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# Stability and periodicity for impulsive neural networks with delays

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**Abstract**

By constructing a suitable Lyapunov functional and using some inequalities, we investigate the existence, uniqueness and global exponential stability of periodic solutions for a class of generalized cellular neural networks with impulses and time-varying delays. Finally, an illustrative example and simulations are given to show the effectiveness of the main results.

**Keywords:** periodicity; stability; impulsive; delays

## 1 Introduction

The dynamics of cellular neural networks has been deeply investigated due to its applicability in solving image processing, signal processing and pattern recognition problems [1, 2]. Recently, the study of the existence and exponential stability of periodic solutions for neural networks has received much attention and many known results have been obtained [3–27]. For example, authors [11] considered the following neural networks with delays:

$$
\begin{aligned}
\dot{x}_i(t) = {}&-a_i(t)x_i(t) + \sum_{j=1}^{n} b_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^{n} c_{ij}(t)f_j(x_j(t - \tau_{ij}(t))) \\
&+ \sum_{j=1}^{n}\sum_{l=1}^{n} d_{ijl}(t)f_j(x_j(t))f_l(x_l(t)) + \sum_{j=1}^{n}\sum_{l=1}^{n} e_{ijl}(t)f_j(t - \sigma_{ijl}(t)) \\
&\times f_l(x_l(t - \sigma_{ijl}(t))) + I_i(t),
\end{aligned}
\tag{1}
$$

when $\tau_{ij}(t) = \tau_{ij}$, $\sigma_{ijl}(t) = \sigma_{ijl}$ are constants, a set of easily verifiable sufficient conditions guaranteeing the existence and globally exponential stability of one periodic solution was derived.

However, in practice, impulsive effects are inevitably encountered in implementation of networks, which can also be found in information science, electronics, automatic control systems and so on (see [12, 13, 17–19, 21, 22, 24–27]). Thus it is necessary to study the impulsive case of system (1).

Motivated by the discussion, in this paper, by employing some inequalities and constructing a suitable Lyapunov functional, we aim to investigate the existence and exponential stability of a periodic solution for a class of nonautonomous neural networks with

impulses and time-varying delays as follows:

$$\dot{x}_i(t) = -a_i(t)x_i(t) + \sum_{j=1}^n b_{ij}(t)f_j\big(x_j(t)\big) + \sum_{j=1}^n c_{ij}(t)f_j\big(x_j\big(t-\tau_{ij}(t)\big)\big)$$

$$+ \sum_{j=1}^n \sum_{l=1}^n d_{ijl}(t)f_j\big(x_j(t)\big)f_l\big(x_l(t)\big) + \sum_{j=1}^n \sum_{l=1}^n e_{ijl}(t)f_j\big(t-\sigma_{ijl}(t)\big) \tag{2}$$

$$\times f_l\big(x_l\big(t-\sigma_{ijl}(t)\big)\big) + I_i(t), \quad t \neq t_k,$$

$$\Delta x_i(t_k) = e_{ik}\big(x_i\big(t_k^-\big)\big), \quad t = t_k,$$

where $i, j, l = 1, 2, \ldots, n$, $n$ corresponds to the number of units in the neural networks. $x_i(t)$ corresponds to the state variable at time $t$, $f_j$ is the activation of the neurons. $I_i(t)$ is the external bias at time $t$. $0 \le \tau_{ij}(t) \le \tau$, $0 \le \sigma_{ijl}(t) \le \sigma$ corresponds to the transmission delay respectively, $\tau = \max \tau_{ij}^+$, $\tau_{ij}^+ = \max_{0 \le t \le \omega} |\tau_{ij}(t)|$, $\sigma = \max \sigma_{ijl}^+$, $\sigma_{ilj}^+ = \max_{0 \le t \le \omega} |\sigma_{ijl}(t)|$. The fixed moments of time $t_k$ satisfy $t_1 < t_2 < \cdots$, $\lim_{k \to \infty} t_k = \infty$. $\Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-)$, $x_i(t_k) = x_i(t_k^+)$ and $x_i(t_k^-)$ exists. $e_{ik}(x_i(t_k^-))$ represents impulsive perturbations of the $i$th unit at $t_k$, $k = 1, 2, \ldots$.

By appropriately choosing coefficients, system (2) contains many models as its special cases, which were studied in [6, 8, 12, 24, 25, 27] respectively.

Throughout this paper, for $i, j, l = 1, 2, \ldots, n$, $k = 1, 2, \ldots$, $a_i(t)$, $b_{ij}(t)$, $c_{ij}(t)$, $d_{ijl}(t)$, $e_{ijl}(t)$, $\tau_{ij}(t)$, $\sigma_{ijl}(t)$, $I_i(t)$ are all continuous $\omega$-periodic functions and $[0, \omega] \cap \{t_k\} = \{t_1, t_2, \ldots, t_q\}$. Further, we suppose that:

(H$_1$) There exist positive constants $M_i$ and $N_i$ such that

$$\big|f_i(u) - f_i(v)\big| \le M_i|u - v|, \qquad \big|f_i(u)\big| \le N_i \quad \text{for all } u, v \in R.$$

(H$_2$) There exists a constant $L_{ik}$ such that $|e_{ik}(u) - e_{ik}(v)| \le L_{ik}|u - v|$ for $u, v \in R$.

(H$_3$) There exists a positive integer $q$ such that $t_{k+q} = t_k + \omega$, $e_{i(k+q)}(\cdot) = e_{ik}(\cdot)$.

