{2n+2},x_{2n+1})\big)d(x_{2n+2},x_{2n+3})\left(\frac{d(x_{2n+1},x_{2n+2})}{1+d(x_{2n+1},x_{2n+2})}\right) \\
&\quad \precsim \alpha\big(d(x_{2n+2},x_{2n+1})\big)d(x_{2n+2},x_{2n+1}) + \beta\big(d(x_{2n+2},x_{2n+1})\big)d(x_{2n+2},x_{2n+3}),
\end{aligned}$$

which implies that

$$\begin{aligned}
d(x_{2n+2},x_{2n+3}) &\precsim \left(\frac{\alpha(d(x_{2n+2},x_{2n+1}))}{1-\beta(d(x_{2n+2},x_{2n+1}))}\right)d(x_{2n+2},x_{2n+1}) \\
&= \gamma\big(d(x_{2n+2},x_{2n+1})\big)d(x_{2n+2},x_{2n+1}).
\end{aligned} \tag{3.3}$$

From (3.2) and (3.3), we have

$$d(x_n,x_{n+1}) \precsim \gamma\big(d(x_{n-1},x_n)\big)d(x_{n-1},x_n)$$

for all $n \in \mathbb{N}$. Therefore, we get

$$\big|d(x_n,x_{n+1})\big| \leq \gamma\big(d(x_{n-1},x_n)\big)\big|d(x_{n-1},x_n)\big| \leq \big|d(x_{n-1},x_n)\big| \tag{3.4}$$

for all $n \in \mathbb{N}$. This implies the sequence $\{|d(x_{n-1}, x_n)|\}_{n\in\mathbb{N}}$ is monotone non-increasing and bounded below. Therefore, $|d(x_{n-1}, x_n)| \to d$ for some $d \geq 0$.

Next, we claim that $d = 0$. Assume to the contrary that $d > 0$. In (3.4), taking $n \to \infty$, we have

$$\gamma\big(d(x_{n-1}, x_n)\big) \to 1.$$

Since $\gamma \in \Gamma$, we get $|d(x_{n-1}, x_n)| \to 0$, which is a contradiction. Therefore, we have $d = 0$, that is,

$$\big|d(x_{n-1}, x_n)\big| \to 0. \tag{3.5}$$

Next, we prove that $\{x_n\}$ is a C-Cauchy sequence. According to (3.5), it is sufficient to show that the subsequence $\{x_{2n}\}$ is a C-Cauchy sequence. On the contrary, assume that $\{x_{2n}\}$ is not a C-Cauchy sequence. By Definition 3.1(1), there is $c \in \mathbb{C}$ with $0 \prec c$ for which, for all $k \in \mathbb{N}$, there exists $m_k > n_k \geq k$ such that

$$d(x_{2n_k}, x_{2m_k}) \succsim c. \tag{3.6}$$

Further, corresponding to n_k, we can choose m_k in such a way that it is the smallest integer with $m_k > n_k \geq k$ satisfying (3.6). Then we have

$$d(x_{2n_k}, x_{2m_k}) \succsim c \tag{3.7}$$

and

$$d(x_{2n_k}, x_{2m_k-2}) \prec c. \tag{3.8}$$

By (3.7), (3.8) and the notion of a complex-valued metric, we have

$$\begin{aligned} c &\precsim d(x_{2n_k}, x_{2m_k}) \\ &\precsim d(x_{2n_k}, x_{2m_k-2}) + d(x_{2m_k-2}, x_{2m_k-1}) + d(x_{2m_k-1}, x_{2m_k}) \\ &\prec c + d(x_{2m_k-2}, x_{2m_k-1}) + d(x_{2m_k-1}, x_{2m_k}). \end{aligned}$$

This implies

$$|c| \leq \big|d(x_{2n_k}, x_{2m_k})\big| \leq |c| + \big|d(x_{2m_k-2}, x_{2m_k-1})\big| + \big|d(x_{2m_k-1}, x_{2m_k})\big|.$$

On taking limit as $k \to \infty$, we have

$$\big|d(x_{2n_k}, x_{2m_k})\big| \to |c|. \tag{3.9}$$

Further, we have

$$\begin{aligned} d(x_{2n_k}, x_{2m_k}) &\precsim d(x_{2n_k}, x_{2m_k+1}) + d(x_{2m_k+1}, x_{2m_k}) \\ &\precsim d(x_{2n_k}, x_{2m_k}) + d(x_{2m_k}, x_{2m_k+1}) + d(x_{2m_k+1}, x_{2m_k}), \end{aligned}$$

and then

$$\begin{aligned}\left|d(x_{2n_k},x_{2m_k})\right| &\le \left|d(x_{2n_k},x_{2m_k+1})\right| + \left|d(x_{2m_k+1},x_{2m_k})\right| \\ &\le \left|d(x_{2n_k},x_{2m_k})\right| + \left|d(x_{2m_k},x_{2m_k+1})\right| + \left|d(x_{2m_k+1},x_{2m_k})\right|.\end{aligned}$$

