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Existence of solutions for a coupled system of nonlinear fractional differential equations with fractional boundary conditions on the half-line

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Abstract

In this article, we study a boundary value problem of a coupled system of nonlinear Riemann-Liouville type fractional differential equations with fractional boundary conditions on the half-line. An appropriate compactness criterion is established to prove the existence of solutions of the problem by means of the Schauder fixed point theorem. An illustrative example is also given.

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Keywords: boundary value problem; fractional differential system; Riemann-Liouville fractional derivative; fixed-point theorem

1 Introduction

In recent years, the subject of fractional differential equations has gained a considerable attention and it has emerged as an interesting and popular field of research. It is mainly due to the fact that the tools of fractional calculus are found to be more practical and effective than the corresponding ones of classical calculus in the mathematical modeling of several phenomena involving fractals and chaos. In fact, fractional calculus has numerous applications in various disciplines of science and engineering such as mechanics, electricity, chemistry, biology, economics, control theory, signal and image processing, polymer rheology, regular variation in thermodynamics, biophysics, blood flow phenomena, aerodynamics, electro-dynamics of complex medium, viscoelasticity and damping, control theory, wave propagation, percolation, identification, fitting of experimental data, *etc.* For theoretical development and methods of solution for fractional differential equations, see the books [1–6] and references therein. For details on the geometric and physical interpretation of the derivatives of non-integer order, see [7–9]. Some recent results on fractional boundary value problems on a finite interval can be found in [10–21] and references therein.

In [10], using the monotone iterative method, Zhang investigated the existence and uniqueness of solutions for the following initial value problem of the fractional differential equations:

$$\begin{cases} D_{0^+}^{\alpha} u(t) = f(t, u(t)), & t \in (0, T], \\ t^{1-\alpha} u(t)|_{t=0} = u_0, \end{cases} \tag{1}$$

where $0 < T < \infty$ and D^{α} is the Riemann-Liouville fractional derivative of order $\alpha \in (0,1)$.

Arara *et al.* [22] studied the existence of bounded solutions for differential equations involving the Caputo fractional derivative on the unbounded domain given by

$$\begin{cases} {}^{c}D_{0^+}^{\alpha} u(t) = f(t, u(t)), & t \in [0, \infty), \\ u(0) = u_0, \\ u \text{ is bounded on } [0, \infty), \end{cases} \tag{2}$$

where $\alpha \in (1,2)$, ${}^{c}D_{0^+}^{\alpha}$ is the Caputo fractional derivative of order α, $u_0 \in R$, and $f : [0, \infty) \times R \to R$ is continuous. Using the Schauder fixed point theorem combined with the diagonalization method, it is proved that BVP (2) has at least one solution on $[0, \infty)$.

Zhao and Ge [23] considered the following boundary value problem for fractional differential equations:

$$\begin{cases} D_{0^+}^{\alpha} u(t) + f(t, u(t)) = 0, & 0 < t < \infty, 1 < \alpha < 2, \\ u(0) = 0, \\ \lim_{t\to\infty} D_{0^+}^{\alpha-1} u(t) = 0, \end{cases} \tag{3}$$

where $0 < \xi < \infty$, $\beta \geq 0$ and f is a given function, $D_{0^+}^{\alpha}$ is the Riemann-Liouville fractional derivative. By using the properties of the Green's function together with the Schauder fixed point theorem, it has been proved that BVP (3) has at least one positive solution subject to the assumptions: $f : [0, \infty) \times R \to [0, \infty)$ is continuous; and there exist a nondecreasing function $\omega \in C([0, \infty), [0, \infty))$ and a function $\phi \in L^1[0, \infty)$ such that $|f(t, (1 + t^{\alpha-1})u)| \leq \phi(t)\omega(u)$ on $[0, \infty) \times [0, \infty)$.

In [24], Liu and Jia investigated the boundary value problem for a fractional differential equation of the form

$$\begin{cases} {}^{c}D_{0^+}^{\alpha} [p(t)u'(t)] + q(t)f(t, u(t)) = 0, & t > 0, \\ p(0)u'(0) = 0, \\ \lim_{t\to\infty} u(t) = \int_0^{\infty} g(s)u(s)\, ds, \end{cases} \tag{4}$$

where ${}^{c}D_{0^+}^{\alpha}$ is the Caputo fractional derivative of order $\alpha \in (0,1)$, f, g, p, q are given functions, $p(t) > 0$ for all $t \geq 0$ with $\int_0^{\infty} \frac{1}{p(s)}\, ds < \infty$ and $k(s) = \int_s^{\infty} \frac{(r-s)^{\alpha-1}}{p(r)}\, dr$ being continuous on $[0, \infty)$, $g \in L^1[0, \infty)$ with $\int_0^{\infty} g(s)\, ds < 1$. The existence of at least three nonnegative solutions of the problem (4) was established by using fixed point theory and the method of upper and lower solutions.

For some more work on boundary value problems of fractional differential equations on a half-line/semi-infinite interval, we refer the reader to the papers [25–29].

On the other hand, the study for coupled systems of fractional differential equations is also important as such systems occur in various problems of applied nature; for instance, see [30–33]. Some recent results on coupled systems of fractional differential equations on a finite interval can be found in [34–37].

In this paper, we discuss the existence of solutions to a boundary value problem of a coupled system of nonlinear fractional differential equations on the half-line given by

$$\begin{cases} D_{0^+}^{\alpha}x(t) = f(t, y(t), D_{0^+}^{p}y(t)), \quad t \in (0,\infty), \\ D_{0^+}^{\beta}y(t) = g(t, x(t), D_{0^+}^{q}x(t)), \quad t \in (0,\infty), \\ a\lim_{t\to 0} t^{2-\alpha}x(t) - b\lim_{t\to 0} D_{0^+}^{\alpha-1}x(t) = x_0, \\ c\lim_{t\to 0} t^{2-\beta}y(t) - d\lim_{t\to 0} D_{0^+}^{\beta-1}x(t) = y_0, \\ \lim_{t\to\infty} D_{0^+}^{\alpha-1}x(t) = x_1, \\ \lim_{t\to\infty} D_{0^+}^{\beta-1}x(t) = y_1, \end{cases} \tag{5}$$

where $a, b, c, d > 0$, $\alpha, \beta \in (1,2)$, $p \in (\beta - 1, \beta)$, $q \in (\alpha - 1, \alpha)$, $x_0, y_0, x_1, y_1 \in R$, D_{0^+} is the standard Riemann-Liouville fractional derivative and $f, g : (0,\infty) \times R^2 \to R$ are continuous functions and f, g may be singular at $t = 0$.

We establish sufficient conditions for the existence of solutions of (5) by applying the Schauder fixed point theorem. Our results are new in the sense that we consider BVP (5) on a half-line with the assumptions on p, q of the form $p \in (\beta - 1, \beta)$, $q \in (\alpha - 1, \alpha)$. Moreover, both the nonlinear functions f and g are allowed to be linear as well as super linear. The paper is organized as follows: the preliminary results are given in Section 2, the main results are presented in Section 3, while an example is discussed in Section 4 to illustrate the main theorems.

2 Preliminary results

Let us begin this section with some basic concepts of fractional calculus [1–3]. For $a > 0$ and $b, c > 0$, denote the gamma function and beta function respectively as

$$\Gamma(a) = \int_0^{+\infty} s^{a-1}e^{-s}\,ds, \qquad \mathbf{B}(b,c) = \int_0^1 (1-x)^{b-1}x^{c-1}\,dx.$$

Definition 2.1 The Riemann-Liouville fractional integral of order $\alpha > 0$ of a continuous function $f : (0,\infty) \to R$ is given by

$$I_{0^+}^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)\,ds,$$

provided that the right-hand side exists.

Definition 2.2 The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0,\infty) \to R$ is given by

$$D_{0^+}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^{n+1}}{dt^{n+1}}\int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}}\,ds,$$

where $n - 1 < \alpha \le n$, provided that the right-hand side is point-wise defined on $(0,\infty)$.

