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On series solutions of $y'' - f(x)y = 0$ and applications

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Abstract

In this paper, we develop explicit expressions for the Taylor series coefficients in the formal Taylor series solution of the second-order linear differential equation $y'' - f(x)y = 0$ for a given arbitrary function $f(x)$ in terms of initial conditions. As applications, we apply our results to $f(x) = e^x$, $\cos(x)$ and Airy's equation and give explicit formulas for the Taylor coefficients of $\exp(\sum_{k=0}^{\infty} a_k x^k)$, which is a long-standing question.

Keywords: Taylor series solutions; the second-order ODE; Maple

1 Introduction

The classical second-order linear differential equation

$$y'' - f(x)y = 0 \tag{1.1}$$

has been the subject of an innumerable amount of papers and of many classical mathematicians. In particular, a challenging problem has been the behavior of $y(x)$ as x tends to infinity. Since no general formula solutions have been established before, one had to use certain indirect ways including numerical approaches (see, for example, [1, 2] and [3]).

In this paper, we present a constructive approach which yields explicit expressions for Taylor series solutions in terms of initial conditions. To the best of our knowledge, this is the first explicit formula giving solutions for differential equations of the type (1.1). In literature, no arbitrary method exists which gives complete explicit solutions for any $f(x)$ for which there is a Taylor series solution as in the case of an ordinary point. It can easily be extended to regular singular points as well. It is well known that recurrence relations by their very nature cannot give analytic expressions for all the coefficients. When finding analytical solutions of differential equations of the type (1.1), for example, Airy's equation, one might express solutions in some forms of Bessel functions or gamma functions, which often does not give much insight into the actual behavior of the solutions. Our results provide direct approaches for some problems. Also, our method can be used for evaluating precisely and very simply up to any number of terms of the series as well as for establishing convergence depending on $f(x)$ and its derivatives. This method is new and novel and has a number of unexpected applications including the relationship between the zeros of a Taylor series and the Taylor coefficients, which is not included to keep the length of the paper to the permitted maximum length.

In the final part of this paper, we use our formulas to give one solution for a long-standing question: finding a Taylor series expansion for $e^{\sum_{k=0}^{\infty} a_k x^k}$. Furthermore, this formula can be easily implemented in mathematical software like Maple [4] which provides more efficient ways to discuss these kinds of questions including parameter calculations.

2 Taylor series solutions

First we use (1.1) to give an expression of the nth derivative $y^{(n)}$ in terms of $f(x)$. For a suitable positive integer k, we have

$$\begin{aligned}
y^{(k+2)} &= \sum_{s_1=0}^{k} \binom{k}{s_1} f^{(k-s_1)} y^{(s_1)} \\
&= \sum_{s_1=0}^{1} \binom{k}{s_1} f^{(k-s_1)} y^{(s_1)} + \sum_{s_1=0}^{k-2} \binom{k}{s_1+2} f^{(k-s_1-2)} y^{(s_1+2)} \\
&= \sum_{s_1=0}^{1} P_1(s_1,k,x) + \sum_{s_1=0}^{k-2} \sum_{s_2=0}^{s_1} \binom{k}{s_1+2} \binom{s_1}{s_2} f^{(k-s_1-2)} f^{(s_1-s_2)} y^{(s_2)} \\
&= \sum_{s_1=0}^{1} P_1(s_1,k,x) + \left(\sum_{s_1=0}^{1} + \sum_{s_1=2}^{k-2} \right) \sum_{s_2=s_1}^{k-2} \binom{k}{s_2+2} \binom{s_2}{s_1} f^{(k-s_2-2)} f^{(s_2-s_1)} y^{(s_1)} \\
&\quad \text{(Interchanging the summations and then interchanging } s_1 \text{ and } s_2\text{)} \\
&= \sum_{s_1=0}^{1} \bigl(P_1(s_1,k,x) + P_2(s_1,k,x) \bigr) \\
&\quad + \sum_{s_1=2}^{k-2} \sum_{s_2=s_1}^{k-2} \binom{k}{s_2+2} \binom{s_2}{s_1+2} f^{(k-s_2-2)} f^{(s_2-s_1-2)} y^{(s_1+2)} \\
&= \sum_{s_1=0}^{1} \sum_{j=1}^{2} P_j(s_1,k,x) + \sum_{s_1=0}^{k-4} \sum_{s_2=s_1}^{k-4} \binom{k}{s_2+4} \binom{s_2+2}{s_1+2} f^{(k-s_2-4)} f^{(s_2-s_1)} y^{(s_1+2)},
\end{aligned}$$

where $P_1(s_1,k,x) = \binom{k}{s_1} f^{(k-s_1)}(x) y^{(s_1)}(x)$ and

$$P_2(s_1,k,x) = \sum_{s_2=s_1}^{k-2} \binom{k}{s_2+2} \binom{s_2}{s_1} f^{(k-s_2-2)} f^{(s_2-s_1)} y^{(s_1)}. \tag{2.1}$$