Passing to the limit when $k \to \infty$ and using (3.5) and (3.9), we get

$$\left|d(x_{2n_k},x_{2m_k+1})\right| \to |c|. \tag{3.10}$$

Now, from the triangle inequality for a complex-valued metric d, we obtain that

$$\begin{aligned}d(x_{2n_k},x_{2m_k+1}) &\precsim d(x_{2n_k},x_{2n_k+1}) + d(x_{2n_k+1},x_{2m_k+2}) + d(x_{2m_k+2},x_{2m_k+1}) \\ &= d(x_{2n_k},x_{2n_k+1}) + d(Sx_{2n_k},Tx_{2m_k+1}) + d(x_{2m_k},x_{2m_k+1}) \\ &\precsim d(x_{2n_k},x_{2n_k+1}) + \alpha\bigl(d(x_{2n_k},x_{2m_k+1})\bigr)d(x_{2n_k},x_{2m_k+1}) \\ &\quad + \frac{\beta(d(x_{2n_k},x_{2m_k+1}))d(x_{2n_k},Sx_{2n_k})d(x_{2m_k+1},Tx_{2m_k+1})}{1+d(x_{2n_k},x_{2m_k+1})} + d(x_{2m_k},x_{2m_k+1}) \\ &= d(x_{2n_k},x_{2n_k+1}) + \alpha\bigl(d(x_{2n_k},x_{2m_k+1})\bigr)d(x_{2n_k},x_{2m_k+1}) \\ &\quad + \frac{\beta(d(x_{2n_k},x_{2m_k+1}))d(x_{2n_k},x_{2n_k+1})d(x_{2m_k+1},x_{2m_k+2})}{1+d(x_{2n_k},x_{2m_k+1})} + d(x_{2m_k},x_{2m_k+1}),\end{aligned}$$

which implies that

$$\begin{aligned}&\left|d(x_{2n_k},x_{2m_k+1})\right| \\ &\quad\le \left|d(x_{2n_k},x_{2n_k+1})\right| + \alpha\bigl(d(x_{2n_k},x_{2m_k+1})\bigr)\left|d(x_{2n_k},x_{2m_k+1})\right| \\ &\qquad + \beta\bigl(d(x_{2n_k},x_{2m_k+1})\bigr)\left|\frac{d(x_{2n_k},x_{2n_k+1})d(x_{2m_k+1},x_{2m_k+2})}{1+d(x_{2n_k},x_{2m_k+1})}\right| + \left|d(x_{2m_k},x_{2m_k+1})\right| \\ &\quad\le \left|d(x_{2n_k},x_{2n_k+1})\right| + \alpha\bigl(d(x_{2n_k},x_{2m_k+1})\bigr)\left|d(x_{2n_k},x_{2m_k+1})\right| \\ &\qquad + \left|\frac{d(x_{2n_k},x_{2n_k+1})d(x_{2m_k+1},x_{2m_k+2})}{1+d(x_{2n_k},x_{2m_k+1})}\right| + \left|d(x_{2m_k},x_{2m_k+1})\right| \\ &\quad\le \left|d(x_{2n_k},x_{2n_k+1})\right| + \frac{\alpha(d(x_{2n_k},x_{2m_k+1}))}{1-\beta(d(x_{2n_k},x_{2m_k+1}))}\left|d(x_{2n_k},x_{2m_k+1})\right| \\ &\qquad + \left|\frac{d(x_{2n_k},x_{2n_k+1})d(x_{2m_k+1},x_{2m_k+2})}{1+d(x_{2n_k},x_{2m_k+1})}\right| + \left|d(x_{2m_k},x_{2m_k+1})\right| \\ &\quad\le \left|d(x_{2n_k},x_{2n_k+1})\right| + \gamma\bigl(d(x_{2n_k},x_{2m_k+1})\bigr)\left|d(x_{2n_k},x_{2m_k+1})\right| \\ &\qquad + \left|\frac{d(x_{2n_k},x_{2n_k+1})d(x_{2m_k+1},x_{2m_k+2})}{1+d(x_{2n_k},x_{2m_k+1})}\right| + \left|d(x_{2m_k},x_{2m_k+1})\right| \\ &\quad\le \left|d(x_{2n_k},x_{2n_k+1})\right| + \left|d(x_{2n_k},x_{2m_k+1})\right| \\ &\qquad + \left|\frac{d(x_{2n_k},x_{2n_k+1})d(x_{2m_k+1},x_{2m_k+2})}{1+d(x_{2n_k},x_{2m_k+1})}\right| + \left|d(x_{2m_k},x_{2m_k+1})\right|.\end{aligned}$$