It is easy to show that for $\varrho \geq 0$ and $\mu > -1$, we have

$$I_{0^+}^{\varrho} t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\varrho+1)} t^{\mu+\varrho}, \qquad D_{0^+}^{\varrho} t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\varrho+1)} t^{\mu-\varrho}.$$

Let $C(0,\infty)$ be the set of all continuous functions on $(0,\infty)$. For $\sigma > \max\{q-\alpha, p-\beta\}$, ones sees from $p \in (\beta-1,\beta)$, $q \in (\alpha-1,\alpha)$ that $\sigma > -1$. We choose

$$X = \left\{ x \in C(0,\infty) : \begin{array}{c} D_{0^+}^{q} x \in C(0,\infty) \\ \frac{t^{2-\alpha}}{1+t^{\sigma+2}} x(t) \text{ is bounded on } (0,\infty) \\ \frac{t^{2+q-\alpha}}{1+t^{\sigma+2}} D_{0^+}^{q} x(t) \text{ is bounded on } (0,\infty) \end{array} \right\}$$

and

$$Y = \left\{ y \in C(0,\infty) : \begin{array}{c} D_{0^+}^{p} y \in C(0,\infty) \\ \frac{t^{2-\beta}}{1+t^{\sigma+2}} x(t) \text{ is bounded on } (0,\infty) \\ \frac{t^{2+p-\beta}}{1+t^{\sigma+2}} D_{0^+}^{p} x(t) \text{ is bounded on } (0,\infty) \end{array} \right\}.$$

For $x \in X$, define the norm by

$$\|x\|_X = \max\left\{ \sup_{t\in(0,\infty)} \frac{t^{2-\alpha}}{1+t^{\sigma+2}} |x(t)|, \sup_{t\in(0,\infty)} \frac{t^{2+q-\alpha}}{1+t^{\sigma+2}} |D_{0^+}^{q} x(t)| \right\}.$$

It is easy to show that X is a real Banach space. For $y \in Y$, define the norm by

$$\|y\|_Y = \max\left\{ \sup_{t\in(0,\infty)} \frac{t^{2-\beta}}{1+t^{\sigma+2}} |y(t)|, \sup_{t\in(0,\infty)} \frac{t^{2+p-\beta}}{1+t^{\sigma+2}} |D_{0^+}^{p} y(t)| \right\}.$$

It is easy to show that Y is a real Banach space. Thus, $(X \times Y, \|\cdot\|)$ is Banach space with the norm defined by

$$\|(x,y)\| = \max\{\|x\|_X, \|y\|_Y\} \quad \text{for } (x,y) \in X \times Y.$$

Lemma 2.1 *Let $1 < \alpha < 2$, $x_0, x_1 \in R$, and let $e : (0,\infty) \to R$ be a given function such that there exist numbers $M > 0$, $\sigma > -1$ and $k > 0$ with $|e(t)| \leq M t^{\sigma} e^{-kt}$. Then $x \in X$ is a solution of the problem*

$$\begin{cases} D_{0^+}^{\alpha} x(t) = e(t), \quad t \in J = (0,\infty), \\ a \lim_{t\to 0} t^{2-\alpha} x(t) - b \lim_{t\to 0} D_{0^+}^{\alpha-1} x(t) = x_0, \\ \lim_{t\to\infty} D_{0^+}^{\alpha-1} x(t) = x_1, \end{cases} \tag{6}$$

if and only if $x \in X$ and

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e(s)\, ds + \frac{x_1 - \int_0^{\infty} e(s)\, ds}{\Gamma(\alpha)} t^{\alpha-1} + \frac{x_0 + b x_1 - b\int_0^{\infty} e(s)\, ds}{a} t^{\alpha-2}. \tag{7}$$

Proof It is easy to see that $\int_0^\infty t^\sigma e^{-kt}\,dt < \infty$. For arbitrary constants c_1, c_2, the general solution of the equation $D_{0^+}^\alpha x(t) = e(t)$ can be written as

$$x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}e(s)\,ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} \tag{8}$$

with

$$D_{0^+}^{\alpha-1}x(t) = \int_0^t e(s)\,ds + \Gamma(\alpha)c_1.$$

Using the boundary conditions of (6), we find that

$$c_1 = \frac{x_1 - \int_0^\infty e(s)\,ds}{\Gamma(\alpha)}, \qquad c_2 = \frac{x_0 + bx_1 - b\int_0^\infty e(s)\,ds}{a}.$$

Substituting the values of c_1 and c_2 in (8), we obtain (7).

Now, we prove $x \in X$. Clearly,

$$\begin{aligned} D_{0^+}^q x(t) ={}& \frac{\int_0^t (t-s)^{\alpha-q-1}e(s)\,ds}{\Gamma(\alpha-q)} + \frac{x_1 - \int_0^\infty e(s)\,ds}{\Gamma(\alpha-q)}t^{\alpha-q-1} \\ &+ \frac{x_0 + bx_1 - b\int_0^\infty e(s)\,ds}{a}\frac{\Gamma(\alpha-1)t^{\alpha-q-2}}{\Gamma(\alpha-q-1)}. \end{aligned} \tag{9}$$

It follows from (7) and (9) together with $|e(t)| \le Mt^\sigma e^{-kt}$, $\sigma > -1$ that $x, D_{0^+}^q x \in C(0,\infty)$.

Observe that

$$\frac{t^{2-\alpha}}{1+t^{\sigma+2}}|x(t)| \le \frac{|x_0 + bx_1|}{a} + \frac{|x_1|}{\Gamma(\alpha)} + \frac{M}{\Gamma(\alpha)}\mathbf{B}(\alpha, \sigma+1) + \left(\frac{1}{\Gamma(\alpha)} + \frac{1}{a}\right)\frac{\Gamma(\sigma+1)}{k^{\sigma+1}} < +\infty$$

and

$$\begin{aligned} \frac{t^{2+q-\alpha}}{1+t^{\sigma+2}}|D_{0^+}^q x(t)| \le{}& \frac{M}{\Gamma(\alpha-q)}\frac{t^{\sigma+2}}{1+t^{\sigma+2}}\int_0^1 (1-w)^{\alpha-q-1}w^\sigma\,dw \\ &+ \frac{|x_1|}{\Gamma(\alpha-q)} + \frac{|x_0+bx_1|}{a}\frac{\Gamma(\alpha-1)}{|\Gamma(\alpha-q-1)|} \\ &+ \left(\frac{1}{\Gamma(\alpha-q)} + \frac{\Gamma(\alpha-1)}{a|\Gamma(\alpha-q-1)|}\right)\frac{\Gamma(\sigma+1)}{k^{\sigma+1}} < +\infty. \end{aligned}$$

Hence $x \in X$.

Conversely, if $x \in X$ satisfies (7), then it can easily be shown that $x \in X$ and satisfies (6). This completes the proof. □

Consider the coupled system of integral equations

$$\begin{cases} x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,y(s),D_{0^+}^p y(s))\,ds + \frac{x_1 - \int_0^\infty f(s,y(s),D_{0^+}^p y(s))\,ds}{\Gamma(\alpha)}t^{\alpha-1} \\ \qquad + \frac{x_0 + bx_1 - \int_0^\infty f(s,y(s),D_{0^+}^p y(s))\,ds}{a}t^{\alpha-2}, \\ y(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\alpha-1}g(s,x(s),D_{0^+}^q x(s))\,ds + \frac{y_1 - \int_0^\infty g(s,x(s),D_{0^+}^q x(s))\,ds}{\Gamma(\alpha)}t^{\beta-1} \\ \qquad + \frac{y_0 + dy_1 - \int_0^\infty g(s,x(s),D_{0^+}^q x(s))\,ds}{c}t^{\beta-2}. \end{cases} \tag{10}$$

For the sequel, we need the following assumptions:

(H) There exist numbers $\sigma_i, \mu_i \in (-1, \sigma)$, $k_i > 0$, $l_i > 0$ $(i = 0, 1, 2)$ and positive numbers A, B, C, A_1, B_1, C_1 such that for $t \in (0, \infty)$, $u_1, u_2, v_1, v_2 \in R$, f and g satisfy the conditions

$$\left|f\left(t, \frac{1+t^{\sigma+2}}{t^{2-\beta}}u_1, \frac{1+t^{\sigma+2}}{t^{2+p-\beta}}u_2\right) - Ct^{\sigma_0}e^{-k_0 t}\right| \leq At^{\sigma_1}e^{-k_1 t}|u_1| + Bt^{\sigma_2}e^{-k_2 t}|u_2|$$

and

$$\left|g\left(t, \frac{1+t^{\sigma+2}}{t^{2-\alpha}}v_1, \frac{1+t^{\sigma+2}}{t^{2+q-\alpha}}v_2\right) - C_1 t^{\mu_0}e^{-l_0 t}\right| \leq A_1 t^{\mu_1}e^{-l_1 t}|v_1| + B_1 t^{\mu_2}e^{-l_2 t}|v_2|;$$

(G) There exist numbers $\sigma_i, \mu_i \in (-1, \sigma)$, $k_i > 0$, $l_i > 0$ $(i = 0, 1, 2)$, $\delta > 1$ and positive numbers A, B, C, A_1, B_1, C_1 such that for $t \in (0, \infty)$, $u_1, u_2, v_1, v_2 \in R$, f and g satisfy the conditions

$$\left|f\left(t, \frac{1+t^{\sigma+2}}{t^{2-\beta}}u_1, \frac{1+t^{\sigma+2}}{t^{2+p-\beta}}u_2\right) - Ct^{\sigma_0}e^{-k_0 t}\right| \leq At^{\sigma_1}e^{-k_1 t}|u_1|^\delta + Bt^{\sigma_2}e^{-k_2 t}|u_2|^\delta$$

and

$$\left|g\left(t, \frac{1+t^{\sigma+2}}{t^{2-\alpha}}v_1, \frac{1+t^{\sigma+2}}{t^{2+q-\alpha}}v_2\right) - C_1 t^{\mu_0}e^{-l_0 t}\right| \leq A_1 t^{\mu_1}e^{-l_1 t}|v_1|^\delta + B_1 t^{\mu_2}e^{-l_2 t}|v_2|^\delta.$$

Lemma 2.2 *Suppose that* (H) *or* (G) *holds. Then* $(x, y) \in X \times Y$ *is a solution of* (5) *if and only if* $(x, y) \in X \times Y$ *is a solution of* (10).