Using induction (we omit the detailed proof due to the page limit), we obtain

$$\begin{aligned}
y^{(2k+1)} &= \sum_{s_1=0}^{1} \sum_{j=1}^{k} P_j(s_1, 2k-1, x), \quad k = 2,3,\ldots, \\
y^{(2k+2)} &= \sum_{s_1=0}^{1} \sum_{j=1}^{k+1} P_j(s_1, 2k, x) \qquad (2.2) \\
&= \sum_{s_1=0}^{1} \sum_{j=1}^{k} P_j(s_1, 2k, x) + [f]^{k+1} y, \quad k = 1,2,\ldots,
\end{aligned}$$

where

$$
\begin{aligned}
&P_q(s_1,k,x)\\
&\quad = \sum_{s_2=s_1}^{k-2(q-1)} \sum_{s_3=s_2}^{k-2(q-1)} \cdots \sum_{s_q=s_{q-1}}^{k-2(q-1)} \biggl[\binom{k}{s_q+2(q-1)} \binom{s_q+2(q-2)}{s_{q-1}+2(q-2)} \cdots \binom{s_2}{s_1}\\
&\qquad \times f^{(k-2(q-1)-s_q)} f^{(s_q-s_{q-1})} f^{(s_{q-1}-s_{q-2})} \cdots f^{(s_2-s_1)} y^{(s_1)} \biggr]. \qquad (2.3)
\end{aligned}
$$

Next, we establish the recursive relation between P_q and P_{q-1}.

Lemma 2.1

$$
P_q(s_1,k,x) = \sum_{s_2=s_1}^{k-2(q-1)} \binom{k}{s_2+2(q-1)} f^{(k-s_2-2(q-1))} P_{q-1}\bigl(s_1, s_2+2(q-2), x\bigr)
$$

for suitable values of k and q.

Proof To establish this result, we will repeatedly make use of the following formula for interchanging finite sums:

$$
\sum_{s_2=s_1}^{k-2q} \sum_{s_3=s_1}^{s_2} a_{s_2,s_3} = \sum_{s_2=s_1}^{k-2q} \sum_{s_3=s_2}^{k-2q} a_{s_3,s_2}. \qquad (2.4)
$$

From (2.3) with $s_2,\ldots,s_q$ replaced by $s_3,\ldots,s_{q+1}$ respectively, we get for the right-hand side (R.S.) of Lemma 2.1 (for simplicity consider P_q) as given by

$$
\text{R.S.} = \sum_{s_2=s_1}^{k-2q} \sum_{s_3=s_1}^{s_2} \sum_{s_4=s_2}^{s_2} \sum_{s_5=s_4}^{s_2} \cdots \sum_{s_{q+1}=s_q}^{s_2} \binom{k}{s_2+2q} f^{(k-2(q-1)-s_{q+1})} a_{s_2,s_3,s_4,\ldots,s_{q+1}},
$$

where

$$
a_{s_2,\ldots,s_{q+1}} = \binom{s_{q+1}+2(q-2)}{s_q+2(q-2)} \binom{s_q+2(q-3)}{s_{q-1}+2(q-3)} \cdots \binom{s_3}{s_1} f^{(s_{q+1}-s_q)} \cdots f^{(s_2-s_1)} y^{(s_1)}.
$$

Now, using (2.4) for the first two items above, we get

$$
\text{R.S.} = \sum_{s_2=s_1}^{k-2q} \sum_{s_3=s_2}^{k-2q} \sum_{s_4=s_2}^{s_3} \sum_{s_5=s_4}^{s_3} \cdots \sum_{s_{q+1}=s_q}^{s_3} \binom{k}{s_3+2q} f^{(k-2(q-1)-s_{q+1})} a_{s_3,s_2,s_4,s_5,\ldots,s_{q+1}}.
$$

Again, using (2.4) for the second and third sums above, we get

$$
\text{R.S.} = \sum_{s_2=s_1}^{k-2q} \sum_{s_3=s_2}^{k-2q} \sum_{s_4=s_3}^{k-2q} \sum_{s_5=s_4}^{s_4} \cdots \sum_{s_{q+1}=s_q}^{s_4} \binom{k}{s_4+2q} f^{(k-2(q-1)-s_{q+1})} a_{s_4,s_2,s_3,s_5,\ldots,s_{q+1}}.
$$