Taking $k \to \infty$, we have

$$|c| \le \Bigl(\lim_{k\to\infty} \gamma\bigl(d(x_{2n_k},x_{2m_k+1})\bigr)\Bigr)|c| \le |c|,$$

that is,

$$\lim_{k\to\infty} \gamma\bigl(d(x_{2n_k}, x_{2m_k+1})\bigr) = 1.$$

Since $\gamma \in \Gamma$, we get $|d(x_{2n_k}, x_{2m_k+1})| \to 0$, which contradicts $0 \prec c$. Therefore, we can conclude that $\{x_{2n}\}$ is a C-Cauchy sequence and hence $\{x_n\}$ is a C-Cauchy sequence. By the completeness of X, there exists a point $z \in X$ such that $x_n \to z$ as $n \to \infty$.

Next, we claim that $Sz = z$. If $Sz \neq z$, then $d(z, Sz) > 0$. By the notion of a complex-valued metric d, we have

$$\begin{aligned}
&d(z, Sz)\\
&\quad\precsim d(z, x_{2n+2}) + d(x_{2n+2}, Sz)\\
&\quad= d(z, x_{2n+2}) + d(Tx_{2n+1}, Sz)\\
&\quad= d(z, x_{2n+2}) + d(Sz, Tx_{2n+1})\\
&\quad\precsim d(x_{2n+2}, z) + \alpha\bigl(d(z, x_{2n+1})\bigr)d(z, x_{2n+1}) + \frac{\beta(d(z, x_{2n+1}))d(z, Sz)d(x_{2n+1}, Tx_{2n+1})}{1 + d(z, x_{2n+1})}\\
&\quad= d(x_{2n+2}, z) + \alpha\bigl(d(z, x_{2n+1})\bigr)d(z, x_{2n+1}) + \frac{\beta(d(z, x_{2n+1}))d(z, Sz)d(x_{2n+1}, x_{2n+2})}{1 + d(z, x_{2n+1})}\\
&\quad\precsim d(x_{2n+2}, z) + d(z, x_{2n+1}) + \frac{d(z, Sz)d(x_{2n+1}, x_{2n+2})}{1 + d(z, x_{2n+1})},
\end{aligned} \tag{3.11}$$

which implies that

$$\bigl|d(z, Sz)\bigr| \le \bigl|d(x_{2n+2}, z)\bigr| + \bigl|d(z, x_{2n+1})\bigr| + \left|\frac{d(z, Sz)d(x_{2n+1}, x_{2n+2})}{1 + d(z, x_{2n+1})}\right|.$$

Taking $n \to \infty$, we have $|d(z, Sz)| = 0$, which is a contradiction. Thus, we get $Sz = z$. It follows similarly that $Tz = z$. Therefore, $z = Sz = Tz$, that is, z is a common fixed point of S and T.

Finally, we show that z is a unique common fixed point of S and T. Assume that there exists another point $\widehat{z}$ such that $\widehat{z} = S\widehat{z} = T\widehat{z}$. Now, we have

$$\begin{aligned}
d(z, \widehat{z}) &= d(Sz, T\widehat{z})\\
&\precsim \alpha\bigl(d(z, \widehat{z})\bigr)d(z, \widehat{z}) + \frac{\beta(d(z, \widehat{z}))d(z, Sz)d(\widehat{z}, T\widehat{z})}{1 + d(z, \widehat{z})}\\
&= \alpha\bigl(d(z, \widehat{z})\bigr)d(z, \widehat{z}).
\end{aligned}$$

Hence $|d(z, \widehat{z})| \le \alpha(d(z, \widehat{z}))|d(z, \widehat{z})|$. Since $0 \le \alpha(d(z, \widehat{z})) < 1$, we get $|d(z, \widehat{z})| = 0$ and then $z = \widehat{z}$. Therefore, z is a unique common fixed point of S and T. This completes the proof. □

Corollary 3.3 *Let (X, d) be a C-complete complex-valued metric space and $S, T : X \to X$ be mappings. If S and T satisfy*

$$d(Sx, Ty) \precsim \lambda d(x, y) + \frac{\mu d(x, Sx)d(y, Ty)}{1 + d(x, y)} \tag{3.12}$$

for all $x, y \in X$, where λ, μ are nonnegative reals with $\lambda + \mu < 1$, then S and T have a unique common fixed point in X.

