○ **Advances in Difference Equations**
a SpringerOpen Journal

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On the reciprocal sums of the generalized Fibonacci sequences

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Abstract

The Fibonacci sequence has been generalized in many ways. One of them is defined by the relation $u_n = au_{n-1} + u_{n-2}$ if n is even, $u_n = bu_{n-1} + u_{n-2}$ if n is odd, with initial values $u_0 = 0$ and $u_1 = 1$, where a and b are positive integers. In this paper, we consider the reciprocal sum of u_n and then establish some identities relating to $\|(\sum_{k=n}^{\infty} \frac{1}{u_k})^{-1}\|$, where $\|x\|$ denotes the nearest integer to x.

MSC: Primary 11B39

Keywords: generalized Fibonacci sequence; infinite sum; reciprocal sum

1 Introduction

For any integer $n \geq 0$, the well-known Fibonacci sequence F_n is defined by the second-order linear recurrence sequence $F_{n+2} = F_{n+1} + F_n$, where $F_0 = 0$ and $F_1 = 1$. The Fibonacci sequence has been generalized in many ways, for example, by changing the initial values, by changing the recurrence relation, and so on. Edson and Yayenie [1] defined a further generalized Fibonacci sequence u_n depending on two real parameters used in a non-linear recurrence relation, namely,

$$u_n = \begin{cases} au_{n-1} + u_{n-2} & \text{if } n \text{ is even and } n \geq 2, \\ bu_{n-1} + u_{n-2} & \text{if } n \text{ is odd and } n \geq 1, \end{cases} \tag{1}$$

with initial values $u_0 = 0$ and $u_1 = 1$, where a, b are positive integers. This new sequence is actually a family of sequences where each new choice of a and b produces a distinct sequence. When $a = b = 1$, we have the classical Fibonacci sequence and when $a = b = 2$, we obtain the Pell numbers. Even further, if we set $a = b = k$ for some positive integer k, we obtain the k-Fibonacci numbers.

Various properties of the Fibonacci numbers and related sequences have been studied by many authors, see [2–4]. Recently, Ohtsuka and Nakamura [5] studied the partial infinite sums of reciprocal Fibonacci numbers and proved that

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2} & \text{if } n \text{ is even and } n \geq 2, \\ F_{n-2} - 1 & \text{if } n \text{ is odd and } n \geq 1, \end{cases}$$

where $\lfloor x \rfloor$ (the floor function) denotes the greatest integer less than or equal to x.

Some related works can also be found in [6–13]. In particular, in [13], the authors studied a problem which is a little different from that of [5], namely that of determining the

nearest integer to $(\sum_{k=n}^{\infty} \frac{1}{v_k})^{-1}$. Specifically, suppose that $\|x\| = \lfloor x + \frac{1}{2} \rfloor$ (the nearest integer function), and $\{v_n\}_{n \geq 0}$ is an integer sequence satisfying the recurrence formula

$$v_n = a_1 v_{n-1} + a_2 v_{n-2} + \cdots + a_s v_{n-s} \quad (s \geq 2)$$

for any positive integer $a_1, a_2, \ldots, a_m$, with the initial conditions $v_0 \geq 0$, $v_k \in \mathbf{N}$, $1 \leq k \leq s - 1$. Then, provided $a_1 \geq a_2 \geq \cdots \geq a_m \geq 1$, we can conclude that there exists a positive integer n_0 such that

$$\left\| \left(\sum_{k=n}^{\infty} \frac{1}{v_k} \right)^{-1} \right\| = v_n - v_{n-1}$$

for all $n > n_0$.

Because the Fibonacci sequence has been generalized to a higher-order recursive sequence, any study on linear recursive sequences has little significance in this context, and we have to consider other non-linear recursive sequences. The main purpose of this paper is concerned with finding expressions for

$$\left\| \left(\sum_{k=n}^{\infty} \frac{1}{u_k} \right)^{-1} \right\|.$$

In fact, this problem is difficult because each item of this sequence relies on the previous relation. In order to resolve the question, we consider the reciprocal sums in two directions: on the one hand to the subsequence u_{pk+q} and on the other to the product form $u_k u_{k+2c+1}$, where p, q, c are non-negative integers and $p \geq 2$. The results are as follows.

Theorem 1 *Let $\{u_n\}$ be a second-order sequence defined by* (1). *For any even $p \geq 2$ and non-negative integer $q < p$, there exists a positive integer n_1 such that*

$$\left\| \left(\sum_{k=n}^{\infty} \frac{1}{u_{pk+q}} \right)^{-1} \right\| = u_{pn+q} - u_{pn-p+q}$$

for all $n \geq n_1$.