For convenience, we introduce the following notations:

$$a_i^- = \min_{t \in [0,\omega]} a_i(t) > 0, \qquad b_{ij}^+ = \max_{0 \le t \le \omega} \big|b_{ij}(t)\big|, \qquad c_{ij}^+ = \max_{0 \le t \le \omega} \big|c_{ij}(t)\big|,$$

$$d_{ijl}^+ = \max_{0 \le t \le \omega} \big|d_{ijl}(t)\big|, \qquad e_{ijl}^+ = \max_{0 \le t \le \omega} \big|e_{ijl}(t)\big|.$$

This paper is organized as follows. In Section 2, preliminaries are introduced. In Section 3, by constructing a suitable Lyapunov functional, the criteria ensuring global exponential stability of a periodic solution for system (2) are established. In Section 4, an example and simulations are shown to illustrate the validity of the main results. Finally, we conclude this paper with a brief discussion in Section 5.

## 2 Preliminaries

Firstly, we introduce some definitions and lemmas. Let $R^n$ be the space of $n$-dimensional real column vectors, and $PC = \{f : [-\tau^*, 0] \to R^n | f(s)$ is continuous for $s \notin (-\tau^*, 0] \cap \{t_k\}, f(s^+) = f(s), f(s^-)$ exists for $s \in (-\tau^*, 0] \cap \{t_k\}, k = 1, 2, \ldots\}$. $PC_t = \{f : [-\tau^*, 0] \to R^n | f(s)$

is continuous for $s \notin (-\tau^*, 0] \cap \{t_k - t\}, f(s^+) = f(s), f(s^-)$ exists for $s \in (-\tau^*, 0] \cap \{t_k - t\}, k = 1, 2, \ldots\}$.

A function $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T : [-\tau^*, +\infty) \to R^n$ is called a solution of (2) with the initial condition given by

$$x(s) = \phi(s) \quad \text{for all } s \in [-\tau^*, 0], \phi \in PC, \tag{3}$$

if $x(t) \in PC[[0, +\infty), R^n]$ satisfies (2) and (3), where $\tau^* = \max_{1 \le i,j,l \le n} \{\max_{t \in [0,\infty)} (|\tau_{ij}(t)|, |\sigma_{ijl}(t)|)\}$. We denote a solution through $\phi$ by $x(t, \phi)$ or $x_t(\phi)$, $x_t(s, \phi) = x(t + s, \phi)$ for all $s \in [-\tau^*, 0]$, $t \ge 0$. Obviously, any solution $x(t, \phi)$ of (2) is continuous at $t \ne t_k$ and right-hand continuous at $t = t_k$, $t \ge 0$. In addition, we define $\|\phi\| = \sup_{-\tau^* \le s \le 0} (\sum_{i=1}^{n} |\phi_i(s)|^r)^{\frac{1}{r}}$, where $\phi = (\phi_1, \phi_2, \ldots, \phi_n) \in PC$ and $r \ge 1$ is a constant.

**Definition 1** System (2) is said to be globally exponentially stable if for any two solutions $x(t, \phi)$ and $y(t, \psi)$, there exist some constants $r > 0$ and $M \ge 1$ such that

$$\left| x(t, \phi) - y(t, \psi) \right| \le M \|\phi - \psi\| e^{-rt}$$

for all $t > 0$.

**Definition 2** Let $f(t) : R \to R$ be a continuous function, then the Dini right derivative of $f(t)$ is defined as

$$\frac{D^+ f(t)}{dt} = \limsup_{h \to 0^+} \frac{f(t + h) - f(t)}{h}.$$

From Definition 2, we can easily obtain the following lemma.

**Lemma 1** [12] *Let $f(t)$ be a continuous function on R. If $f(t)$ is differentiable at $t_0$, then*

$$D^+ \left| f(t_0) \right| = \begin{cases} \dot{f}(t_0) & \text{if } f(t_0) > 0 \text{ or } f(t_0) = 0 \text{ and } \dot{f}(t_0) > 0; \\ -\dot{f}(t_0) & \text{if } f(t_0) < 0 \text{ or } f(t_0) = 0 \text{ and } \dot{f}(t_0) < 0; \\ 0 & \text{if } f(t_0) = 0 \text{ and } \dot{f}(t_0) = 0. \end{cases}$$

Next, we introduce two important inequalities which play a key role in obtaining the main results.

**Lemma 2** (Beckenbach and Bellman) [12, 28] *For $a \ge 0$, $b_k \ge 0$, $k = 1, 2, \ldots, m$, the following inequality holds*:

$$a \prod_{k=1}^{m} b_k^{q_k} \le \frac{1}{r} \sum_{k=1}^{m} q_k b_k^r + \frac{1}{r} a^r,$$

*where $q_k > 0$ are some constants and $\sum_{k=1}^{m} q_k = r - 1$, $r > 1$.*

**Lemma 3** (Halanay inequality) [12, 24] *Assume that $p, q$ are constants satisfying $p > q > 0$. $g(t)$ is a continuous nonnegative function on $[t_0 - \tau, t_0]$ satisfying $D^+ g(t) \le -pg(t) + q\|g_t\|$*

*for all $t \geq t_0$, where $\|g_t\| = \sup_{-\tau \leq s \leq 0} |g(t + s)|$. Then, for all $t \geq t_0$, we have*

$$g(t) \leq \|g_{t_0}\| e^{-\lambda(t-t_0)},$$

*where $\lambda$ is the unique positive root of the equation $\lambda - p + qe^{\lambda\tau} = 0$.*

## 3 Main results

In this section, we construct a suitable Lyapunov functional to study the existence and global exponential stability of periodic solutions of system (2).