Proof We can prove this result by applying Theorem 3.2 by setting $\alpha(x) = \lambda$ and $\beta(x) = \mu$. □

Corollary 3.4 *Let (X, d) be a C-complete complex-valued metric space and $T : X \to X$ be a mapping. If there exist two mappings $\alpha, \beta : \mathbb{C}_+ \to [0,1)$ such that, for all $x, y \in X$,*

(a) $\alpha(x) + \beta(x) < 1$;
(b) *the mapping $\gamma : \mathbb{C}_+ \to [0,1)$ defined by $\gamma(x) := \frac{\alpha(x)}{1-\beta(x)}$ belongs to Γ;*
(c) $d(Tx, Ty) \precsim \alpha(d(x,y))d(x,y) + \frac{\beta(d(x,y))d(x,Tx)d(y,Ty)}{1+d(x,y)}$.

Then T has a unique fixed point in X.

Proof We can prove this result by applying Theorem 3.2 with $S = T$. □

Corollary 3.5 *Let (X, d) be a C-complete complex-valued metric space and $T : X \to X$ be a mapping. If T satisfies*

$$d(Tx, Ty) \precsim \lambda d(x,y) + \frac{\mu d(x,Tx)d(y,Ty)}{1 + d(x,y)} \tag{3.13}$$

for all $x, y \in X$, where λ, μ are nonnegative reals with $\lambda + \mu < 1$, then T has a unique fixed point in X.

Proof We can prove this result by applying Corollary 3.4 with $\alpha(x) = \lambda$ and $\beta(x) = \mu$. □

Theorem 3.6 *Let (X, d) be a C-complete complex-valued metric space and $T : X \to X$. If there exist two mappings $\alpha, \beta : \mathbb{C}_+ \to [0,1)$ such that, for all $x, y \in X$,*

(a) $\alpha(x) + \beta(x) < 1$;
(b) *the mapping $\gamma : \mathbb{C}_+ \to [0,1)$ defined by $\gamma(x) := \frac{\alpha(x)}{1-\beta(x)}$ belongs to Γ;*
(c) $d(T^n x, T^n y) \precsim \alpha(d(x,y))d(x,y) + \frac{\beta(d(x,y))d(x,T^n x)d(y,T^n y)}{1+d(x,y)}$ *for some $n \in \mathbb{N}$.*

Then T has a unique fixed point in X.

Proof From Corollary 3.4, we get T^n has a unique fixed point z. Since

$$T^n(Tz) = T(T^n z) = Tz,$$

we know that Tz is a fixed point of T^n. Therefore, $Tz = z$ by the uniqueness of a fixed point of T^n. Therefore, z is also a fixed point of T. Since the fixed point of T is also a fixed point of T^n, the fixed point of T is also unique. □

Corollary 3.7 *Let (X, d) be a C-complete complex-valued metric space and $S, T : X \to X$ be mappings. If T satisfy*

$$d(T^n x, T^n y) \precsim \lambda d(x,y) + \frac{\mu d(x,T^n x)d(y,T^n y)}{1 + d(x,y)} \tag{3.14}$$

for all $x, y \in X$ for some $n \in \mathbb{N}$, where λ, μ are nonnegative reals with $\lambda + \mu < 1$, then T has a unique fixed point in X.

Proof We can prove this result by applying Theorem 3.6 with $\alpha(x) = \lambda$ and $\beta(x) = \mu$. □

Remark 3.8 It is easy to see that Corollaries 3.3, 3.5 and 3.7 hold in complete complex-valued metric spaces. Therefore, Corollaries 3.3, 3.5 and 3.7 become Theorem 4, Corollary 5 and Corollary 6 of Azam *et al.* [14] in complete complex-valued metric spaces.

4 Common fixed points (II)

In this section, we prove a common fixed point theorem for weakly compatible mappings in C-complete complex-valued metric spaces.

Since Banach's fixed point theorem, many authors have improved, extended and generalized Banach's fixed point theorem in several ways. Especially, in [18], Jungck generalized Banach's fixed point theorem by using the concept of commuting mappings as follows.