Proof Let $(x, y) \in X \times Y$. In view of the assumption (H), it follows that

$$\begin{aligned} \left|f\left(t, y(t), D_{0^+}^p y(t)\right)\right| &= \left|f\left(t, \frac{1+t^{\sigma+2}}{t^{2-\beta}}\frac{t^{2-\beta}}{1+t^{\sigma+2}}y(t), \frac{1+t^{\sigma+2}}{t^{2+p-\beta}}\frac{t^{2+p-\beta}}{1+t^{\sigma+2}}D_{0^+}^p y(t)\right)\right| \\ &\leq Ct^{\sigma_0}e^{-k_0 t} + At^{\sigma_1}e^{-k_1 t}\|y\|_Y + Bt^{\sigma_2}e^{-k_2 t}\|y\|_Y \end{aligned}$$

and

$$\left|g\left(t, x(t), D_{0^+}^q px(t)\right)\right| \leq C_1 t^{\mu_0}e^{-l_0 t} + A_1 t^{\mu_1}e^{-l_1 t}\|x\|_X + B_1 t^{\mu_2}e^{-l_2 t}\|x\|_X.$$

The rest of the proof follows from Lemma 2.1. Similarly, we can show that the result holds if (G) holds. This completes the proof. □

Let us define an operator $F : X \times Y \to X \times Y$ as

$$F(x, y)(t) = \left((F_1 y)(t), (F_2 x)(t)\right),$$

where

$$\begin{aligned} (F_1 y)(t) = {} & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f\left(s, y(s), D_{0^+}^p y(s)\right)ds + \frac{x_1 - \int_0^\infty f(s, y(s), D_{0^+}^p y(s))\,ds}{\Gamma(\alpha)}t^{\alpha-1} \\ & + \frac{x_0 + bx_1 - \int_0^\infty f(s, y(s), D_{0^+}^p y(s))\,ds}{a}t^{\alpha-2} \end{aligned}$$

and

$$(F_2x)(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\alpha-1} g\big(s, x(s), D_{0^+}^q x(s)\big)\,ds + \frac{y_1 - \int_0^\infty g(s, x(s), D_{0^+}^q x(s))\,ds}{\Gamma(\alpha)} t^{\beta-1}$$
$$+ \frac{y_0 + dy_1 - \int_0^\infty g(s, x(s), D_{0^+}^q x(s))\,ds}{c} t^{\beta-2}.$$

Lemma 2.3 *Suppose that* (H) *or* (G) *holds. Then the fixed point of the operator F coincides with the solution of* (5) *and $F: X \times Y \to X \times Y$ is completely continuous.*

Proof It follows from Lemma 2.2 that the fixed point of the operator F coincides with the solution of (5). Suppose that (H) holds. The remaining proof consists of the following five steps.

Step 1. We show that $F: X \times Y \to X \times Y$ is well defined and maps bounded sets into bounded sets.

For $(x, y) \in X \times Y$, we get

$$\begin{aligned} r &= \max\big\{\|x\|_X, \|y\|_Y\big\} \\ &= \max\bigg\{\sup_{t\in(0,\infty)} \frac{t^{2-\alpha}}{1+t^{\sigma+2}}\big|x(t)\big|, \sup_{t\in(0,\infty)} \frac{t^{2-\beta}}{1+t^{\sigma+2}}\big|y(t)\big|, \\ &\quad \sup_{t\in(0,\infty)} \frac{t^{2+q-\alpha}}{1+t^{\sigma+2}}\big|D_{0^+}^q x(t)\big|, \sup_{t\in(0,\infty)} \frac{t^{2+p-\beta}}{1+t^{\sigma+2}}\big|D_{0^+}^p y(t)\big|\bigg\} < \infty. \end{aligned}$$

By the definition of F, we have

$$(F_1y), D_{0^+}^q (F_1y) \in C(0,\infty).$$

By the method used in Lemma 2.2, we get (H) implies that

$$\begin{aligned} &\big|f\big(t, y(t), D_{0^+}^p y(t)\big)\big| \\ &\quad = \left|f\left(t, \frac{1+t^{\sigma+2}}{t^{2-\beta}}\frac{t^{2-\beta}}{1+t^{\sigma+2}}y(t), \frac{1+t^{\sigma+2}}{t^{2+p-\beta}}\frac{t^{2+p-\beta}}{1+t^{\sigma+2}}D_{0^+}^p y(t)\right)\right| \\ &\quad \le Ct^{\sigma_0}e^{-k_0t} + At^{\sigma_1}e^{-k_1t}\left|\frac{t^{2-\beta}}{1+t^{\sigma+2}}y(t)\right| + Bt^{\sigma_2}e^{-k_2t}\left|\frac{t^{2+p-\beta}}{1+t^{\sigma+2}}D_{0^+}^p y(t)\right| \\ &\quad \le Ct^{\sigma_0}e^{-k_0t} + Art^{\sigma_1}e^{-k_1t} + Brt^{\sigma_2}e^{-k_2t} \end{aligned}$$

and

$$\big|g\big(t, x(t), D_{0^+}^q x(t)\big)\big| \le C_1t^{\mu_0}e^{-l_0t} + A_1rt^{\mu_1}e^{-l_1t} + B_1rt^{\mu_2}e^{-l_2t}.$$

Hence

$$\begin{aligned} &\frac{t^{2-\alpha}}{1+t^{\sigma+2}}\big|(F_1y)(t)\big| \\ &\quad \le \frac{1}{\Gamma(\alpha)}C\int_0^1 (1-w)^{\alpha-1}w^{\sigma_0}\,dw + Ar\int_0^1 (1-w)^{\alpha-1}w^{\sigma_1}\,dw \end{aligned}$$

$$+ Br\int_0^1 (1-w)^{\alpha-1}w^{\sigma_2}\,dw + \frac{|x_1|}{\Gamma(\alpha)} + \frac{|x_0+bx_1|}{a}$$
$$+ \left(\frac{1}{\Gamma(\alpha)} + \frac{1}{a}\right)\left[\frac{C}{k_0^{\sigma_0+1}}\Gamma(\sigma_0+1) + \frac{Ar}{k_1^{\sigma_1+1}}\Gamma(\sigma_1+1) + \frac{Br}{k_2^{\sigma_2+1}}\Gamma(\sigma_2+1)\right] < +\infty.$$

Furthermore, we have

$$\begin{aligned} D_{0^+}^q(F_1y)(t) = {}& \frac{\int_0^t (t-s)^{\alpha-q-1}f(s,y(s),D_{0^+}^p y(s))\,ds}{\Gamma(\alpha-q)} \\ &+ \left(x_1 - \int_0^\infty f\left(s,y(s),D_{0^+}^p y(s)\right)ds\right)\frac{t^{\alpha-q-1}}{\Gamma(\alpha-q)} \\ &+ \left(x_0 + bx_1 - \int_0^\infty f\left(s,y(s),D_{0^+}^p y(s)\right)ds\right)\frac{\Gamma(\alpha-1)t^{\alpha-q-2}}{\Gamma(\alpha-q-1)}. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} &\frac{t^{2+q-\alpha}}{1+t^{\sigma+2}}\left|D_{0^+}^q(F_1y)(t)\right| \\ &\quad\le \frac{1}{\Gamma(\alpha-q)}C\int_0^1 (1-w)^{\alpha-p-1}w^{\sigma_0}\,dw + Ar\int_0^1 (1-w)^{\alpha-q-1}w^{\sigma_1}\,dw \\ &\qquad + Br\int_0^1 (1-w)^{\alpha-q-1}w^{\sigma_2}\,dw + \frac{|x_1|}{\Gamma(\alpha-q)} + \frac{|x_0+bx_1|\Gamma(\alpha-1)}{|\Gamma(\alpha-q-1)|} \\ &\qquad + \left(\frac{1}{\Gamma(\alpha-q)}\frac{t}{1+t^{\sigma+2}} + \frac{\Gamma(\alpha-q)}{|\Gamma(\alpha-q-1)|}\frac{1}{1+t^{\sigma+2}}\right) \\ &\qquad \times \left[\frac{C}{k_0^{\sigma_0+1}}\Gamma(\sigma_0+1) + \frac{Ar}{k_1^{\sigma_1+1}}\Gamma(\sigma_1+1) + \frac{Br}{k_2^{\sigma_2+1}}\Gamma(\sigma_2+1)\right] < +\infty. \end{aligned}$$

Then $F_1y \in X$. Similarly, we can prove that $F_2x \in Y$. Thus $F : X \times Y \to X \times Y$ is well defined.