**Theorem 1** *Suppose that* $(H_1)$-$(H_3)$ *hold. Further,*

$(H_4)$ *There exist $r > 1$, $q_k > 0$, $\lambda_i > 0$, $\alpha_{ki}$, $\beta_{ki}$, $\gamma_{ki}$ and $\eta_{ki} \in R$ $(i = 1, 2, \ldots, n, k = 1, 2, \ldots, m)$ such that $p > q > 0$, $\sum_{k=1}^{m} q_k = r - 1$, where*

$$
\begin{aligned}
p = \max_{1 \leq i \leq n} \Bigg[ & ra_i^- - \sum_{j=1}^{n} b_{ji}^+ \frac{\lambda_j}{\lambda_i} M_i^{r(1-\sum_{k=1}^{m}\alpha_{ki})} - \sum_{j=1}^{n} b_{ij}^+ \sum_{k=1}^{m} q_k M_j^{\frac{r\alpha_{kj}}{q_k}} - \sum_{j=1}^{n} c_{ij}^+ \sum_{k=1}^{m} q_k M_j^{\frac{r\beta_{kj}}{q_k}} \\
& - \sum_{j=1}^{n} \sum_{l=1}^{n} (d_{ijl}^+ + d_{ilj}^+) N_l \sum_{k=1}^{m} q_k M_j^{\frac{r\gamma_{kj}}{q_k}} - \sum_{j=1}^{n} \sum_{l=1}^{n} (d_{jil}^+ + d_{jli}^+) \frac{\lambda_j}{\lambda_i} N_l M_i^{1-\sum_{k=1}^{m}\gamma_{ki}} \\
& - \sum_{j=1}^{n} \sum_{l=1}^{n} (e_{ijl}^+ + e_{ilj}^+) N_l \sum_{k=1}^{m} q_k M_j^{\frac{r\eta_{kj}}{q_k}} \Bigg],
\end{aligned}
$$

$$
q = \max_{1 \leq i \leq n} \Bigg[ \sum_{j=1}^{n} c_{ji}^+ \frac{\lambda_j}{\lambda_i} M_i^{r(1-\sum_{k=1}^{m}\beta_{ki})} + \sum_{j=1}^{n} \sum_{l=1}^{n} \frac{\lambda_j}{\lambda_i} (e_{jil}^+ + e_{jli}^+) N_l M_i^{r(1-\sum_{k=1}^{m}\eta_{ki})} \Bigg].
$$

$(H_5)$ *There is $0 < \mu < \lambda$ such that $\ln[(1 + L_k)e^{\frac{\lambda\tau}{r}}] \leq \frac{\mu(t_k - t_{k-1})}{r}$, $k = 1, 2, \ldots, L_k = \max_{1 \leq i \leq n}\{L_{ik}\}$, $\lambda$ is the unique positive solution of the equation $\lambda - p + qe^{\lambda\tau} = 0$.*

*Then system (2) has a unique $\omega$-periodic solution which is globally exponentially stable.*

*Proof* Let $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ and $y(t) = (y_1(t), y_2(t), \ldots, y_n(t))^T$ be two solutions of (2) through $\phi$ and $\psi$, respectively, where $\phi, \psi \in PC$, then we have

$$
\begin{aligned}
\frac{d(x_i(t) - y_i(t))}{dt} \\
= & -a_i(t)\big(x_i(t) - y_i(t)\big) \\
& + \sum_{j=1}^{n} b_{ij}(t)\big(f_j(x_j(t)) - f_j(y_j(t))\big) + \sum_{j=1}^{n} c_{ij}(t)\big(f_j(x_j(t - \tau_{ij}(t))) - f_j(y_j(t - \tau_{ij}(t)))\big) \\
& + \sum_{j=1}^{n} \sum_{l=1}^{n} d_{ijl}(t)\big(f_j(x_j(t))f_l(x_l(t)) - f_j(y_j(t))f_l(y_l(t))\big) \\
& + \sum_{j=1}^{n} \sum_{l=1}^{n} e_{ijl}(t)\big(f_j(x_j(t - \sigma_{ijl}(t)))f_l(x_l(t - \sigma_{ill}(t))) \\
& - f_j(y_j(t - \sigma_{ijl}(t)))f_l(y_l(t - \sigma_{ijl}(t)))\big)x_i(t_k) - y_i(t_k) \\
= & \; x_i(t_k^-) - y_i(t_k^-) + e_{ik}(x_i(t_k^-)) - e_{ik}(y_i(t_k^-)).
\end{aligned}
\tag{4}
$$

Let $z_i(t) = x_i(t) - y_i(t)$. Consider the following Lyapunov functional:

$$V(t) = \sum_{i=1}^{n} \lambda_i \left| x_i(t) - y_i(t) \right|^r = \sum_{i=1}^{n} \lambda_i \left| z_i(t) \right|^r.$$