Theorem J *Let* (X, d) *be a complete metric space. Then a continuous mapping* $S : X \to X$ *has a fixed point in* X *if and only if there exists a number* $\alpha \in (0, 1)$ *and a mapping* $T : X \to X$ *such that* $T(X) \subset S(X)$, S *and* T *are commuting* (*i.e.*, $TSx = STx$ *for all* x *in* X),

$$d(Tx, Ty) \leq \alpha d(Sx, Sy)$$

for all $x, y \in X$. *Further,* S *and* T *have a unique common fixed point in* X (*i.e.*, *there exists a unique point* z *in* X *such that* $Sz = Tz = z$).

Note that if we put $S = I_X$ (the identity mapping on X) in Theorem J, we have Banach's fixed point theorem.

Since Theorem J, in 1986, Jungck [18] introduced more generalized commuting mappings in metric spaces, called compatible mappings, which also are more general than weakly commuting mappings (that is, the mappings $S, T : X \to X$ are said to be *weakly commuting* if $d(STx, TSx) \leq d(Sx, Tx)$ for all $x \in X$) introduced by Sessa [19] as follows.

Definition 4.1 Let S and T be mappings from a metric space (X, d) into itself. The mappings S and T are said to be *compatible* if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

In general, commuting mappings are weakly commuting and weakly commuting mappings are compatible, but the converse is not necessarily true; some examples can be found in [18, 20–22].

Also, some authors introduced some kind of generalizations of compatible mappings in metric spaces and other spaces (see [21–24]) and they proved common fixed point theorems using these kinds of compatible mappings in metric spaces and other spaces.

In [25], Jungck and Rhoades introduced the concept of weakly compatible mappings in symmetric spaces (X, d) and proved some common fixed point theorems for these mappings in symmetric spaces as follows.

Definition 4.2 Let S and T be mappings from a metric space (X,d) into itself. The mappings S and T are said to be *weakly compatible* if they commute at coincidence points of S and T.

In Djoudi and Nisse [26], we can find an example to show that there exist weakly compatible mappings which are not compatible mappings in metric spaces.

Now, we give the main result in this section.

Theorem 4.3 *Let (X,d) be a complex-valued metric space and $S,T:X\to X$ be such that $T(X)\subseteq S(X)$ and $S(X)$ is C-complete. If there exist two mappings $\alpha,\beta:\mathbb{C}_+\to[0,1)$ such that, for all $x,y\in X$,*

(a) $\alpha(x)+\beta(x)<1$;

(b) *the mapping $\gamma:\mathbb{C}_+\to[0,1)$ defined by $\gamma(x):=\frac{\alpha(x)}{1-\beta(x)}$ belongs to Γ;*

(c) $d(Tx,Ty)\precsim\alpha(d(Sx,Sy))d(Sx,Sy)+\frac{\beta(d(Sx,Sy))d(Sx,Tx)d(Sy,Ty)}{1+d(Sx,Sy)}$.

Then S and T have a unique point of coincidence in X. Moreover, S and T have a unique common fixed point in X if S and T are weakly compatible.

Proof Consider the mapping $S:X\to X$. By Lemma 2.10, there exists $E\subseteq X$ such that $S(E)=S(X)$ and $S:E\to X$ is one-to-one.

Next, we define a mapping $\mathcal{W}:S(E)\to S(E)$ by $\mathcal{W}(Sx)=Tx$ for all $Sx\in S(E)$. Therefore, $\mathcal{W}$ is well defined since S is one-to-one on E. Since $\mathcal{W}\circ S=T$, using (c), we get

$$d\big(\mathcal{W}(Sx),\mathcal{W}(Sy)\big)\precsim\alpha\big(d(Sx,Sy)\big)d(Sx,Sy)+\frac{\beta(d(Sx,Sy))d(Sx,\mathcal{W}(Sx))d(Sy,\mathcal{W}(Sy))}{1+d(Sx,Sy)} \tag{4.1}$$

for all $Sx,Sy\in S(E)$. Since $S(E)=S(X)$ is C-complete and (4.1) holds, we can apply Corollary 3.4 with a mapping $\mathcal{W}$. Therefore, there exists a unique fixed point $z\in S(X)$ such that $\mathcal{W}z=z$. It follows from $z\in S(X)$ that $z=Sz'$ for some $z'\in X$. So, $\mathcal{W}(Sz')=Sz'$, that is, $Tz'=Sz'$. Therefore, T and S have a unique point of coincidence.