Proceeding similarly,

$$\begin{aligned}
\text{R.S.} &= \sum_{s_2=s_1}^{k-2q}\sum_{s_3=s_2}^{k-2q}\cdots\sum_{s_{q+1}=s_q}^{k-2q}\binom{k}{s_q+2q} f^{(k-2(q-1)-s_{q+1})} a_{s_{q+1},s_q,\dots,s_2} \\
&= \sum_{s_2=s_1}^{k-2q}\sum_{s_3=s_2}^{k-2q}\cdots\sum_{s_{q+1}=s_q}^{k-2q}\binom{k}{s_q+2q} f^{(k-2(q-1)-s_{q+1})} a_{s_2,s_3,s_4,\dots,s_{q+1}} \\
&= P_{q+1}(s_1,k,x). \qquad \square
\end{aligned}$$

From the above discussion and Lemma 2.1, the formal Taylor series solution at $x = a$ for $y(x)$ can be written as follows.

Theorem 2.2

$$\begin{aligned}
y(x) &= \left\{ y(a) + \sum_{k=0}^{\infty} \frac{1}{(2k+2)!}\left[\sum_{s_1=0}^{1}\sum_{j=1}^{k+1} P_j(s_1,2k,a)\right](x-a)^{2k+2}\right\} \\
&\quad + \left\{ y'(a)(x-a) + \sum_{k=0}^{\infty} \frac{1}{(2k+3)!}\left[\sum_{s_1=0}^{1}\sum_{j=1}^{k} P_j(s_1,2k+1,a)\right](x-a)^{2k+3}\right\}.
\end{aligned}$$

3 Applications

In the first part of this section, we discuss three typical examples using our formulas, and in the second part, we give one solution for a long-standing question.

Example 3.1 Consider $y'' + e^x y = 0$ with $y(0) = 1$, $y'(0) = 0$. From (2.2), we have $y^{(2k+2)}(0) = \sum_{j=1}^{k}(-1)^j P_j(0,2k,0) + (-1)^{k+1}$. Now, using Theorem 2.2 and the results in Section 2 of [5], we obtain that $y(x)$ tends to zero as $x \to \infty$.

Example 3.2 Consider the initial value problem $y'' + \cos(x)y = 0$, $y(0) = 1$, $y'(0) = 0$. It is easy to get the general Taylor series solution by our formula as follows:

$$y(x) = 1 - \frac{1}{2}x^2 + \frac{1}{12}x^4 - \frac{1}{80}x^6 + \frac{11}{8{,}064}x^8 + O(x^{10}).$$

Example 3.3 It is well known that the general solution for Airy's equation $y'' = xy$ is given by $y = x^{\frac{1}{2}}[c_1 f_1(\frac{2}{3}ix^{\frac{3}{2}}) + c_2 f_2(\frac{2}{3}ix^{\frac{3}{2}})]$, where f_1 and f_2 are linearly independent solutions of the Bessel equation of order $\frac{1}{3}$. When we consider the general recurrence relation $y = \sum_{n=0}^{\infty} a_n(x-1)^n$ at $x = 1$, we have $(n+2)(n+1)a_{n+2} = a_n + a_{n-1}$, $n \geq 1$. It is well known that although we can determine as many coefficients a_n in terms of a_0 and a_1, there is no known 'formula' for a_n (see, for example, [6], pp.246-247). From our theory, the solution at $x = a$ can be written as follows:

$$\begin{aligned}
y(x) &= \left\{ y(a) + \sum_{k=0}^{\infty} \frac{1}{(2k+2)!}\left[\sum_{s_1=0}^{1}\sum_{j=1}^{k+1} P_j(s_1,2k,a)\right](x-a)^{2k+2}\right\} \\
&\quad + \left\{ y'(a)(x-a) + \sum_{k=0}^{\infty} \frac{1}{(2k+3)!}\left[\sum_{s_1=0}^{1}\sum_{j=1}^{k} P_j(s_1,2k+1,a)\right](x-a)^{2k+3}\right\}.
\end{aligned}$$

Clearly, all coefficients are just in terms of $y(a)$ and $y'(a)$, that is, a_0 and a_1.