It is easy to show similarly that F maps bounded sets into bounded sets.

Step 2. We show that F is continuous.

Let $(u_n, v_n) \in X \times Y$ with $(u_n, v_n) \to (u_0, v_0)$ as $n \to \infty$. We will prove that $(F_1v_n, F_2u_n) \to (F_1v_0, F_2u_0)$ as $n \to \infty$. It is easy to see that there exists $r > 0$ such that

$$\begin{aligned} \left\|(u_n,v_n)\right\| &= \max\left\{\|u_n\|_X, \|v_n\|_Y\right\} \\ &= \max\left\{\sup_{t\in(0,\infty}\frac{t^{2-\alpha}}{1+t^{\sigma+2}}\left|u_n(t)\right|, \sup_{t\in(0,\infty}\frac{t^{2-\beta}}{1+t^{\sigma+2}}\left|v_n(t)\right|,\right. \\ &\qquad \left.\sup_{t\in(0,\infty)}\frac{t^{2+q-\alpha}}{1+t^{\sigma+2}}\left|D_{0^+}^q u_n(t)\right|, \sup_{t\in(0,\infty)}\frac{t^{2+p-\beta}}{1+t^{\sigma+2}}\left|D_{0^+}^p v_n(t)\right|\right\} \le r < \infty. \end{aligned}$$

Then (H) implies that

$$\begin{aligned} \left|f\left(t,v_n(t),D_{0^+}^p v_n(t)\right)\right| &= \left|f\left(t, \frac{1+t^{\sigma+2}}{t^{2-\beta}}\frac{t^{2-\beta}}{1+t^{\sigma+2}}v_n(t), \frac{1+t^{\sigma+2}}{t^{2+p-\beta}}\frac{t^{2+p-\beta}}{1+t^{\sigma+2}}D_{0^+}^p v_n(t)\right)\right| \\ &\le Ct^{\sigma_0}e^{-k_0t} + Art^{\sigma_1}e^{-k_1t} + Brt^{\sigma_2}e^{-k_2t} \end{aligned}$$

and

$$\left|g\big(t,u_n(t),D_{0^+}^q u_n(t)\big)\right| \le C_1 t^{\mu_0} e^{-l_0 t} + A_1 r t^{\mu_1} e^{-l_1 t} + B_1 r t^{\mu_2} e^{-l_2 t}.$$

Observe that

$$\begin{aligned}
(F_1 v_n)(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\big(s,v_n(s),D_{0^+}^p v_n(s)\big)\,ds \\
&\quad + \frac{x_1 - \int_0^\infty f(s,v_n(s),D_{0^+}^p v_n(s))\,ds}{\Gamma(\alpha)} t^{\alpha-1} \\
&\quad + \frac{x_0 + bx_1 - \int_0^\infty f(s,v_n(s),D_{0^+}^p v_n(s))\,ds}{a} t^{\alpha-2}, \\
(F_2 u_n)(t) &= \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\alpha-1} g\big(s,u_n(s),D_{0^+}^q u_n(s)\big)\,ds \\
&\quad + \frac{y_1 - \int_0^\infty g(s,u_n(s),D_{0^+}^p u_n(s))\,ds}{\Gamma(\alpha)} t^{\beta-1} \\
&\quad + \frac{y_0 + dy_1 - \int_0^\infty g(s,u_n(s),D_{0^+}^p u_n(s))\,ds}{c} t^{\beta-2}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{t^{2-\alpha}}{1+t^{\sigma+2}} \left|(F_1 v_n)(t) - (F_1 v_0)(t)\right| \\
&\quad \le \frac{2}{\Gamma(\alpha)} C \int_0^1 (1-w)^{\alpha-1} w^{\sigma_0}\,dw + 2Ar \int_0^1 (1-w)^{\alpha-1} w^{\sigma_1}\,dw + 2Br \int_0^1 (1-w)^{\alpha-1} w^{\sigma_2}\,dw \\
&\qquad + \left(\frac{1}{\Gamma(\alpha)} + \frac{1}{a}\right)\left[\frac{C}{k_0^{\sigma_0+1}}\Gamma(\sigma_0+1) + \frac{Ar}{k_1^{\sigma_1+1}}\Gamma(\sigma_1+1) + \frac{Br}{k_2^{\sigma_2+1}}\Gamma(\sigma_2+1)\right] < +\infty.
\end{aligned}$$

It follows from the Lebesgue dominated convergence theorem that

$$\lim_{n\to\infty} \frac{t^{2-\alpha}}{1+t^{\sigma+2}} \left|(F_1 v_n)(t) - (F_1 v_0)(t)\right| = 0.$$

Furthermore, we have

$$\begin{aligned}
D_{0^+}^q (F_1 v_n)(t) &= \frac{\int_0^t (t-s)^{\alpha-q-1} f(s,v_n(s),D_{0^+}^p v_n(s))\,ds}{\Gamma(\alpha-q)} \\
&\quad + \left(x_1 - \int_0^\infty f\big(s,v_n(s),D_{0^+}^p v_n(s)\big)\,ds\right) \frac{t^{\alpha-q-1}}{\Gamma(\alpha-q)} \\
&\quad + \left(x_0 + bx_1 - \int_0^\infty f\big(s,v_n(s),D_{0^+}^p v_n(s)\big)\,ds\right) \frac{\Gamma(\alpha-1)t^{\alpha-q-2}}{\Gamma(\alpha-q-1)}.
\end{aligned}$$

In a similar manner, we find that

$$\lim_{n\to\infty} \frac{t^{2+q-\alpha}}{1+t^{\sigma+2}} \left|D_{0^+}^q (F_1 v_n)(t) - D_{0^+}^q (F_1 v_0)(t)\right| = 0.$$

Thus,

$$\lim_{n\to\infty}\frac{t^{2-\beta}}{1+t^{\sigma+2}}\big|(F_2u_n)(t)-(F_2u_0)(t)\big|=0,$$
$$\lim_{n\to\infty}\frac{t^{2+p-\eta}}{1+t^{\sigma+2}}\big|D_{0^+}^p(F_2u_n)(t)-D_{0^+}^p(F_2u_0)(t)\big|=0.$$

Hence we get

$$\lim_{n\to\infty}(F_1v_n,F_2u_n)=(F_1v_0,F_2u_0),$$

which shows that F is continuous.

In order to show that F maps bounded sets of $X\times Y$ to relatively compact sets of $X\times Y$, it suffices to prove that both F_1 and F_2 map bounded sets to relatively compact sets.

Recall $W\subset X$ is relatively compact if

(i) it is bounded,
(ii) both $\frac{t^{2-\alpha}}{1+t^{\sigma+2}}W$ and $\frac{t^{2+q-\alpha}}{1+t^{\sigma+2}}W$ are equicontinuous on any closed subinterval $[a,b]$ of $(0,\infty)$,
(iii) both $\frac{t^{2-\alpha}}{1+t^{\sigma+2}}W$ and $\frac{t^{2+q-\alpha}}{1+t^{\sigma+2}}W$ are equiconvergent at $t=0$,
(iv) both $\frac{t^{2-\alpha}}{1+t^{\sigma+2}}W$ and $\frac{t^{2+q-\alpha}}{1+t^{\sigma+2}}W$ are equiconvergent at $t=\infty$.

$W\subset Y$ is relatively compact if

(i) it is bounded,
(ii) both $\frac{t^{2-\beta}}{1+t^{\sigma+2}}W$ and $\frac{t^{2+p-\beta}}{1+t^{\sigma+2}}W$ are equicontinuous on any closed subinterval $[a,b]$ of $(0,\infty)$,
(iii) both $\frac{t^{2-\beta}}{1+t^{\sigma+2}}W$ and $\frac{t^{2+p-\beta}}{1+t^{\sigma+2}}W$ are equiconvergent at $t=0$,
(iv) both $\frac{t^{2-\beta}}{1+t^{\sigma+2}}W$ and $\frac{t^{2+p-\beta}}{1+t^{\sigma+2}}W$ are equiconvergent at $t=\infty$.

Step 3. We prove that both $F_1:\Omega_1\to Y$ and $F_2:\Omega_2\to X$ are equicontinuous on a finite closed interval of $(0,\infty)$.