Calculating the Dini upper right derivative of $V(t)$ along the solution of (2) at a continuous point $t \neq t_k$ and applying Lemma 2, we have

$$D^+(V(t) = \sum_{i=1}^{n} \lambda_i D^+ \left| x_i(t) - y_i(t) \right|^r = \sum_{i=1}^{n} \lambda_i r \left| x_i(t) - y_i(t) \right|^{r-1} D^+ \left| x_i(t) - y_i(t) \right|$$

$$= \sum_{i=1}^{n} \lambda_i r \left| x_i(t) - y_i(t) \right|^{r-1} \operatorname{sgn}\left( x_i(t) - y_i(t) \right)$$

$$\times \Bigg\{ -a_i(t)\left( x_i(t) - y_i(t) \right) + \sum_{j=1}^{n} b_{ij}(t) \left( f_j(x_j(t)) - f_j(y_j(t)) \right)$$

$$+ \sum_{j=1}^{n} c_{ij}(t) \left( f_j\left( x_j\left( t - \tau_{ij}(t) \right) \right) - f_j\left( y_j\left( t - \tau_{ij}(t) \right) \right) \right)$$

$$+ \sum_{j=1}^{n} \sum_{l=1}^{n} d_{ijl}(t) \left( f_j(x_j(t)) f_l(x_l(t)) - f_j(y_j(t)) f_l(y_l(t)) \right)$$

$$+ \sum_{j=1}^{n} \sum_{l=1}^{n} e_{ijl}(t) \left( f_j\left( x_j\left( t - \sigma_{ijl}(t) \right) \right) f_l\left( x_l\left( t - \sigma_{ijl}(t) \right) \right) \right.$$

$$\left. - f_j\left( y_j\left( t - \sigma_{ijl}(t) \right) \right) f_l\left( y_l\left( t - \sigma_{ijl}(t) \right) \right) \right) \Bigg\}$$

$$\leq \sum_{i=1}^{n} \lambda_i r \Bigg\{ -a_i(t) \left| x_i(t) - y_i(t) \right|^r + \sum_{j=1}^{n} b_{ij}^+ M_j \left| x_i(t) - y_i(t) \right|^{r-1} \left| x_j(t) - y_j(t) \right|$$

$$+ \sum_{j=1}^{n} c_{ij}^+ M_j \left| x_i(t) - y_i(t) \right|^{r-1} \left| x_j\left( t - \tau_{ij}(t) \right) - y_j\left( t - \tau_{ij}(t) \right) \right|$$

$$+ \sum_{j=1}^{n} \sum_{l=1}^{n} e_{ijl}^+ \left[ N_l M_j \left| x_i(t) - y_i(t) \right|^{r-1} \left| x_j\left( t - \sigma_{ijl}(t) \right) - y_j\left( t - \sigma_{ijl}(t) \right) \right| \right.$$

$$\left. + N_j M_l \left| x_i(t) - y_i(t) \right|^{r-1} \left| x_l\left( t - \sigma_{ijl}(t) \right) - y_l\left( t - \sigma_{ijl}(t) \right) \right| \right]$$

$$+ \sum_{j=1}^{n} \sum_{l=1}^{n} d_{ijl}^+ \left[ N_l M_j \left| x_i(t) - y_i(t) \right|^{r-1} \left| x_j(t) - y_j(t) \right| \right.$$

$$\left. + N_j M_l \left| x_i(t) - y_i(t) \right|^{r-1} \left| x_l(t) - y_l(t) \right| \right] \Bigg\}$$

$$= \sum_{i=1}^{n} \lambda_i r \Bigg\{ -a_i(t) \left| z_i(t) \right|^r + \sum_{j=1}^{n} b_{ij}^+ M_j \left| z_i(t) \right|^{r-1} \left| z_j(t) \right|$$

$$+ \sum_{j=1}^{n} c_{ij}^+ M_j \left| z_i(t) \right|^{r-1} \left| z_j\left( t - \tau_{ij}(t) \right) \right|$$

$$+ \sum_{j=1}^{n} \sum_{l=1}^{n} e_{ijl}^{+} \left[ N_l M_j |z_i(t)|^{r-1} |z_j(t - \sigma_{ijl}(t))| + N_j M_l |z_i(t)|^{r-1} |z_l(t - \sigma_{ijl}(t))| \right]$$

$$+ \sum_{j=1}^{n} \sum_{l=1}^{n} d_{ijl}^{+} \left[ N_l M_j |z_i(t)|^{r-1} |z_j(t)| + N_j M_l |z_i(t)|^{r-1} |z_l(t)| \right] \Bigg\}$$

$$= \sum_{i=1}^{n} \lambda_i r \Bigg\{ -a_i(t) |z_i(t)|^{r} + \sum_{j=1}^{n} b_{ij}^{+} M_j^{1-\sum_{k=1}^{m} \alpha_{kj}} |z_j(t)| \prod_{k=1}^{m} M_j^{\alpha_{kj}} |z_i(t)|^{q_k}$$

$$+ \sum_{j=1}^{n} c_{ij}^{+} M_j^{1-\sum_{k=1}^{m} \beta_{kj}} |z_j(t - \tau_{ij}(t))| \prod_{k=1}^{m} M_j^{\beta_{kj}} |z_i(t)|^{q_k}$$

$$+ \sum_{j=1}^{n} \sum_{l=1}^{n} (d_{ijl}^{+} + d_{ilj}^{+}) N_l M_j^{1-\sum_{k=1}^{m} \gamma_{kj}} |z_j(t)| \prod_{k=1}^{m} M_j^{\gamma_{kj}} |z_i(t)|^{q_k}$$