Next, we show that S and T have a common fixed point. Now, we have $z=Tz'=Sz'$. Since S and T are weakly compatible, we get

$$Sz=STz'=TSz'=Tz.$$

This implies $Sz=Tz$ is a point of coincidence of S and T. But z is a unique point of coincidence of S and T. Therefore, we conclude that $z=Sz=Tz$, which implies that z is a common fixed point of S and T.

Finally, we prove the uniqueness of a common fixed point of S and T. Assume that $\overline{z}$ is another common fixed point of S and T. So, $\overline{z}=S\overline{z}=T\overline{z}$, and then $\overline{z}$ is also a point of coincidence of S and T. However, we know that z is a unique point of coincidence of S and T. Therefore, we get $\overline{z}=z$, that is, z is a unique common fixed point of S and T. This completes the proof. □

5 Urysohn integral equations

In this section, we show that Theorem 3.2 can be applied to the existence of a common solution of the system of the Urysohn integral equations.

Theorem 5.1 *Let $X = C([a,b],\mathbb{R}^n)$, $a > 0$, and $d : X \times X \to \mathbb{C}$ be defined by*

$$d(x,y) = \max_{t\in[a,b]} \big\|x(t) - y(t)\big\|_\infty \sqrt{1+a^2} e^{i\tan^{-1} a}.$$

Consider the Urysohn integral equations

$$x(t) = \int_a^b K_1\big(t,s,x(s)\big)\,ds + g(t), \tag{5.1}$$

$$x(t) = \int_a^b K_2\big(t,s,x(s)\big)\,ds + h(t), \tag{5.2}$$

where $t \in [a,b] \subseteq \mathbb{R}$, $x,g,h \in X$ and $K_1, K_2 : [a,b] \times [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$.

Suppose that K_1, K_2 are such that $F_x, G_x \in X$ for all $x \in X$, where

$$F_x(t) = \int_a^b K_1\big(t,s,x(s)\big)\,ds,$$

$$G_x(t) = \int_a^b K_2\big(t,s,x(s)\big)\,ds$$

for all $t \in [a,b]$.

If there exist two mappings $\alpha, \beta : \mathbb{C}_+ \to [0,1)$ such that for all $x,y \in X$ the following hold:

(a) $\alpha(x) + \beta(x) < 1$;

(b) *the mapping $\gamma : \mathbb{C}_+ \to [0,1)$ defined by $\gamma(x) := \frac{\alpha(x)}{1-\beta(x)}$ belongs to Γ;*

(c) $\|F_x(t) - G_y(t) + g(t) - h(t)\|_\infty \sqrt{1+a^2} e^{i\tan^{-1} a} \precsim \alpha(\max_{t\in[a,b]} A(x,y)(t))A(x,y)(t) + \beta(\max_{t\in[a,b]} A(x,y)(t))B(x,y)(t)$, *where*

$$A(x,y)(t) = \big\|x(t) - y(t)\big\|_\infty \sqrt{1+a^2} e^{i\tan^{-1} a},$$

$$B(x,y)(t) = \frac{\|F_x(t) + g(t) - x(t)\|_\infty \|G_y(t) + h(t) - y(t)\|_\infty}{1 + d(x,y)} \sqrt{1+a^2} e^{i\tan^{-1} a},$$

then the system of integral equations (5.1) *and* (5.2) *has a unique common solution.*

Proof Define two mappings $S, T : X \to X$ by $Sx = F_x + g$ and $Tx = G_x + h$. Then we have

$$d(Sx, Ty) = \max_{t\in[a,b]} \big\|F_x(t) - G_y(t) + g(t) - h(t)\big\|_\infty \sqrt{1+a^2} e^{i\tan^{-1} a},$$

$$d(x, Sx) = \max_{t\in[a,b]} \big\|F_x(t) + g(t) - x(t)\big\|_\infty \sqrt{1+a^2} e^{i\tan^{-1} a}$$

and

$$d(y, Ty) = \max_{t\in[a,b]} \big\|G_y(t) + h(t) - y(t)\big\|_\infty \sqrt{1+a^2} e^{i\tan^{-1} a}.$$

We can show easily that for all $x,y \in X$,

$$d(Sx, Ty) \precsim \alpha\big(d(x,y)\big)d(x,y) + \frac{\beta(d(x,y))d(x,Sx)d(y,Ty)}{1 + d(x,y)}.$$

Now, we can apply Theorem 3.2. Therefore, we get the Urysohn integral equations (5.1) and (5.2) have a unique common solution. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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