Let $\Omega_1\subset Y$ and $\Omega_2\subset X$ be bounded sets. Then there exists $r>0$ such that

$$\begin{aligned}\big\|(u,v)\big\| &= \max\big\{\|u\|_X,\|v\|_Y\big\}\\ &= \max\bigg\{\sup_{t\in(0,\infty}\frac{t^{2-\alpha}}{1+t^{\sigma+2}}\big|u(t)\big|,\sup_{t\in(0,\infty}\frac{t^{2-\beta}}{1+t^{\sigma+2}}\big|v(t)\big|,\\ &\quad\sup_{t\in(0,\infty)}\frac{t^{2+q-\alpha}}{1+t^{\sigma+2}}\big|D_{0^+}^q u(t)\big|,\sup_{t\in(0,\infty)}\frac{t^{2+p-\beta}}{1+t^{\sigma+2}}\big|D_{0^+}^p v(t)\big|\bigg\}\\ &\le r<\infty,\quad u\in\Omega_2,v\in\Omega_1.\end{aligned}$$

Then (H) implies that

$$\big|f\big(t,v(t),D_{0^+}^p v(t)\big)\big|\le Ct^{\sigma_0}e^{-k_0t}+Art^{\sigma_1}e^{-k_1t}+Brt^{\sigma_2}e^{-k_2t}$$

and

$$\big|g\big(t,u(t),D_{0^+}^q u(t)\big)\big|\le C_1t^{\mu_0}e^{-l_0t}+A_1rt^{\mu_1}e^{-l_1t}+B_1rt^{\mu_2}e^{-l_2t}.$$

For $[a,b] \subset (0,\infty)$ with $t_1, t_2 \in [a,b]$ with $t_1 < t_2$ and $v \in \Omega_1$, we have

$$\begin{aligned}
&\left|\frac{t_1^{2-\alpha}}{1+t_1^{\sigma+2}}(F_1 v)(t_1) - \frac{t_2^{2-\alpha}}{1+t_2^{\sigma+2}}(F_1 v)(t_2)\right| \\
&\quad \le \frac{1}{\Gamma(\alpha)}\left|\frac{t_1^{2-\alpha}}{1+t_1^{\sigma+2}}\int_0^{t_1}(t_1-s)^{\alpha-1} f\bigl(s, v(s), D_{0^+}^p v(s)\bigr)\,ds \right.\\
&\qquad \left. - \frac{t_2^{2-\alpha}}{1+t_2^{\sigma+2}}\int_0^{t_2}(t_2-s)^{\alpha-1} f\bigl(s, v(s), D_{0^+}^p v(s)\bigr)\,ds\right| \\
&\qquad + \frac{|x_1 - \int_0^\infty f(s, v(s), D_{0^+}^p v(s))\,ds|}{\Gamma(\alpha)}\left|\frac{t_1}{1+t_1^{\sigma+2}} - \frac{t_2}{1+t_2^{\sigma+2}}\right| \\
&\qquad + \left|x_0 + bx_1 - \int_0^\infty f\bigl(s, v(s), D_{0^+}^p v(s)\bigr)\,ds\right|\left|\frac{1}{1+t_1^{\sigma+2}} - \frac{1}{1+t_2^{\sigma+2}}\right|.
\end{aligned}$$

Since $|a^\nu - b^\nu| \le |a-b|^\nu$ for all $a, b \ge 0$ and $\nu \in (0,1)$, therefore, we get

$$\begin{aligned}
&\left|\frac{t_1^{2-\alpha}}{1+t_1^{\sigma+2}}\int_0^{t_1}(t_1-s)^{\alpha-1} f\bigl(s, v(s), D_{0^+}^p v(s)\bigr)\,ds \right.\\
&\qquad \left. - \frac{t_2^{2-\alpha}}{1+t_2^{\sigma+2}}\int_0^{t_2}(t_2-s)^{\alpha-1} f\bigl(s, v(s), D_{0^+}^p v(s)\bigr)\,ds\right| \\
&\quad \le \left|\frac{t_1^{2-\alpha}}{1+t_1^{\sigma+2}} - \frac{t_2^{2-\alpha}}{1+t_2^{\sigma+2}}\right| \\
&\qquad \times \biggl[Cb^{\alpha+\sigma_0}\int_0^1 (1-w)^{\alpha-1}w^{\sigma_0}\,dw + Arb^{\alpha+\sigma_1}\int_0^1 (1-w)^{\alpha-1}w^{\sigma_1}\,dw \\
&\qquad + Brb^{\alpha+\sigma_2}\int_0^1 (1-w)^{\alpha-1}w^{\sigma_2}\,dw\biggr] \\
&\qquad + Cb^{\alpha+\sigma_0}\int_{t_1/t_2}^1 (1-w)^{\alpha-1}w^{\sigma_0}\,dw + Arb^{\alpha+\sigma_1}\int_{t_1/t_2}^1 (1-w)^{\alpha-1}w^{\sigma_1}\,dw \\
&\qquad + Brb^{\alpha+\sigma_2}\int_{t_1/t_2}^1 (1-w)^{\alpha-1}w^{\sigma_2}\,dw + |t_1-t_2|^{\alpha-1}\left[\frac{Cb^{\sigma_0+1}}{\sigma_0+1} + \frac{Arb^{\sigma_1+1}}{\sigma+1} + \frac{Brb^{\sigma_2+1}}{\sigma_2+1}\right] \\
&\quad \to 0 \quad \text{uniformly in } \Omega_1 \text{ as } t_2 \to t_1.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\left|x_1 - \int_0^\infty f\bigl(s, v(s), D_{0^+}^p v(s)\bigr)\,ds\right| \\
&\quad \le |x_1| + \frac{C}{k_0^{\sigma_0+1}}\Gamma(\sigma_0+1) + \frac{Ar}{k_1^{\sigma_1+1}}\Gamma(\sigma_1+1) + \frac{Br}{k_2^{\sigma_2+1}}\Gamma(\sigma_2+1).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left|x_0 + bx_1 - \int_0^\infty f\bigl(s, v(s), D_{0^+}^p v(s)\bigr)\,ds\right| \\
&\quad \le |x_0 + bx_1| + \frac{C\Gamma(\sigma_0+1)}{k_0^{\sigma_0+1}} + \frac{Ar\Gamma(\sigma_1+1)}{k_1^{\sigma_1+1}} + \frac{Br\Gamma(\sigma_2+1)}{k_2^{\sigma_2+1}}.
\end{aligned}$$

Hence

$$\left|\frac{t_1^{2-\alpha}}{1+t_1^{\sigma+2}}(F_1v)(t_1)-\frac{t_2^{2-\alpha}}{1+t_2^{\sigma+2}}(F_1v)(t_2)\right|\to 0 \quad \text{uniformly in } \Omega_1 \text{ as } t_2\to t_1. \tag{11}$$

On the other hand, we have

$$\begin{aligned}
&\left|\frac{t_1^{2+q-\alpha}}{1+t_1^{\sigma+2}}D_{0^+}^q(F_1v)(t_1)-\frac{t_2^{2+q-\alpha}}{1+t_2^{\sigma+2}}D_{0^+}^q(F_1v)(t_2)\right| \\
&\quad\le \frac{1}{\Gamma(\alpha-q)}\left|\frac{t_1^{2+q-\alpha}}{1+t_1^{\sigma+2}}\int_0^{t_1}(t_1-s)^{\alpha-q-1}f\bigl(s,v(s),D_{0^+}^p v(s)\bigr)\,ds\right. \\
&\qquad\left.-\frac{t_2^{2+q-\alpha}}{1+t_2^{\sigma+2}}\int_0^{t_2}(t_2-s)^{\alpha-q-1}f\bigl(s,v(s),D_{0^+}^p v(s)\bigr)\,ds\right| \\
&\qquad+\frac{|x_1-\int_0^\infty f(s,v(s),D_{0^+}^p v(s))\,ds|}{\Gamma(\alpha-q)}\left|\frac{t_1}{1+t_1^{\sigma+2}}-\frac{t_2}{1+t_2^{\sigma+2}}\right| \\
&\qquad+\left|x_0+bx_1-\int_0^\infty f\bigl(s,v(s),D_{0^+}^p v(s)\bigr)\,ds\right|\frac{\Gamma(\alpha-1)}{|\Gamma(\alpha-q-1)|}\left|\frac{1}{1+t_1^{\sigma+2}}-\frac{1}{1+t_2^{\sigma+2}}\right|.
\end{aligned}$$

Let

$$M_0=\max\bigl\{a^{\alpha-q+\sigma_0},b^{\alpha-q+\sigma_0},a^{\alpha-q+\sigma_1},b^{\alpha-q+\sigma_1},a^{\alpha-q+\sigma_2},b^{\alpha-q+\sigma_2}\bigr\}.$$

