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Statistical approximation by Kantorovich-type discrete q-Beta operators

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Abstract
The aim of the present paper is to introduce a Kantorovich-type modification of the q-discrete beta operators and to investigate their statistical and weighted statistical approximation properties. Rates of statistical convergence by means of the modulus of continuity and the Lipschitz-type function are also established for operators. Finally, we construct a bivariate generalization of the operator and also obtain the statistical approximation properties.
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1 Introduction

Gupta *et al.* [1] introduced discrete q-Beta operators as follows:

$$V_{n,q}\big(f(t);x\big) = V_n(f;q;x) = \frac{1}{[n]_q}\sum_{k=0}^{\infty} p_{n,k}(q;x) f\left(\frac{[k]_q}{[n+1]_q q^{k-1}}\right), \tag{1.1}$$

where

$$p_{n,k}(q;x) = \frac{q^{k(k-1)/2}}{B_q(k+1,n)} \frac{x^k}{(1+x)_q^{n+k+1}}.$$

In the above paper, Gupta *et al.* [1] introduced and studied some approximation properties of these operators. They also obtained some global direct error estimates for the above operators using the second-order Ditzian-Totik modulus of smoothness and defined and studied the limit discrete q-Beta operator. Also, they gave the following equalities:

$$V_n(1;q;x) = 1, \qquad V_n(t;q;x) = x \quad \text{for every } n \in \mathbb{N} \quad \text{and}$$
$$V_n\big(t^2;q;x\big) = \left(\frac{1}{q[n+1]_q} + 1\right)x^2 + \frac{x}{[n+1]_q}.$$

In the recent years, applications of q-calculus in approximation theory is one of the interesting areas of research. Several authors have proposed the q analogues of Kantorovich-type modification of different linear positive operators and studied their statistical approximation behaviors.

In 1974, Khan [2] studied approximation of functions in various classes using different types of operators.

On the other hand, statistical convergence was first introduced by Fast [3], and it has become an area of active research. Also, statistical convergence was introduced by Gadjiev and Orhan [4], Doğru [5], Duman [6], Gupta and Radu [7], Ersan and Doğru [8] and Doğru and Örkcü [9].

In 2011, Örkcü and Doğru obtained weighted statistical approximation properties of Kantorovich-type q-Szász-Mirakjan operators [10].

Recently, Doğru and Kanat [11] defined the Kantorovich-type modification of Lupas operators as follows:

$$\tilde{R}_n(f;q;x) = [n+1]\sum_{k=0}^{n}\left(\int_{[k]/[n+1]}^{[k+1]/[n+1]} f(t)\,d_qt\right)\binom{n}{k}\frac{q^{-k}q^{k(k-1)/2}x^k(1-x)^{n-k}}{(1-x+qx)\cdots(1-x+q^{n-1}x)}. \tag{1.2}$$

In [11], Doğru and Kanat proved the following statistical Korovkin-type approximation theorem for operators (1.2).

Theorem 1 *Let $q := (q_n)$, $0 < q < 1$, be a sequence satisfying the following conditions*:

$$st\text{-}\lim_n q_n = 1, \qquad st\text{-}\lim_n q_n^n = a \quad (a < 1) \quad \textit{and} \quad st\text{-}\lim_n \frac{1}{[n]} = 0, \tag{1.3}$$

then if f is any monotone increasing function defined on $[0,1]$, *for the positive linear operator $\tilde{R}_n(f;q;x)$, then*

$$st\text{-}\lim_n \left\|\tilde{R}_n(f;q;\cdot) - f\right\|_{C[0,1]} = 0$$

holds.

In [5], Doğru gave some example so that (q_n) is statistically convergent to 1 in ordinary case. Throughout the present paper, we consider $0 < q < 1$. Following [12, 13], for each non-negative integer n, we have

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & q \neq 1, \\ n, & q = 1, \end{cases}$$

$$[n]_q! = \begin{cases} [n]_q[n-1]_q[n-2]_q\cdots[1]_q, & n = 1,2,\ldots, \\ 1, & n = 0, \end{cases}$$

and

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Further, we use the q-Pochhammer symbol, which is defined as

$$(1+x)_q^n = \prod_{j=0}^{n-1}\left(1+q^jx\right).$$

The q-derivative $\mathcal{D}_q f$ of a function f is defined by

$$(\mathcal{D}_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x} \quad \text{if } x \neq 0.$$