$$+ \sum_{j=1}^{n} \sum_{l=1}^{n} (e_{ijl}^{+} + e_{ilj}^{+}) N_l M_j^{1-\sum_{k=1}^{m} \eta_{kj}} |z_j(t - \sigma_{ijl}(t))| \prod_{k=1}^{m} M_j^{\eta_{kj}} |z_i(t)|^{q_k} \Bigg\}$$

$$\leq \sum_{i=1}^{n} \lambda_i \Bigg\{ -ra_i^{-} |z_i(t)|^{r} + \sum_{j=1}^{n} b_{ij}^{+} M_j^{r(1-\sum_{k=1}^{m} \alpha_{kj})} |z_j(t)|^{r} + \sum_{j=1}^{n} b_{ij}^{+} \sum_{k=1}^{m} q_k M_j^{\frac{r\alpha_{kj}}{q_k}} |z_i(t)|^{r}$$

$$+ \sum_{j=1}^{n} c_{ij}^{+} M_j^{r(1-\sum_{k=1}^{m} \beta_{kj})} |z_j(t - \tau_j(t))|^{r} + \sum_{j=1}^{n} c_{ij}^{+} \sum_{k=1}^{m} q_k M_j^{\frac{r\beta_{kj}}{q_k}} |z_i(t)|^{r}$$

$$+ \sum_{j=1}^{n} \sum_{l=1}^{n} (d_{ijl}^{+} + d_{ilj}^{+}) N_l \left[ \sum_{k=1}^{m} q_k M_j^{\frac{r\gamma_{kj}}{q_k}} |z_i(t)|^{r} + M_j^{r(1-\sum_{k=1}^{m} \gamma_{kj})} |z_j(t)|^{r} \right]$$

$$+ \sum_{j=1}^{n} \sum_{l=1}^{n} (e_{ijl}^{+} + e_{ilj}^{+}) N_l \left[ \sum_{k=1}^{m} q_k M_j^{\frac{r\eta_{kj}}{q_k}} |z_i(t)|^{r} + M_j^{r(1-\sum_{k=1}^{m} \eta_{kj})} |z_j(t - \sigma_{ijl}(t))|^{r} \right] \Bigg\}$$

$$\leq \sum_{i=1}^{n} \lambda_i \Bigg[ -ra_i^{-} + \sum_{j=1}^{n} b_{ji}^{+} \frac{\lambda_j}{\lambda_i} M_i^{r(1-\sum_{k=1}^{m} \alpha_{ki})} + \sum_{j=1}^{n} b_{ij}^{+} \sum_{k=1}^{m} q_k M_j^{\frac{r\alpha_{kj}}{q_k}}$$

$$+ \sum_{j=1}^{n} c_{ij}^{+} \sum_{k=1}^{m} q_k M_j^{\frac{r\beta_{kj}}{q_k}} + \sum_{j=1}^{n} \sum_{l=1}^{n} (d_{ijl}^{+} + d_{ilj}^{+}) N_l \sum_{k=1}^{m} q_k M_j^{\frac{r\gamma_{kj}}{q_k}}$$

$$+ \sum_{j=1}^{n} \sum_{l=1}^{n} (d_{jil}^{+} + d_{jli}^{+}) \frac{\lambda_j}{\lambda_i} N_l M_i^{r(1-\sum_{k=1}^{m} \gamma_{ki})}$$

$$+ \sum_{j=1}^{n} \sum_{l=1}^{n} (e_{ijl}^{+} + e_{ilj}^{+}) N_l \sum_{k=1}^{m} q_k M_j^{\frac{r\eta_{kj}}{q_k}} \Bigg] |z_i(t)|^{r}$$

$$+ \sum_{i=1}^{n} \lambda_i \Bigg[ \sum_{j=1}^{n} c_{ji}^{+} \frac{\lambda_j}{\lambda_i} M_i^{r(1-\sum_{k=1}^{m} \beta_{ki})} + \sum_{j=1}^{n} \sum_{l=1}^{n} \frac{\lambda_j}{\lambda_i} (e_{jil}^{+} + e_{jli}^{+}) N_l M_i^{r(1-\sum_{k=1}^{m} \eta_{ki})} \Bigg] \|z_i\|^{r}$$

$$\leq -pV(t) + q\|V_t\|. \tag{5}$$

For any $t \in [t_0, t_1)$, by (H$_4$) and Lemma 3, we have

$$|V(t)| \leq \|V_{t_0}\| e^{-\lambda(t-t_0)}, \tag{6}$$

*i.e.*,

$$\left|V(t)\right|^{1/r} \le \|V_{t_0}\|^{1/r} e^{-\lambda(t-t_0)/r}.$$

Also, it follows from $(H_2)$ that

$$
\begin{aligned}
\left|V(t_1)\right|^{1/r} &\le \left\{\sum_{i=1}^{n} \lambda_i \big(\left|x_i(t_1^-) - y_i(t_1^-)\right| + L_{i1}\left|x_i(t_1^-) - y_i(t_1^-)\right|\big)^r\right\}^{1/r} \\
&= \left\{\sum_{i=1}^{n} \lambda_i (1+L_{i1})^r \left|x_i(t_1^-) - y_i(t_1^-)\right|^r\right\}^{1/r} \\
&\le (1+L_1)\|V_{t_0}\|^{1/r} e^{-\lambda(t_1-t_0)/r}.
\end{aligned}
\tag{7}
$$