Note that $q\in(\alpha-1,\alpha-1/2)$ and $|a^\nu-b^\nu|\le|a-b|^\nu$ for all $a,b\ge 0$ and $\nu\in(0,1)$. Note $\sigma>\max\{q-\alpha,p-\beta\}$. Then

$$\begin{aligned}
&\left|\frac{t_1^{2+q-\alpha}}{1+t_1^{\sigma+2}}\int_0^{t_1}(t_1-s)^{\alpha-q-1}f\bigl(s,v(s),D_{0^+}^p v(s)\bigr)\,ds\right. \\
&\qquad\left.-\frac{t_2^{2+q-\alpha}}{1+t_2^{\sigma+2}}\int_0^{t_2}(t_2-s)^{\alpha-q-1}f\bigl(s,v(s),D_{0^+}^p v(s)\bigr)\,ds\right| \\
&\quad\le M_0\left|\frac{t_1^{2+q-\alpha}}{1+t_1^{\sigma+2}}-\frac{t_2^{2+q-\alpha}}{1+t_2^{\sigma+2}}\right| \\
&\qquad\times\bigl[C\mathbf{B}(\alpha-q,\sigma_0+1)+Ar\mathbf{B}(\alpha-q,\sigma_1+1)+Br\mathbf{B}(\alpha-q,\sigma_2+1)\bigr] \\
&\qquad+M_0\left[C\int_{t_1/t_2}^1(1-w)^{\alpha-q-1}w^{\sigma_0}\,dw+Ar\int_{t_1/t_2}^1(1-w)^{\alpha-q-1}w^{\sigma_1}\,dw\right. \\
&\qquad\left.+Br\int_{t_1/t_2}^1(1-w)^{\alpha-q-1}w^{\sigma_2}\,dw\right] \\
&\qquad+\left|t_2^{\alpha+\sigma_0-q}-t_1^{\alpha+\sigma_0-q}\right|\bigl[C\mathbf{B}(\alpha-q,\sigma_0+1)+Ar\mathbf{B}(\alpha-q,\sigma_1+1)+Br\mathbf{B}(\alpha-q,\sigma_2+1)\bigr] \\
&\qquad+M_0\left[C\int_{t_1/t_2}^1(1-w)^{\alpha-q-1}w^{\sigma_0}\,dw+Ar\int_{t_1/t_2}^1(1-w)^{\alpha-q-1}w^{\sigma_1}\,dw\right. \\
&\qquad\left.+Br\int_{t_2/t_2}^1(1-w)^{\alpha-q-1}w^{\sigma_2}\,dw\right] \\
&\quad\to 0 \quad \text{uniformly in } \Omega_1 \text{ as } t_2\to t_1.
\end{aligned}$$

Similarly, it can be shown that both

$$\frac{|x_1 - \int_0^\infty f(s, v(s), D_{0^+}^p v(s))\, ds|}{\Gamma(\alpha - q)}$$

and

$$\left| x_0 + bx_1 - \int_0^\infty f\big(s, v(s), D_{0^+}^p v(s)\big)\, ds \right| \frac{\Gamma(\alpha - 1)}{|\Gamma(\alpha - q - 1)|}$$

are uniformly bounded. Then

$$\begin{aligned} &\left| \frac{t_1^{2+q-\alpha}}{1 + t_1^{\sigma+2}} \int_0^{t_1} (t_1 - s)^{\alpha-q-1} f\big(s, v(s), D_{0^+}^p v(s)\big)\, ds \right. \\ &\quad \left. - \frac{t_2^{2+q-\alpha}}{1 + t_2^{\sigma+2}} \int_0^{t_2} (t_2 - s)^{\alpha-q-1} f\big(s, v(s), D_{0^+}^p v(s)\big)\, ds \right| \\ &\to 0 \quad \text{uniformly in } \Omega_1 \text{ as } t_2 \to t_1. \end{aligned} \tag{12}$$

From (11) and (12), we infer that $F_1 : \Omega_1 \to Y$ is equicontinuous on a finite closed interval of $(0, \infty)$. Similarly, we can show that $F_2 : \Omega_2 \to X$ is equicontinuous on a finite closed interval on $(0, \infty)$.

Step 4. Now we prove that both $F_1 : \Omega_1 \to Y$ and $F_2 : \Omega_2 \to X$ are equiconvergent as $t \to 0$. By the assumption (H), we have

$$\begin{aligned} &\left| \frac{t^{2-\alpha}}{1 + t^{\sigma+2}} (F_1 y)(t) - \frac{x_0 + bx_1 - \int_0^\infty f(s, y(s), D_{0^+}^p y(s))\, ds}{a} \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \frac{1}{1 + t^{\sigma+2}} \left[Ct^{2+\sigma_0} \int_0^1 (1 - w)^{\alpha-1} w^{\sigma_0}\, dw \right. \\ &\quad \left. + At^{2+\sigma_1} \int_0^1 (1 - w)^{\alpha-1} w^{\sigma_1}\, dw + Bt^{2+\sigma_2} \int_0^1 (1 - w)^{\alpha-1} w^{\sigma_2}\, dw \right] \\ &\quad + \frac{|x_1|}{\Gamma(\alpha)} \frac{t}{1 + t^{\sigma+2}} + |x_0| \frac{t^{\sigma+2}}{1 + t^{\sigma+2}} \\ &\to 0 \quad \text{uniformly in } \Omega_1 \text{ as } t \to 0. \end{aligned}$$

Furthermore, for $\sigma_i \in (-1, \sigma)$, we have

$$\begin{aligned} &\left| \frac{t^{2+q-\alpha}}{1 + t^{\sigma+2}} D_{0^+}^q (F_1 v)(t) - \frac{x_0 + bx_1 - \int_0^\infty f(s, v(s), D_{0^+}^p v(s))\, ds}{a} \frac{\Gamma(\alpha - 1)}{\Gamma(\alpha - q - 1)} \right| \\ &\leq \frac{1}{\Gamma(\alpha - q)} \left[C \frac{t^{2+\sigma_0}}{1 + t^{\sigma+2}} \mathbf{B}(\alpha - q, \sigma_0 + 1) + A \frac{t^{2+\sigma_1}}{1 + t^{\sigma+2}} \mathbf{B}(\alpha - q, \sigma_1 + 1) \right. \\ &\quad \left. + B \frac{t^{2+\sigma_2}}{1 + t^{\sigma+2}} \mathbf{B}(\alpha - q, \sigma_2 + 1) \right] \\ &\quad + \frac{|x_1| + [C\Gamma(\sigma_0 + 1) + A\Gamma(\sigma_1 + 1) + B\Gamma(\sigma_2 + 1)]}{\Gamma(\alpha - q)} \frac{t}{1 + t^{\sigma+2}} \\ &\quad + \frac{|x_0 + bx_1| + [C\Gamma(\sigma_0 + 1) + A\Gamma(\sigma_1 + 1) + B\Gamma(\sigma_2 + 1)]}{a} \frac{\Gamma(\alpha - 1)}{\Gamma(\alpha - q - 1)} \frac{t^{\sigma+2}}{1 + t^{\sigma+2}} \\ &\to 0 \quad \text{uniformly in } \Omega_1 \text{ as } t \to 0. \end{aligned}$$

Hence $F_1 : \Omega_1 \to Y$ is equiconvergent as $t \to 0$. Similarly, we can prove that $F_2 : \Omega_2 \to Y$ is equiconvergent as $t \to 0$.

Step 5. Finally, we show that both $F_1 : \Omega_1 \to Y$ and $F_2 : \Omega_2 \to X$ are equiconvergent as $t \to \infty$. By the assumption (H), we have

$$\begin{aligned}\left|\frac{t^{2-\alpha}}{1+t^{\sigma+2}}(F_1 v)(t)\right| \le{}& \frac{r}{\Gamma(\alpha)}\left[\frac{Ct^{2+\sigma_0}}{1+t^{\sigma+2}}\int_0^1 (1-w)^{\alpha-1}w^{\sigma_0}\,dw\right.\\&\left.+\frac{At^{2+\sigma_1}}{1+t^{\sigma+2}}\int_0^1 (1-w)^{\alpha-1}w^{\sigma_1}\,dw+\frac{Bt^{2+\sigma_2}}{1+t^{\sigma+2}}\int_0^1 (1-w)^{\alpha-1}w^{\sigma_2}\,dw\right]\\&+\frac{|x_1|}{\Gamma(\alpha)}\frac{t}{1+t^{\sigma+2}}+\frac{|x_0|}{1+t^{\sigma+2}}\\ \to{}& 0 \quad \text{uniformly in } \Omega_1 \text{ as } t\to\infty.\end{aligned}$$

Furthermore, for $\sigma_i \in (-1, \sigma)$, we have

$$\begin{aligned}&\left|\frac{t^{2+q-\alpha}}{1+t^{\sigma+2}}D_{0^+}^q(F_1 v)(t)\right|\\&\quad\le \frac{r}{\Gamma(\alpha-q)}\left[C\frac{t^{2+\sigma_0}}{1+t^{\sigma+2}}\mathbf{B}(\alpha-q,\sigma_0+1)+A\frac{t^{2+\sigma_1}}{1+t^{\sigma+2}}\mathbf{B}(\alpha-q,\sigma_1+1)\right.\\&\qquad\left.+B\frac{t^{2+\sigma_2}}{1+t^{\sigma+2}}\mathbf{B}(\alpha-q,\sigma_2+1)\right]\\&\qquad+\frac{|x_1|+r[C\Gamma(\sigma_0+1)+A\Gamma(\sigma_1+1)+B\Gamma(\sigma_2+1)]}{\Gamma(\alpha-q)}\frac{t}{1+t^{\sigma+2}}\\&\qquad+\frac{|x_0+bx_1|+r[C\Gamma(\sigma_0+1)+A\Gamma(\sigma_1+1)+B\Gamma(\sigma_2+1)]}{a}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-1)}\frac{1}{1+t^{\sigma+2}}\\&\quad\to 0 \quad \text{uniformly in } \Omega_1 \text{ as } t\to\infty.\end{aligned}$$

Hence $F_1 : \Omega_1 \to Y$ is equiconvergent as $t \to \infty$. Similarly, we can prove that $F_2 : \Omega_2 \to Y$ is equiconvergent as $t \to \infty$.