Theorem 2 *Let $\{u_n\}$ be a second-order sequence defined by* (1). *For any integer $c \geq 0$, there exists a positive integer n_2 such that*

$$\left\| \left(\sum_{k=n}^{\infty} \frac{a^k b^{k+2c+1}}{u_k u_{k+2c+1}} \right)^{-1} - \left(\frac{u_n u_{n+2c+1}}{a^n b^{n+2c+1}} - \frac{u_{n-1} u_{n+2c}}{a^{n-1} b^{n+2c}} \right) \right\| = 0$$

for all $n \geq n_2$.

Open problem In the light of our investigation, for any positive integer $s \geq 2$ and l, whether there exist identities for

$$\left(\sum_{k=n}^{\infty} \frac{1}{u_k^s} \right)^{-1} \quad \text{and} \quad \left(\sum_{k=n}^{\infty} \frac{1}{u_k u_{k+l}} \right)^{-1}$$

represent two interesting, albeit challenging, open problems.

2 Proofs of the theorems

We need the following lemma.

Lemma (Generalized Binet's formula) *The terms of the generalized Fibonacci sequence u_n are given by*

$$u_n = \frac{a^{2\lfloor \frac{n}{2} \rfloor - n + 1}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \cdot \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where $\alpha = \frac{ab + \sqrt{a^2 b^2 + 4ab}}{2}$, $\beta = \frac{ab - \sqrt{a^2 b^2 + 4ab}}{2}$.

Proof See Theorem 2 of [4]. □

Proof of Theorem 1 From the geometric series as $\epsilon \to 0$, we have

$$\frac{1}{1 \pm \epsilon} = 1 \mp \epsilon + O(\epsilon^2) = 1 + O(\epsilon).$$

From Lemma and the identity $\alpha\beta = -ab$, we have

$$u_{pk+q} = \begin{cases} \frac{\alpha^{pk+q} - \beta^{pk+q}}{a^{\frac{pk+q-2}{2}} b^{\frac{pk+q}{2}} (\alpha-\beta)} & \text{if } q \text{ is even (so that } pk + q \text{ is even)}, \\ \frac{\alpha^{pk+q} - \beta^{pk+q}}{ab^{\frac{pk+q-1}{2}} (\alpha-\beta)} & \text{if } q \text{ is odd (so that } pk + q \text{ is odd)}. \end{cases}$$

Let

$$A = \begin{cases} \frac{1}{a^{\frac{pk+q-2}{2}} b^{\frac{pk+q}{2}} (\alpha-\beta)} & \text{if } q \text{ is even}, \\ \frac{1}{ab^{\frac{pk+q-1}{2}} (\alpha-\beta)} & \text{if } q \text{ is odd}. \end{cases}$$

Thus,

$$u_{pk+q} = A\alpha^{pk+q} + O\left(\frac{|\beta|^{\frac{pk}{2}}}{\alpha^{\frac{pk}{2}}} \right).$$

Hence,

$$\frac{1}{u_{pk+q}} = \frac{1}{A\alpha^{pk+q}(1 + O(\frac{|\beta|^{\frac{pk}{2}}}{\alpha^{\frac{3pk}{2}}}))} = \frac{1}{A\alpha^{pk+q}} \left(1 + O\left(\frac{|\beta|^{\frac{pk}{2}}}{\alpha^{\frac{3pk}{2}}} \right) \right)$$

$$= \frac{1}{A\alpha^{pk+q}} + O\left(\frac{|\beta|^{\frac{pk}{2}}}{\alpha^{\frac{5pk}{2}}} \right).$$

Thus,

$$\sum_{k=n}^{\infty} \frac{1}{u_{pk+q}} = \frac{\frac{1}{\alpha^{pn+q}}}{A(1 - \frac{1}{\alpha^p})} + O\left(\frac{|\beta|^{\frac{pn}{2}}}{\alpha^{\frac{5pn}{2}}} \right) = \frac{\alpha^p}{A\alpha^{pn+q}(\alpha^p - 1)} \left(1 + O\left(\frac{|\beta|^{\frac{pn}{2}}}{\alpha^{\frac{3pn}{2}}} \right) \right).$$

Taking the reciprocal of this expression yields

$$
\left(\sum_{k=n}^{\infty} \frac{1}{u_{pk+q}} \right)^{-1} = \frac{A\alpha^{pn+q}(\alpha^p - 1)}{\alpha^p} + O\left(\frac{|\beta|^{\frac{pn}{2}}}{\alpha^{\frac{pn}{2}}} \right)
$$

$$
= A\alpha^{pn+q} - A\alpha^{pn-p+q} + O\left(\frac{|\beta|^{\frac{pn}{2}}}{\alpha^{\frac{pn}{2}}} \right)
$$

$$
= u_{pn+q} - u_{pn-p+q} + O\left(\frac{|\beta|^{\frac{pn}{2}}}{\alpha^{\frac{pn}{2}}} \right). \tag{2}
$$