Therefore, from (6) and (7), for any $t \in [t_0, t_1]$, we have

$$\left|V(t)\right|^{1/r} \le (1+L_1)\|V_{t_0}\|^{1/r} e^{-\lambda(t-t_0)/r}. \tag{8}$$

Similar to (6), for all $t \in [t_1, t_2)$, we can derive that

$$
\begin{aligned}
\left|V(t)\right|^{1/r} &\le \left\|V(t_1)\right\|^{1/r} e^{-\lambda(t-t_1)/r} \\
&= \left\{\sup_{-\tau^* \le s \le 0} \left|V(t_1+s)\right|\right\}^{1/r} e^{-\lambda(t-t_1)/r} \\
&\le (1+L_1)\|V_{t_0}\|^{1/r} e^{-\lambda(t_1-\tau-t_0)/r} e^{-\lambda(t-t_1)/r}, \\
&\le (1+L_1)e^{\lambda\tau}\|V_{t_0}\|^{1/r} e^{-\lambda(t-t_0)/r}.
\end{aligned}
$$

Again by $(H_2)$, then

$$
\begin{aligned}
\left|V(t_2)\right|^{1/r} &\le \left\{\sum_{i=1}^{n} \lambda_i \big(\left|x_i(t_2^-) - y_i(t_2^-)\right| + L_{i2}\left|x_i(t_2^-) - y_i(t_2^-)\right|\big)^r\right\}^{1/r} \\
&\le (1+L_1)(1+L_2)e^{\lambda\tau}\|V_{t_0}\|^{1/r} e^{-\lambda(t_2-t_0)/r}.
\end{aligned}
$$

Thus, for any $t \in (t_0, t_2]$,

$$\left|V(t)\right|^{1/r} \le (1+L_1)(1+L_2)e^{\lambda\tau}\|V_{t_0}\|^{1/r} e^{-\lambda(t-t_0)/r}.$$

By mathematical induction, for all $t \in [t_k, t_{k-1})$, we have

$$\left|V(t)\right|^{1/r} \le (1+L_1)(1+L_2)\cdots(1+L_k)e^{k\lambda\tau}\|V_{t_0}\|^{1/r} e^{-\lambda(t-t_0)/r}.$$

According to $(H_5)$, $(1+L_k)e^{\lambda\tau/r} \le e^{\mu(t_k-t_{k-1})/r}$ for all $k = 1, 2, \ldots$, hence

$$
\begin{aligned}
\left|V(t)\right|^{1/r} &\le e^{\mu(t_1-t_0)/r} e^{\mu(t_2-t_1)/r} \cdots e^{\mu(t_k-t_{k-1})/r}\|V_{t_0}\|^{1/r} e^{-\lambda(t-t_0)/r} \\
&\le \|V_{t_0}\|^{1/r} e^{-(\lambda-\mu)(t-t_0)/r}.
\end{aligned}
$$

Therefore,

$$|V(t)|^{1/r} \leq \|V_{t_0}\|^{1/r} e^{-(\lambda-\mu)(t-t_0)/r} \quad \text{for all } t \geq t_0,$$

*i.e.*,

$$|x(t,\phi) - y(t,\psi)| \leq \sqrt[r]{\frac{\lambda_{\max}}{\lambda_{\min}}} e^{-(\lambda-\mu)(t-t_0)/r} \|\phi - \psi\| \quad \text{for all } t \geq t_0,$$

where $\lambda_{\max} = \max_{1 \leq i \leq n}\{\lambda_i\}$, $\lambda_{\min} = \min_{1 \leq i \leq n}\{\lambda_i\}$. This implies that system (2) is globally exponentially stable.

Next, we prove the existence of a periodic solution of (2). Define a Poincáre mapping $T : PC_0 \rightarrow PC_\omega$ as follows:

$$T(\phi) = x_\omega(\phi) \quad \text{for any } \phi \in PC_0.$$

By the periodicity of (2), we can derive that $T^m(\phi) = x_{k\omega}(\phi)$ for any integer $m > 0$. From the above conclusion, we can choose a positive integer $m$ such that $\sqrt[r]{\frac{\lambda_{\max}}{\lambda_{\min}}} e^{-(\lambda-\mu)(t-m\omega)/r} < 1$. According to the periodicity of $\{t_k\}$ and $PC_0 = PC_\omega$, we have

$$\|T^m\phi - T^m\psi\| = \|x_{m\omega}(\phi) - x_{m\omega}(\psi)\| < \|\phi - \psi\|.$$

Then operator $T^m$ is a construction mapping in $PC_0$. Obviously, $PC_0$ is a Banach space. By virtue of the Banach fixed point theorem, $T^m$ has a unique fixed point $\phi^* \in PC_0$. On the other hand, $T^m(T\phi^*) = T(T^m\phi^*) = T(\phi^*)$, then $T\phi^*$ is also a fixed point of $T^m$. By the uniqueness of fixed point, we obtain that $T\phi^* = \phi^*$. Therefore, system (2) has a unique $\omega$-periodic solution which is globally exponentially stable. The proof is complete. $\square$

If there is no impulse, system (2) reduces to (1) studied in [11]. From the proof of Theorem 1, we have the following corollary.