Thus, F_1 and F_2 are completely continuous. Hence F is completely continuous.

Similarly, we can show that the results hold if (G) holds. These complete the proofs. □

3 Main results

In this section, we present the main results of the paper. For the sake of convenience, let us set

$$\begin{aligned}M_1 = \max\Bigg\{&\frac{A}{\Gamma(\alpha)}\mathbf{B}(\alpha,\sigma_1+1)+\frac{B}{\Gamma(\alpha)}\mathbf{B}(\alpha,\sigma_2+1)\\&+\left(\frac{1}{\Gamma(\alpha)}+\frac{1}{a}\right)\left(\frac{A}{k_1^{\sigma_1+1}}\Gamma(\sigma_1+1)+\frac{B}{k_2^{\sigma_2+1}}\Gamma(\sigma_2+1)\right),\\&\frac{A}{\Gamma(\alpha-q)}\mathbf{B}(\alpha-q,\sigma_1+1)+\frac{B}{\Gamma(\alpha-q)}\mathbf{B}(\alpha-q,\sigma_2+1)\\&+\left(\frac{1}{\Gamma(\alpha-q)}+a\frac{\Gamma(\alpha-1)}{|\Gamma(\alpha-q-1)|}\right)\left(\frac{A}{k_1^{\sigma_1+1}}\Gamma(\sigma_1+1)+\frac{B}{k_2^{\sigma_2+1}}\Gamma(\sigma_2+1)\right)\Bigg\},\end{aligned}$$

$$\begin{aligned} M_2 = \max\biggl\{ & \frac{A_1}{\Gamma(\beta)}\mathbf{B}(\beta,\mu_1+1) + \frac{B_1}{\Gamma(\beta)}\mathbf{B}(\beta,\mu_2+1) \\ & + \biggl(\frac{1}{\Gamma(\beta)} + \frac{1}{c}\biggr)\biggl(\frac{A_1}{l_1^{\mu_1+1}}\Gamma(\mu_1+1) + \frac{B_1}{l_2^{\mu_2+1}}\Gamma(\mu_2+1)\biggr), \\ & \frac{A_1}{\Gamma(\beta-p)}\mathbf{B}(\beta-p,\mu_1+1) + \frac{B_1}{\Gamma(\beta-p)}\mathbf{B}(\beta-p,\mu_2+1) \\ & + \biggl(\frac{1}{\Gamma(\beta-p)} + c\frac{\Gamma(\beta-1)}{|\Gamma(\beta-p-1)|}\biggr)\biggl(\frac{A_1}{l_1^{\mu_1+1}}\Gamma(\mu_1+1) + \frac{B_1}{l_2^{\mu_2+1}}\Gamma(\mu_2+1)\biggr)\biggr\}. \end{aligned}$$

Theorem 3.1 *Suppose that* (H) *holds. Then* (5) *has at least one solution* $(x,y) \in X$ *if*

$$\max\{M_1, M_2\} < 1. \tag{13}$$

Proof Let $X \times Y$ be the Banach space equipped with the norm $\|\cdot\|$ (defined in Section 2). We seek the solutions of (5) by obtaining the fixed point of F in $X \times Y$. Note that F is well defined and completely continuous by Lemma 2.3.

Let

$$\begin{aligned} \Psi(t) &= \frac{C}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\sigma_0}e^{-k_0 s}\,ds + \frac{x_1 - C\int_0^\infty s^{\sigma_0}e^{-k_0 s}\,ds}{\Gamma(\alpha)}t^{\alpha-1} \\ &\quad + \frac{x_0 + bx_1 - C\int_0^\infty s^{\sigma_0}e^{-k_0 s}\,ds}{a}t^{\alpha-2}, \\ \Phi(t) &= \frac{C_1}{\Gamma(\beta)}\int_0^t (t-s)^{\alpha-1}s^{\mu_0}e^{-l_0 s}\,ds + \frac{y_1 - C_1\int_0^\infty s^{\mu_0}e^{-l_0 s}\,ds}{\Gamma(\alpha)}t^{\beta-1} \\ &\quad + \frac{y_0 + dy_1 - C_1\int_0^\infty s^{\mu_0}e^{-l_0 s}\,ds}{c}t^{\beta-2}. \end{aligned}$$

It is easy to show that $(\Psi,\Phi) \in X \times Y$. For $r > 0$, we define

$$M_r = \bigl\{(x,y) \in X \times Y : \bigl\|(x,y) - (\Psi,\Phi)\bigr\| \le r\bigr\}. \tag{14}$$

For $(x,y) \in M_r$, we have $\|(x,y) - (\Psi,\Phi)\| \le r$. Then

$$\begin{aligned} &\bigl\|(x,y)\bigr\| \le \bigl\|(x,y) - (\Psi,\Phi)\bigr\| + \bigl\|(\Psi,\Phi)\bigr\| \le r + \bigl\|(\Psi,\Phi)\bigr\|, \\ &\bigl\|(x,y)\bigr\| = \max\bigl\{\|x\|_X, \|y\|_Y\bigr\} \le r + \bigl\|(\Psi,\Phi)\bigr\|. \end{aligned}$$

Using the condition (H) together with the method employed in Step 1 of the proof of Lemma 2.3, we find that

$$\begin{aligned} &\bigl|f\bigl(t, v_n(t), D_{0^+}^p v_n(t)\bigr) - Cs^{\sigma_0}e^{-k_0 s}\bigr| \\ &\quad \le \bigl[At^{\sigma_1}e^{-k_1 t} + Bt^{\sigma_2}e^{-k_2 t}\bigr]\bigl\|(x,y)\bigr\| \end{aligned}$$

and

$$\bigl|g\bigl(t, u_n(t), D_{0^+}^q u_n(t)\bigr) - C_1 t^{\mu_0}e^{-l_0 t}\bigr| \le \bigl[A_1 t^{\mu_1}e^{-l_1 t} + B_1 t^{\mu_2}e^{-l_2 t}\bigr]\bigl\|(x,y)\bigr\|.$$

Then

$$\begin{aligned}&\frac{t^{2-\alpha}}{1+t^{\sigma+2}}\left|(F_1y)(t)-\Psi(t)\right|\\&\quad\le\left[r+\left\|(\Psi,\Phi)\right\|\right]\left[\frac{A}{\Gamma(\alpha)}\mathbf{B}(\alpha,\sigma_1+1)+\frac{B}{\Gamma(\alpha)}\mathbf{B}(\alpha,\sigma_2+1)\right.\\&\qquad\left.+\left(\frac{1}{\Gamma(\alpha)}+\frac{1}{a}\right)\left(\frac{A}{k_1^{\sigma_1+1}}\Gamma(\sigma_1+1)+\frac{B}{k_2^{\sigma_2+1}}\Gamma(\sigma_2+1)\right)\right].\end{aligned}$$

Furthermore, we have

$$\begin{aligned}&\frac{t^{2+q-\alpha}}{1+t^{\sigma+2}}\left|D_{0^+}^q(F_1y)(t)-D_{0^+}^q\Psi(t)\right|\\&\quad\le\left[r+\left\|(\Psi,\Phi)\right\|\right]\left[\frac{A}{\Gamma(\alpha-q)}\mathbf{B}(\alpha-q,\sigma_1+1)+\frac{B}{\Gamma(\alpha-q)}\mathbf{B}(\alpha-q,\sigma_2+1)\right.\\&\qquad\left.+\left(\frac{1}{\Gamma(\alpha-q)}+a\frac{\Gamma(\alpha-1)}{|\Gamma(\alpha-q-1)|}\right)\left(\frac{A}{k_1^{\sigma_1+1}}\Gamma(\sigma_1+1)+\frac{B}{k_2^{\sigma_2+1}}\Gamma(\sigma_2+1)\right)\right].\end{aligned}$$

Thus, it follows that

$$\|F_1y-\Psi\|_X\le\left[r+\left\|(\Psi,\Phi)\right\|\right]M_1.$$

Similarly, one can obtain

$$\|F_2x-\Phi\|_Y\le\left[r+\left\|(\Psi,\Phi)\right\|\right]M_2.$$

Hence

$$\left\|F(x,y)-(\Psi,\Phi)\right\|\le\left[r+\left\|(\Psi,\Phi)\right\|\right]\max\{M_1,M_2\}.$$

We choose

$$r\ge\frac{\|(\Psi,\Phi)\|\max\{M_1,M_2\}}{1-\max\{M_1,M_2\}}.$$

Then, for $(x,y)\in M_r$, we have

$$\left\|F(x,y)-(\Psi,\Phi)\right\|\le r.$$

Then the Schauder fixed point theorem implies that F has a fixed point $(x,y)\in M_r$, which is a bounded solution of (5). The proof is complete. □