The q-Jackson integral is defined as (see [14])

$$\int_0^b f(x)\, d_q x = (1-q)b \sum_{n=0}^{\infty} f\bigl(bq^n\bigr) q^n, \quad 0 < q < 1, \tag{1.4}$$

and over a general interval $[a,b]$, one defines

$$\int_a^b f(x)\, d_q x = \int_0^b f(x)\, d_q x - \int_0^a f(x)\, d_q x. \tag{1.5}$$

Now, let us consider the following Kantorovich-type modification of discrete q-Beta operators for each positive integer n and $q \in (0,1)$:

$$V_n^*(f;q;x) = \frac{[n+1]_q}{[n]_q} \sum_{k=0}^{\infty} \left(\int_{[k]_q/[n+1]_q}^{[k+1]_q/[n+1]_q} f(t)\, d_q t \right) \frac{p_{n,k}(q;x)}{q^{2k-1}}, \tag{1.6}$$

where f is a continuous and non-decreasing function on the interval $[0,\infty)$, $x \in [0,\infty)$.

It is seen that the operators V_n^* are linear from the definition of q-integral, and since f is a non-decreasing function, q-integral is positive, so V_n^* are positive.

To obtain the statistical convergence of operators (1.6), we need the following basic result.

2 Basic result

Lemma 1 *The following equalities hold*:

(i) $V_n^*(1;q;x) = 1$,

(ii) $V_n^*(t;q;x) = x + \frac{q}{[2]_q[n+1]_q}$,

(iii) $V_n^*(t^2;q;x) = \frac{q^{n-2}[n+2]_q x^2}{[n+1]_q} x^2 + (\frac{q^{n-1}}{[n+1]_q} + \frac{(2q+1)}{[n+1]_q[3]_q})x + \frac{q}{[n+1]_q^2[3]_q}$.

Proof By using (1.4), (1.5) and the equality $[k+1]_q = 1 + [k]_q$, we have

$$\int_{[k]_q/[n+1]_q}^{[k+1]_q/[n+1]_q} d_q t = \frac{q^k}{[n+1]_q}, \tag{2.1}$$

$$\int_{[k]_q/[n+1]_q}^{[k+1]_q/[n+1]_q} t\, d_q t = \frac{q^k}{[n+1]_q^2} \left([k]_q + \frac{1}{[2]_q} \right), \tag{2.2}$$

$$\int_{[k]_q/[n+1]_q}^{[k+1]_q/[n+1]_q} t^2\, d_q t = \frac{q^k}{[n+1]_q^3} \left([k]_q^2 + \frac{2q+1}{[3]_q}[k]_q + \frac{1}{[3]_q} \right). \tag{2.3}$$

Hence, by using $V_n(1;q;x) = 1$ and (2.2), we get

$$V_n^*(1;q;x) = 1.$$

Similarly, using (2.2), $V_n(1;q;x) = 1$ and $V_n(t;q;x) = x$, we obtain

$$\begin{aligned} V_n^*(t;q;x) &= V_n(t;q;x) + V_n(1;q;x)\frac{q}{[n+1]_q^2[3]_q} \\ &= x + \frac{q}{[2]_q[n+1]_q}. \end{aligned}$$

Finally, using (2.3), $V_n(1;q;x) = 1$, $V_n(t;q;x) = x$ and $V_n(t^2;q;x) = (\frac{1}{q[n+1]_q} + 1)x^2 + \frac{x}{[n+1]_q}$, we obtain

$$\begin{aligned} V_n^*(t;q;x) &= q^{n-1}V_n(t^2;q;x) + \frac{(2q+1)}{[n+1]_q[3]_q}V_n(t,q;x) + \frac{q}{[n+1]_q^2[3]_q}V_n(1,q;x) \\ &= \frac{q^{n-2}[n+2]_q}{[n+1]_q}x^2 + \left(\frac{q^{n-1}}{[n+1]_q} + \frac{(2q+1)}{[n+1]_q[3]_q}\right)x + \frac{q}{[n+1]_q^2[3]_q}. \end{aligned}$$