Therefore, for any even $p \geq 2$ and integer $0 < q < p$, there exists $n \geq n_1$ sufficiently large such that the modulus of the last error term of identity (2) becomes less than $1/2$. This completes the proof of Theorem 1. □

Proof of Theorem 2 In the first place, suppose that $k \geq 2$ is even. From Lemma we have

$$
u_k = \frac{1}{a^{\frac{k}{2}-1}b^{\frac{k}{2}}} \cdot \frac{\alpha^k - \beta^k}{\alpha - \beta},
$$

and

$$
u_{k+2c+1} = \frac{1}{a^{\frac{k}{2}+c}b^{\frac{k}{2}+c}} \cdot \frac{\alpha^{k+2c+1} - \beta^{k+2c+1}}{\alpha - \beta}.
$$

The identities $(\alpha - \beta)^2 = a^2b^2 + 4ab$ and $\alpha\beta = -ab$ now yield

$$
u_k u_{k+2c+1} = \frac{\alpha^{2k+2c+1} + \beta^{2k+2c+1} - (\alpha\beta)^k(\alpha^{2c+1} + \beta^{2c+1})}{a^{k+c-1}b^{k+c}(\alpha - \beta)^2}
$$

$$
= \frac{\alpha^{2k+2c+1}}{(ab^2 + 4b)(ab)^{k+c}} + O\left(\left(\frac{\alpha\beta}{ab} \right)^k \right)
$$

$$
= \frac{\alpha^{2k+2c+1}}{(ab^2 + 4b)(ab)^{k+c}} + O(1).
$$

Further, if $k \geq 1$ is odd, the same identity is similarly obtained. Thus, in both cases we have

$$
\frac{1}{u_k u_{k+2c+1}} = \frac{1}{\frac{\alpha^{2k+2c+1}}{(ab^2+4b)(ab)^{k+c}} + O(1)} = \frac{1}{\frac{\alpha^{2k+2c+1}}{(ab^2+4b)(ab)^{k+c}}(1 + O(\frac{(ab)^k}{\alpha^{2k}}))}
$$

$$
= \frac{(ab^2 + 4b)(ab)^{k+c}}{\alpha^{2k+2c+1}} + O\left(\frac{(ab)^{2k}}{\alpha^{4k}} \right).
$$

Hence,

$$
\frac{a^k b^{k+2c+1}}{u_k u_{k+2c+1}} = \frac{(ab^3 + 4b^2)a^{2k+c}b^{2k+3c}}{\alpha^{2k+2c+1}} + O\left(\frac{(ab)^{3k}}{\alpha^{4k}} \right)
$$

$$
= \frac{a^{c+1}b^{3c+3} + 4a^c b^{3c+2}}{\alpha^{2c+1}} \cdot \frac{(ab)^{2k}}{\alpha^{2k}} + O\left(\frac{(ab)^{3k}}{\alpha^{4k}} \right).
$$

Let $B = \frac{a^{c+1}b^{3c+3} + 4a^c b^{3c+2}}{\alpha^{2c+1}}$, then

$$\frac{a^k b^{k+2c+1}}{u_k u_{k+2c+1}} = \frac{B(ab)^{2k}}{\alpha^{2k}} + O\left(\frac{(ab)^{3k}}{\alpha^{4k}}\right).$$

Consequently,

$$\sum_{k=n}^{\infty} \frac{a^k b^{k+2c+1}}{u_k u_{k+2c+1}} = B\sum_{k=n}^{\infty} \frac{(ab)^{2k}}{\alpha^{2k}} + O\left(\sum_{k=n}^{\infty} \frac{(ab)^{3k}}{\alpha^{4k}}\right)$$

$$= B \cdot \frac{(\frac{ab}{\alpha})^{2n}}{1 - (\frac{ab}{\alpha})^2} + O\left(\frac{(ab)^{3n}}{\alpha^{4n}}\right).$$