**Corollary 1** *Suppose that* $(H_1)$-$(H_4)$ *hold. Then system* (1) *has a unique $\omega$-periodic solution which is globally exponentially stable.*

**Remark 1** In [11], by using the Mawhin continuation theorem and constructing Lyapunov functionals, the authors studied the existence and stability of system (1), where the delays are constant. However, in this paper, for time-varying delays, by employing many different analysis techniques from [11], we establish the criteria ensuring the existence and exponential stability of a periodic solution of (1). Furthermore, if we take $r = 2$, $m = 1$, $q_k = 1$, $\lambda_i = 1$, $\alpha_{ki} = \beta_{ki} = \gamma_{ki} = \eta_{ki} = 1/2$, then condition $(H_4)$ is transformed into $(H_4)'$ as follows:

$$(H_4)' \quad a_i^- - \sum_{i=1}^{n} b_{ji}^+ M_i - \sum_{j=1}^{n} c_{ji}^+ M_i - \sum_{j=1}^{n}\sum_{l=1}^{n} (d_{jil}^+ + d_{jli}^+) N_l M_i - \sum_{j=1}^{n}\sum_{l=1}^{n} (e_{jil}^+ + e_{jli}^+) N_l M_i > 0,$$

which is just the corresponding condition in [11]. One can derive the same results as [11]. That is, the criteria in [11] are the special case of Theorem 1. Therefore, we extend and improve the earlier results in this sense.

If $d_{ijl}(t) = 0$, then system (2) is transformed into the following model studied in [12].

$$
\dot{x}_i(t) = -a_i(t)x_i(t) + \sum_{j=1}^{n} b_{ij}(t)f_j\big(x_j(t)\big)
$$

$$
+ \sum_{j=1}^{n} c_{ij}(t)f_j\big(x_j\big(t - \tau_{ij}(t)\big)\big) + \sum_{j=1}^{n}\sum_{l=1}^{n} e_{ijl}(t)
$$

$$
\times f_j\big(t - \sigma_{ijl}(t)\big)f_l\big(x_l\big(t - \sigma_{ijl}(t)\big)\big) + I_i(t), \quad t \neq t_k, \tag{9}
$$

$$
\Delta x_i(t_k) = e_{ik}\big(x_i\big(t_k^-\big)\big), \quad t = t_k.
$$

Using a similar proof as above, one can obtain the main result of [12]. It is omitted here.

If $c_{ij}(t) = d_{ijl}(t) = e_{ijl}(t) = 0$, then (2) is transformed into the following model:

$$
\dot{x}_i(t) = -a_i(t)x_i(t) + \sum_{j=1}^{n} b_{ij}(t)f_j\big(x_j(t)\big) + I_i(t), \quad t \neq t_k, \tag{10}
$$

$$
\Delta x_i(t_k) = e_{ik}\big(x_i\big(t_k^-\big)\big), \quad t = t_k.
$$

Similarly, we can obtain the sufficient conditions of the existence and exponential stability of a periodic solution to system (10), we omit it.

If one takes $r = 2$, $m = 2$, $q_k = 1$, $\lambda_i = 1$, $\alpha_{ki} = 1/2$, then (H$_4$) is reduced to

$$
(\mathrm{H}_4)^* \quad a_i^- - \sum_{j=1}^{n} b_{ji}^+ M_i > 0.
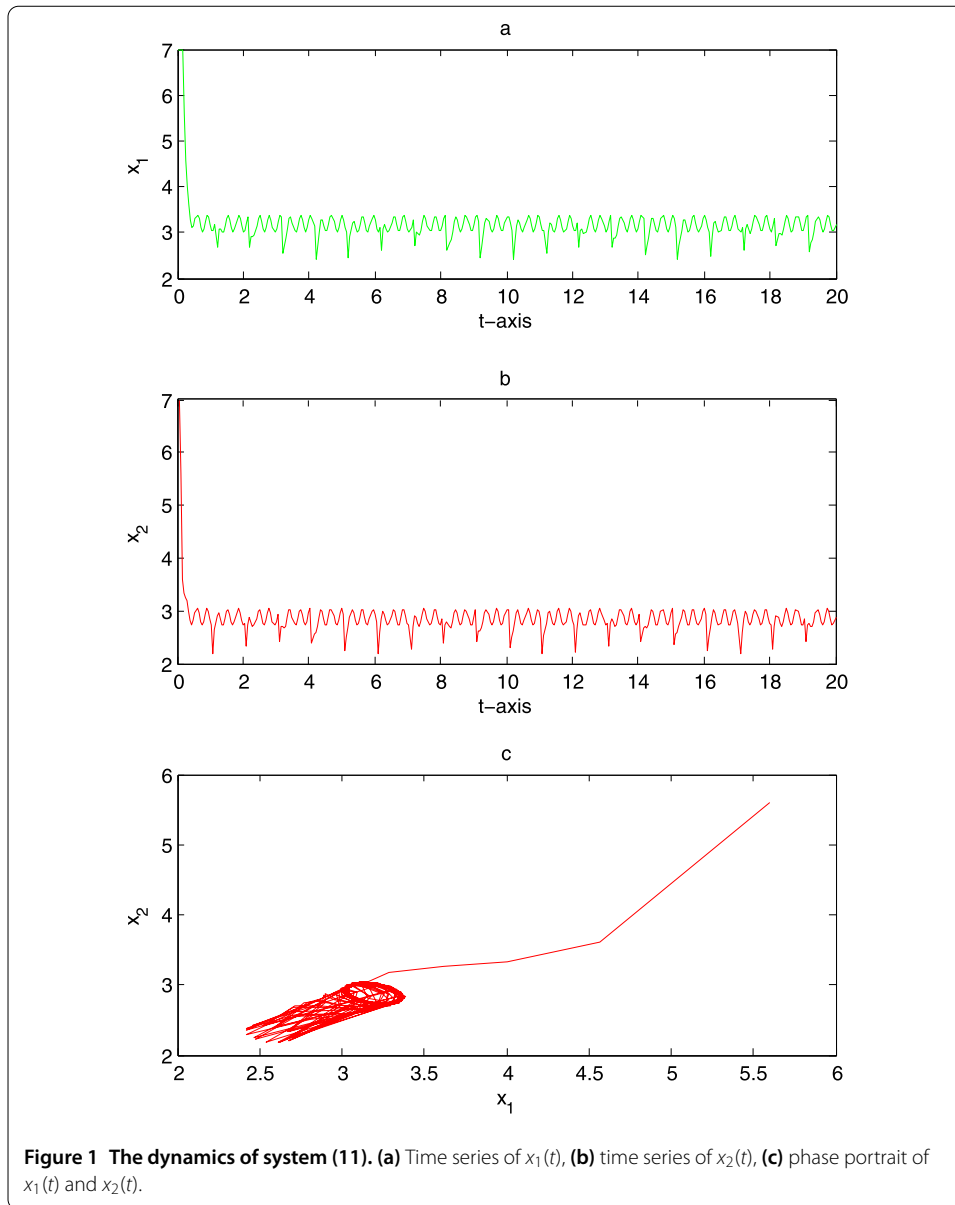$$