□

Remark 1 From Lemma 1, we have

$$\begin{aligned} \alpha_n(x) &= V_n^*(t-x;q;x) = \frac{q}{[2]_q[n+1]_q}, \\ \delta_n(x) &= V_n^*\big((t-x)^2;q;x\big) = V_n^*\big(t^2;q;x\big) - 2xV_n^*(t;q;x) + x^2 \\ &= \left(\frac{q^{n-2}[n+2]_q}{[n+1]_q} - 1\right)x^2 + \left(\frac{q^{n-1}}{[n+1]_q} + \frac{(2q+1)}{[n+1]_q[3]_q} - \frac{2q}{[n+1]_q[2]_q}\right)x \\ &\quad + \frac{q}{[n+1]_q^2[3]_q}. \end{aligned}$$

Remark 2 If we put $q = 1$, we get the moments of Kantorovich-type modification of discrete beta operators as

$$\begin{aligned} V_n^*(t;1;x) &= x + \frac{1}{2(n+1)}, \\ V_n^*\big(t^2;1;x\big) &= \frac{(n+2)}{(n+1)}x^2 + \frac{2x}{(n+1)} + \frac{1}{3(n+1)^2}, \\ V_n^*(t-x;1;x) &= \frac{1}{2(n+1)}, \end{aligned}$$

and

$$\begin{aligned} V_n^*\big((t-x)^2;1;x\big) &= V_n^*\big(t^2;1;x\big) - 2xV_n^*(t;1;x) + x^2 \\ &= \frac{x^2}{(n+1)} + \frac{x}{(n+1)} + \frac{1}{3(n+1)^2}. \end{aligned}$$

3 Korovkin-type statistical approximation properties

The study of Korovkin-type statistical approximation theory is a well-established area of research, which deals with the problem of approximating a function with the help of a sequence of positive linear operators (see [9, 15, 16] for details). The usual Korovkin theorem is devoted to approximation by positive linear operators on finite intervals. The main aim of this paper is to obtain the Korovkin-type statistical approximation properties of our operators defined in (1.6), with the help of Theorem 1.

Let us recall the concept of a limit of a sequence extended to a statistical limit by using the natural density δ of a set K of positive integers:

$$\delta(K) = \lim_n \frac{1}{n}\{\text{the number } k \leq n \text{ such that } k \in K\}$$

whenever the limit exists (see [17]). So, the sequence $x = (x_k)$ is said to be statistically convergent to a number L, meaning that for every $\varepsilon > 0$,

$$\delta\bigl\{k : |x_k - L| \geq \varepsilon\bigr\} = 0$$

and it is denoted by $st\text{-}\lim_k x_k = L$.

In this part, we will use the notation $\|f\|$ instead of $\|f\|_{C[0,\mu]}$ for abbreviation.

Theorem 2 *Let $q = (q_n)$ be a sequence satisfying* (1.3) *for $0 < q_n \leq 1$ and a Kantorovich-type modification of discrete q-Beta operators given by* (1.6). *Then, for any function $f \in C[0,\mu] \subset C[0,\infty)$ and $x \in [0,\mu] \subset [0,\infty)$, where $\mu > 0$, we have*

$$st\text{-}\lim_n \bigl\| V_n^*(f;q_n;\cdot) - f \bigr\| = 0,$$

where $C[0,\mu]$ denotes the space of all real bounded functions f which are continuous in $[0,\mu]$.

Proof Using $V_n^*(1;q_n;x) = 1$, it is clear that

$$st\text{-}\lim_n \bigl\| V_n^*(1;q_n;x) - 1 \bigr\| = 0.$$

Now, by Lemma 1(ii), we have

$$\bigl\| V_n^*(t;q_n;x) - x \bigr\| = \left\| x + \frac{q_n}{[2]_{q_n}[n+1]_{q_n}} - x \right\| \leq \frac{q_n}{[2]_{q_n}[n+1]_{q_n}}. \tag{3.1}$$

For a given $\varepsilon > 0$, we define the following sets:

$$U = \bigl\{k : \bigl\| V_n^*(t;q_k;x) - x \bigr\| \geq \varepsilon\bigr\}$$

and

$$U_1 = \left\{k : \frac{q_k}{[2]_{q_k}[k+1]_{q_k}} \geq \varepsilon\right\}.$$

From (3.1), one can see that $U \subset U_1$. So, we get

$$\delta\bigl\{k \leq n : \bigl\| V_n^*(t,q_k;x) - x \bigr\| \geq \varepsilon\bigr\} \leq \delta\left\{k \leq n : \frac{q_k}{[2]_{q_k}[k+1]_{q_k}} \geq \varepsilon\right\}.$$