Taking the reciprocal of this expression yields

$$\left(\sum_{k=n}^{\infty} \frac{a^k b^{k+2c+1}}{u_k u_{k+2c+1}}\right)^{-1} = \frac{1}{B \cdot \frac{(\frac{ab}{\alpha})^{2n}}{1-(\frac{ab}{\alpha})^2}(1 + O(\frac{|\beta|^n}{\alpha^n}))}$$

$$= \frac{1 - (\frac{ab}{\alpha})^2}{B \cdot (\frac{ab}{\alpha})^{2n}}\left(1 + O\left(\frac{|\beta|^n}{\alpha^n}\right)\right)$$

$$= \frac{1 - (\frac{ab}{\alpha})^2}{B \cdot (\frac{ab}{\alpha})^{2n}} + O\left(\frac{1}{\alpha^n|\beta|^n}\right)$$

$$= \frac{1}{B} \cdot \left(\frac{\alpha}{ab}\right)^{2n} - \frac{1}{B} \cdot \left(\frac{\alpha}{ab}\right)^{2n-2} + O\left(\frac{1}{\alpha^n|\beta|^n}\right). \tag{3}$$

On the other hand,

$$\frac{u_n u_{n+2c+1}}{a^n b^{n+2c+1}} - \frac{u_{n-1}u_{n+2c}}{a^{n-1}b^{n+2c}}$$

$$= \frac{\alpha^{2c+1}}{a^{c+1}b^{3c+3} + 4a^c b^{3c+2}} \cdot \left(\frac{\alpha}{ab}\right)^{2n} - \frac{\alpha^{2c+1}}{a^{c+1}b^{3c+3} + 4a^c b^{3c+2}} \cdot \left(\frac{\alpha}{ab}\right)^{2n-2} + O\left(\frac{1}{a^n b^n}\right)$$

$$= \frac{1}{B} \cdot \left(\frac{\alpha}{ab}\right)^{2n} - \frac{1}{B} \cdot \left(\frac{\alpha}{ab}\right)^{2n-2} + O\left(\frac{1}{a^n b^n}\right). \tag{4}$$

Combining (3) and (4), finally we have

$$\left(\sum_{k=n}^{\infty} \frac{a^k b^{k+2c+1}}{u_k u_{k+2c+1}}\right)^{-1} - \left(\frac{u_n u_{n+2c+1}}{a^n b^{n+2c+1}} - \frac{u_{n-1}u_{n+2c}}{a^{n-1}b^{n+2c}}\right) = O\left(\frac{1}{a^n b^n}\right). \tag{5}$$

It follows that for any integer $c \geq 0$, there exists $n \geq n_2$ sufficiently large such that the modulus of the last error term of identity (5) becomes less than 1/2. This completes the proof of Theorem 2. □

Authors' contributions

HZ obtained the theorems and completed the proof. ZW corrected and improved the final version. Both authors read and approved the final manuscript.

References

1. Edson, M, Yayenie, O: A new generalization of Fibonacci sequence and extended Binet's formula. Integers **9**, 639-654 (2009)
2. Falcón, S, Plaza, Á: On the Fibonacci *k*-numbers. Chaos Solitons Fractals **32**, 1615-1624 (2007)
3. Ma, R, Zhang, W: Several identities involving the Fibonacci numbers and Lucas numbers. Fibonacci Q. **45**, 164-170 (2007)
4. Yi, Y, Zhang, W: Some identities involving the Fibonacci polynomials. Fibonacci Q. **40**, 314-318 (2002)
5. Ohtsuka, H, Nakamura, S: On the sum of reciprocal Fibonacci numbers. Fibonacci Q. **46/47**, 153-159 (2008/2009)
6. Zhang, W, Wang, T: The infinite sum of reciprocal Pell numbers. Appl. Math. Comput. **218**, 6164-6167 (2012)
7. Xu, Z, Wang, T: The infinite sum of the cubes of reciprocal Pell numbers. Adv. Differ. Equ. **2013**, Article ID 184 (2013)
8. Wu, Z, Zhang, W: The sums of the reciprocal of Fibonacci polynomials and Lucas polynomials. J. Inequal. Appl. **2012**, Article ID 134 (2012)
9. Wu, Z, Zhang, W: Several identities involving Fibonacci polynomials and Lucas polynomials. J. Inequal. Appl. **2013**, Article ID 205 (2013)
10. Holliday, S, Komatsu, T: On the sum of reciprocal generalized Fibonacci numbers. Integers **11**, 441-455 (2011)
11. Komatsu, T, Laohakosol, V: On the sum of reciprocals of numbers satisfying a recurrence relation of order *s*. J. Integer Seq. **13**, Article ID 10.5.8 (2010)
12. Kilic, E, Arikan, T: More on the infinite sum of reciprocal usual Fibonacci, Pell and higher order recurrences. Appl. Math. Comput. **219**, 7783-7788 (2013)
13. Wu, Z, Zhang, H: On the reciprocal sums of higher order sequences. Adv. Differ. Equ. **2013**, Article ID 189 (2013)