The following corollary can be derived.

**Corollary 2** *Suppose that* (H$_1$)-(H$_3$), (H$_4$)$^*$, (H$_5$) *hold, then system* (10) *has a unique exponentially stable $\omega$-periodic solution.*

**Remark 2** Corollary 2 is just the result of [27]. In [27], the impulses are required to be linear functions. However, without the above restrictions, we also establish the criteria ensuring the existence of an $\omega$-periodic solution which is exponentially stable. Thus we extend and generalize the earlier results.

In addition, if $d_{ijl}(t) = e_{ijl}(t) = 0$, (2) reduces to the model studied in [8]. Similar results can be derived. In [8], the delay functions are required to be differential, but we do not need the restriction here.

On the other hand, in real world, by appropriately choosing parameters $r$, $m$, $q_k$, $\alpha_{ki}$, $\beta_{ki}$, $\gamma_{ki}$, $\eta_{ki}$, one can see that many known assumptions can be included as special cases of (H$_4$), hence the discussion is interesting and valuable.

**Figure 1 The dynamics of system (11). (a)** Time series of $x_1(t)$, **(b)** time series of $x_2(t)$, **(c)** phase portrait of $x_1(t)$ and $x_2(t)$.

## 4 An illustrative example and simulations

In this section, we give an example and simulations to show the validity of the main results.

**Example** Let

$$\dot{x}_i(t) = -a_i(t)x_i(t) + \sum_{j=1}^{n} b_{ij}(t)f_j\big(x_j(t)\big) + \sum_{j=1}^{n} c_{ij}(t)f_j\big(x_j\big(t - \tau_{ij}(t)\big)\big)$$

$$+ \sum_{j=1}^{n} \sum_{l=1}^{n} d_{ijl}(t)f_j\big(x_j(t)\big)f_l\big(x_l(t)\big) + \sum_{j=1}^{n} \sum_{l=1}^{n} e_{ijl}(t)f_j\big(t - \sigma_{ijl}(t)\big) \qquad (11)$$

$$\times f_l\big(x_l\big(t - \sigma_{ijl}(t)\big)\big) + I_i(t), \quad t \neq t_k,$$

$$\Delta x_i(t_k) = e_{ik}\big(x_i\big(t_k^-\big)\big), \quad t = t_k,$$

where

$$n = 2, \qquad f_j(x) = \frac{|x+2| - |x-2|}{4}, \qquad a_1(t) = 10 + \sin t,$$

$$a_2(t) = 12 + \cos t, \qquad \tau_{ij}(t) = \pi, \qquad \sigma_{ijl}(t) = 2\pi, \qquad e_{ik}(x_i(t_k^-)) = -0.4x_i(t_k),$$

$$B(t) = (b_{ij}(t))_{2\times 2} = \begin{pmatrix} \sin t & \cos t \\ \cos t/4 & \sin t/4 \end{pmatrix}, \qquad C(t) = (c_{ij}(t))_{2\times 2} = \begin{pmatrix} 2\cos t & 2\sin t \\ \sin t/2 & \cos t/2 \end{pmatrix},$$

$$d_{ijl}(t) = e_{ijl}(t) = 4, \qquad t_k = k\pi.$$

Take $r = 4$, $m = 1$, $q_k = 3$, $\alpha = \beta_{ki} = \gamma_{ki} = \delta_{ki} = 1/3$. It is easy to verify that conditions of Theorem (1) hold. Therefore, by Theorem 1, system (11) has a unique $\pi$-periodic solution, which is globally exponentially stable. By numerical analysis, the conclusion can be showed clearly, see Figure 1.

## 5 Conclusion

In this paper, by using the Halanay inequality and constructing Lyapunov functions, we address the existence and exponential stability of periodic solutions for a class of generalized cellular neural networks with impulses and time-varying delays. Easily verifiable sufficient conditions are obtained. The main results extend and improve some previously known results [8, 11, 12, 27]. The criteria possess many adjustable parameters which provide flexibility for the design and analysis of a dynamical system. It is interesting and valuable.

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