By using (1.3), it is clear that

$$st\text{-}\lim_n \left(\frac{q_n}{[2]_{q_n}[n+1]_{q_n}}\right) = 0.$$

So,

$$\delta\left\{k \leq n : \frac{q_k}{[2]_{q_k}[k+1]_{q_k}} \geq \varepsilon\right\} = 0,$$

then

$$st\text{-}\lim_n \left\| V_n^*(t;q_n;x) - x \right\| = 0.$$

Finally, by Lemma 1(iii), we have

$$\begin{aligned}
&\left\| V_n^*\left(t^2;q_n;x\right) - x^2 \right\| \\
&\quad = \left\| \frac{q_n^{n-2}[n+2]_{q_n}}{[n+1]_{q_n}} x^2 + \left(\frac{q_n^{n-1}}{[n+1]_{q_n}} + \frac{(2q_n+1)}{[n+1]_{q_n}[3]_{q_n}}\right) x + \frac{q_n}{[n+1]_{q_n}^2[3]_{q_n}} - x^2 \right\| \\
&\quad \leq \left| \frac{q_n^{n-2}[n+2]_{q_n}}{[n+1]_{q_n}} - 1 \right| \mu^2 + \left| \frac{q_n^{n-1}}{[n+1]_{q_n}} + \frac{(2q_n+1)}{[n+1]_{q_n}[3]_{q_n}} \right| \mu + \left| \frac{q_n}{[n+1]_{q_n}^2[3]_{q_n}} \right| \\
&\quad = A_2\left(\left(\frac{1}{q_n} - 1\right) + \frac{1}{[n+1]_{q_n}}\left(q_n^{n-2} + q_n^{n-1} + \frac{(2q_n+1)}{[3]_{q_n}}\right) + \frac{q_n}{[n+1]_{q_n}^2[3]_{q_n}}\right),
\end{aligned}$$

where $A_2 = \max\{\mu^2, \mu, 1\} = \mu^2$.

If we choose $\alpha_n = (\frac{1}{q_n} - 1)$, $\beta_n = \frac{1}{[n+1]_{q_n}}(q_n^{n-2} + q_n^{n-1} + \frac{(2q_n+1)}{[3]_{q_n}})$, $\gamma_n = \frac{q_n}{[n+1]_{q_n}^2[3]_{q_n}}$, then one can write

$$st\text{-}\lim_n \alpha_n = st\text{-}\lim_n \beta_n = st\text{-}\lim_n \gamma_n = 0, \tag{3.2}$$

by (1.3). Now, given $\varepsilon > 0$, we define the following four sets:

$$\begin{aligned}
&S = \left\{k : \left\| V_n^*\left(t^2;q_k;x\right) - x^2 \right\| \geq \frac{\varepsilon}{A_2}\right\}, \\
&S_1 = \left\{k : \alpha_k \geq \frac{\varepsilon}{3A_2}\right\}, \qquad S_2 = \left\{k : \beta_k \geq \frac{\varepsilon}{3A_2}\right\}, \qquad S_3 = \left\{k : \gamma_k \geq \frac{\varepsilon}{3A_2}\right\}.
\end{aligned}$$

It is obvious that $S \subseteq S_1 \cup S_2 \cup S_3$. So, we get

$$\begin{aligned}
\delta\left\{k \leq n : \left\| V_n^*(t;q_k;x) - x \right\| \geq \frac{\varepsilon}{A_2}\right\} &\leq \delta\left\{k \leq n : \alpha_k \geq \frac{\varepsilon}{3A_2}\right\} + \delta\left\{k \leq n : \beta_k \geq \frac{\varepsilon}{3A_2}\right\} \\
&\quad + \delta\left\{k \leq n : \gamma_k \geq \frac{\varepsilon}{3A_2}\right\}.
\end{aligned}$$