Theorem 3.2 *Suppose that* (G) *holds. Then* (5) *has at least one solution* $(x,y)\in X\times Y$ *if*

$$\frac{\|(\Psi,\Phi)\|^{1-\delta}(\delta-1)^{\delta-1}}{\delta^\delta}\ge\max\{M_1,M_2\}.\tag{15}$$

Proof With Ψ and Φ defined in the proof of Theorem 3.1, it is easy to show that $(\Psi,\Phi) \in X \times Y$. For $(x,y) \in M_r$ (defined in the proof of Theorem 3.1), using (G) and the method of the proof for Theorem 3.1, we find that

$$\left\|F(x,y) - (\Psi,\Phi)\right\| \le \left[r + \left\|(\Psi,\Phi)\right\|\right]^{\delta} \max\{M_1, M_2\}.$$

Let $r = r_0 = \frac{\|(\Psi,\Phi)\|}{\delta-1}$, $\delta > 1$. Then

$$\frac{r_0}{(r_0 + \|(\Psi,\Phi)\|)^{\delta}} = \frac{\|(\Psi,\Phi)\|^{1-\delta}(\delta-1)^{\delta-1}}{\delta^{\delta}} \ge \max\{M_1, M_2\}.$$

Thus, for $(x,y) \in M_{r_0}$, we have

$$\left\|F(x,y) - (\Psi,\Phi)\right\| \le r_0.$$

Hence, we obtain a bounded subset $M_{r_0} \subseteq X \times Y$ such that $T(M_{r_0}) \subseteq M_{r_0}$. In consequence, by the Schauder fixed point theorem, F has a fixed point $(x,y) \in M_{r_0}$. Hence, (x,y) is a bounded solution of (5). This completes the proof. □

4 An example

Consider the fractional boundary value problem given by

$$\begin{cases} D_{0^+}^{\frac{3}{2}} x(t) = Ct^{-\frac{3}{4}} e^{-t} + A\frac{t^{-\frac{1}{2}} e^{-2t}}{1+t^{\frac{3}{2}}} y(t) + B\frac{t^{\frac{9}{10}} e^{-3t}}{1+t^{\frac{3}{2}}} D_{0^+}^{\frac{6}{5}} y(t), & t \in (0,\infty), \\ D_{0^+}^{\frac{7}{4}} y(t) = Ct^{-\frac{3}{4}} e^{-t} + A\frac{t^{-\frac{1}{4}} e^{-2t}}{1+t^{\frac{3}{2}}} x(t) + B\frac{t^{\frac{7}{10}} e^{-3t}}{1+t^{\frac{3}{2}}} D_{0^+}^{\frac{3}{4}} x(t), & t \in (0,\infty), \\ \lim_{t\to 0} t^{2-\alpha} x(t) - \lim_{t\to 0} D_{0^+}^{\alpha-1} x(t) = x_0, \\ \lim_{t\to 0} t^{2-\beta} y(t) - \lim_{t\to 0} D_{0^+}^{\beta-1} x(t) = y_0, \\ \lim_{t\to\infty} D_{0^+}^{\alpha-1} x(t) = x_1, \\ \lim_{t\to\infty} D_{0^+}^{\beta-1} x(t) = y_1, \end{cases} \tag{16}$$

where $A, B, C > 0$ and $x_0, x_1, y_0, y_1 \in R$ are constants, $\alpha = \frac{3}{2}$, $\beta = \frac{7}{4}$, $p = \frac{6}{5}$, $q = \frac{3}{4}$, $a = b = c = d = 1$ and

$$f(t,x,y) = Ct^{-\frac{3}{4}} e^{-t} + A\frac{t^{-\frac{1}{2}} e^{-2t}}{1+t^{\frac{3}{2}}} x + B\frac{t^{\frac{9}{10}} e^{-3t}}{1+t^{\frac{3}{2}}} y,$$

$$g(t,x,y) = Ct^{-\frac{3}{4}} e^{-t} + A\frac{t^{-\frac{1}{4}} e^{-2t}}{1+t^{\frac{3}{2}}} x + B\frac{t^{\frac{7}{10}} e^{-3t}}{1+t^{\frac{3}{2}}} y.$$

Note that $p \in (\beta-1, \beta)$ and $q \in (\alpha-1, \alpha)$.

Choose $\sigma = -\frac{1}{2} > \max\{q-\alpha, p-\beta\}$, $\sigma_0 = \mu_0 = -\frac{3}{4}$, $\sigma_1 = \mu_1 = -\frac{3}{4}$, $\sigma_2 = \mu_2 = -\frac{11}{20}$, $k_0 = l_0 = 1$, $k_1 = l_1 = 2$, $k_2 = l_2 = 3$. One sees that $\sigma_i, \mu_i \in (-1, \sigma)$, $k_i > 0$, $l_i > 0$ $(i = 0,1,2)$.

Thus,

$$f\left(t, \frac{1+t^{\frac{3}{2}}}{t^{\frac{1}{4}}} x, \frac{1+t^{\frac{3}{2}}}{t^{\frac{29}{20}}} y\right) = Ct^{-\frac{3}{4}} e^{-t} + At^{-\frac{3}{4}} e^{-2t} x + Bt^{-\frac{11}{20}} e^{-3t} y,$$

$$g\left(t, \frac{1+t^{\frac{3}{2}}}{t^{\frac{1}{2}}} x, \frac{1+t^{\frac{3}{2}}}{t^{\frac{5}{4}}} y\right) = Ct^{-\frac{3}{4}} e^{-t} + At^{-\frac{3}{4}} e^{-2t} x + Bt^{-\frac{11}{20}} e^{-3t} y.$$

It is easy to show that (H) holds. By direct computation, we get

$$\begin{aligned} M_1 = \max\biggl\{ & \frac{\mathbf{B}(3/2,1/4)}{\Gamma(3/2)}A + \frac{\mathbf{B}(3/2,9/20)}{\Gamma(3/2)}B + \frac{1+\Gamma(3/2)}{\Gamma(3/2)}\left(\frac{\Gamma(1/4)}{2^{1/4}}A + \frac{\Gamma(9/20)}{3^{9/20}}B\right), \\ & \frac{\mathbf{B}(3/4,1/4)}{\Gamma(3/4)}A + \frac{\mathbf{B}(3/4,9/20)}{\Gamma(3/4)}B \\ & + \left(\frac{1}{\Gamma(3/4)} + \frac{\Gamma(1/2)}{|\Gamma(-1/4)|}\right)\left(\frac{\Gamma(1/4)}{2^{1/4}}A + \frac{\Gamma(9/20)}{3^{9/20}}B\right)\biggr\}, \\ M_2 = \max\biggl\{ & \frac{\mathbf{B}(7/4,1/4)}{\Gamma(7/4)}A + \frac{\mathbf{B}(7/4,9/20)}{\Gamma(7/4)}B + \frac{1+\Gamma(7/4)}{\Gamma(7/4)}\left(\frac{\Gamma(1/4)}{2^{1/4}}A + \frac{\Gamma(9/20)}{3^{9/20}}B\right), \\ & \frac{\mathbf{B}(11/20,1/4)}{\Gamma(11/20)}A + \frac{\mathbf{B}(11/20,9/20)}{\Gamma(11/20)}B \\ & + \left(\frac{1}{\Gamma(11/20)} + \frac{\Gamma(3/4)}{|\Gamma(-9/20)|}\right)\left(\frac{\Gamma(1/4)}{2^{1/4}}A + \frac{\Gamma(9/20)}{3^{9/20}}B\right)\biggr\}. \end{aligned}$$

Thus, Theorem 3.1 applies and BVP (16) has at least one solution $(x,y) \in X \times Y$ if $\max\{M_1, M_2\} < 1$. This solution satisfies that

$$\frac{t^{\frac{1}{2}}}{1+t^{\frac{3}{2}}}\bigl|x(t)\bigr|, \qquad \frac{t^{\frac{1}{4}}}{1+t^{\frac{3}{2}}}\bigl|y(t)\bigr|, \qquad \frac{t^{\frac{5}{4}}}{1+t^{\frac{3}{2}}}\bigl|D_{0^+}^{\frac{3}{4}}x(t)\bigr|, \qquad \frac{t^{\frac{29}{20}}}{1+t^{\frac{3}{2}}}\bigl|D_{0^+}^{\frac{6}{5}}y(t)\bigr|$$

are bounded on $(0,\infty)$.

Remark 4.1 It is easy to see that $\max\{M_1, M_2\} < 1$ holds for sufficiently small $A > 0$ and $B > 0$. One notes that $\alpha = \frac{3}{2}$, $\beta = \frac{7}{4}$, $p = \frac{6}{5}$, $q = \frac{3}{4}$ in the mentioned example. It is easy to see that $\alpha - q \geq 1$ and $\beta - p \geq 1$ do not hold. Hence theorems in [34, 35] cannot be applied to solve this example.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors, YL, BA and RPA, contributed to each part of this study equally and read and approved the final version of the manuscript.

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