Moreover, using our recursive formula, we can give the general term for the initial conditions $y(0) = 1$, $y'(0) = 0$. By (2.1), it is easy to get

$$P_1(0,1,0) = 1 \quad \text{and} \quad P_1(0,k,0) = 0 \quad \text{for } k = 2,3,\ldots. \tag{3.1}$$

Using the initial conditions, the only non-zero term on the right-hand side of $y(x)$ is when $x^{(k-s_2-2(q-1))} = 1$, that is, $k - s_2 - 2(q-1) = 1$. Then we have the following recursive relation:

$$\begin{aligned} P_q(0,k,0) &= \binom{k}{s_2 + 2(q-1)} P_{q-1}\bigl(0, s_2 + 2(q-2), 0\bigr) \\ &= kP_{q-1}(0,k-3,0) = k(k-3)P_{q-2}(0,k-6,0). \end{aligned} \tag{3.2}$$

Note that $y'(0) = 0$. All terms with $s_1 = 1$ are zero. Combining (3.1) and (3.2) in $y(x)$, there is only one non-zero term in $\sum_{j=1}^{k}$ or $\sum_{j=1}^{k+1}$. Furthermore, the coefficient c_n of x^n in $y(x)$ is non-zero if and only if $n = 3m$, and then it is easy to get that $c_n = \frac{1}{(3m)(3m-1)(3m-3)(3m-4)\cdots(6\cdot 5)(3\cdot 2)}$. Hence

$$\lim_{m\to\infty} \frac{c_{3(m+1)}}{c_{3m}} |x|^3 = \lim_{m\to\infty} \frac{1}{(3m+3)(3m+2)} |x|^3 = 0.$$

Therefore, the radius of convergence equals ∞.

In the rest of this section, we describe a long-standing question: finding a Taylor series expansion for $e^{\sum_{k=0}^{\infty} a_k x^k}$ given by $h(x) = \sum_{k=0}^{\infty} c_k x^k$. For the history of this problem, see, *e.g.*, Pounaltmadi [7]. In [7] a recursive formula for c_k in terms of a_k is given by $c_{k+1} = \sum_{j=0}^{k} (1 - \frac{j}{k+1}) a_{k+1-j} c_j$, $k = 0,1,\ldots$ with $c_0 = e^{a_0}$. There are no methods of finding c_k explicitly for all k, although one can find it by actual computation for a finite number of c_k's. This process is tedious and difficult, and also not feasible for large k. Now we can use our formulas to solve it.

Denote $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $y = e^{f(x)}$. Then

$$y'' = \bigl\{f''(x) + \bigl[f'(x)\bigr]^2\bigr\} e^{f(x)} = \bigl\{\bigl[f'(x)\bigr]^2 + f''(x)\bigr\} y(x) := g(x)y(x),$$

where $g(x)$ is given by

$$\sum_{k=0}^{\infty} \left\{ \left[\sum_{j=0}^{k} (j+1)(k-j+1) a_{j+1} a_{k-j+1} \right] + (k+2)(k+1) a_{k+2} \right\} x^k := \sum_{k=0}^{\infty} d_k x^k.$$

For $y'' = g(x)y$ at $x = 0$, $d_0 = g(0) = a_1^2 + 2a_2$ and $d_1 = g'(0) = 4a_1 a_2 + 6a_3$. The general Taylor series solution is given by

$$\begin{aligned} y(x) &= \exp\left(\sum_{k=0}^{\infty} a_k x^k \right) \\ &= \left\{ y(0) + \sum_{k=1}^{\infty} \frac{1}{(2k+2)!} \left[\sum_{s_0=0}^{1} \sum_{j=1}^{k} P_j(s_0, 2k, 0) + \bigl[g(0)\bigr]^{k+1} y(0) \right] x^{2k+2} \right\} \\ &\quad + \left\{ y'(0)x + \sum_{k=1}^{\infty} \frac{1}{(2k+3)!} \left[\sum_{s_0=0}^{1} \sum_{j=1}^{k} P_j(s_0, 2k+1, 0) \right] x^{2k+3} \right\}, \end{aligned} \tag{3.3}$$

where all coefficients of x in the power series are in terms of d_k's and hence in terms of a_k's. In particular, if $f(x)$ is a polynomial of degree m and $[f'(x)]^2$ is a polynomial of degree $2m-2$ yielding $g^{(k)}(x)=0$, $k=2m-1, 2m, \dots$.

Our method can also be applied to other forms of functions. For example, consider $y = e^{\int_0^x f(t)\,\mathrm{d}t}$. Then $y'' = [f^2(x)+f'(x)]y(x)$. Let $g(x)=f^2(x)+f'(x)$. Applying above formula (3.3) gives a Taylor series expansion $y(x)=\sum_{k=0}^{\infty} b_k x^k$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors, YZ and PNS, contributed to each part of this study equally and read and approved the final version of the manuscript.

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