So, the right-hand side of the inequalities is zero by (3.2), then

$$st\text{-}\lim_n \left\| V_n^*\left(t^2;q_n;x\right) - x^2 \right\| = 0.$$

So, the proof is completed. □

4 Weighted statistical approximation

Let $B_{x^2}[0,\infty)$ be the set of all functions f defined on $[0,\infty)$ satisfying the condition $f(x) \leq M_f(1+x^2)$, where M_f is a constant depending only on f. By $C_{x^2}[0,\infty)$, we denote the subspace of all continuous functions belonging to $B_{x^2}[0,\infty)$. Also, let $C^*_{x^2}[0,\infty)$ be the subspace of all functions $f \in C_{x^2}[0,\infty)$, for which $\lim_{x\to\infty}\frac{f(x)}{1+x^2}$ is finite. The norm on $C^*_{x^2}[0,\infty)$ is $\|f\|_{x^2} = \sup_{x\in[0,\infty)}\frac{|f(x)|}{1+x^2}$.

Theorem 3 *Let $q=(q_n)$ be a sequence satisfying* (1.3) *for* $0 < q_n \leq 1$. *Then, for all nondecreasing functions $f \in C^*_{x^2}[0,\infty)$, we have*

$$st\text{-}\lim_n \left\| V_n^*(f;q_n;\cdot) - f \right\|_{x^2} = 0.$$

Proof As a consequence of Lemma 1, since $V_n^*(x^2;q_n;x) \leq C_1x^2$, where C_1 is a positive constant, $V_n^*(f;q_n;x)$ is a sequence of linear positive operators acting from $C^*_{x^2}[0,\infty)$ to $B_{x^2}[0,\infty)$. Using $V_n^*(1;q_n;x)=1$, it is clear that

$$st\text{-}\lim_n \left\| V_n^*(1;q_n;x) - 1 \right\|_{x^2} = 0.$$

Now, by Lemma 1(ii), we have

$$\left\| V_n^*(t;q_n;x) - x \right\|_{x^2} = \sup_{x\in[0,\infty)} \frac{|V_n^*(t;q_n;x) - x|}{1+x^2} \leq \frac{q_n}{[2]_{q_n}[n+1]_{q_n}}. \tag{4.1}$$

By using (1.3), it is clear that

$$st\text{-}\lim_n \left(\frac{q_n}{[2]_{q_n}[n+1]_{q_n}} \right) = 0,$$

then

$$st\text{-}\lim_n \left\| V_n^*(t;q_n;x) - x \right\|_{x^2} = 0.$$

Finally, by Lemma 1(iii), we have

$$\begin{aligned}
&\left\| V_n^*\left(t^2;q_n;x\right) - x^2 \right\|_{x^2} \\
&\quad\leq \left(\frac{q_n^{n-2}[n+2]_{q_n}}{[n+1]_{q_n}} - 1 \right) \sup_{x\in[0,\infty)} \frac{x^2}{1+x^2} + \left(\frac{q_n^{n-1}}{[n+1]_{q_n}} + \frac{(2q_n+1)}{[n+1]_{q_n}[3]_{q_n}} \right) \\
&\qquad \times \sup_{x\in[0,\infty)} \frac{x}{1+x^2} + \frac{q_n}{[n+1]^2_{q_n}[3]_{q_n}} \\
&\quad\leq \left(\frac{q_n^{n-2}[n+2]_{q_n}}{[n+1]_{q_n}} - 1 \right) + \left(\frac{q_n^{n-1}}{[n+1]_{q_n}} + \frac{(2q_n+1)}{[n+1]_{q_n}[3]_{q_n}} \right) + \frac{q_n}{[n+1]^2_{q_n}[3]_{q_n}} \\
&\quad= \left(\frac{1}{q_n} - 1 \right) + \frac{1}{[n+1]_{q_n}} \left(q_n^{n-2} + q_n^{n-1} + \frac{(2q_n+1)}{[3]_{q_n}} \right) + \frac{q_n}{[n+1]^2_{q_n}[3]_{q_n}}.
\end{aligned}$$

If we choose $\alpha_n = (\frac{1}{q_n} - 1)$, $\beta_n = \frac{1}{[n+1]_{q_n}}(q_n^{n-2} + q_n^{n-1} + \frac{(2q_n+1)}{[3]_{q_n}})$, $\gamma_n = \frac{q_n}{[n+1]^2_{q_n}[3]_{q_n}}$, then one can write

$$st\text{-}\lim_n \alpha_n = st\text{-}\lim_n \beta_n = st\text{-}\lim_n \gamma_n = 0, \tag{4.2}$$

by (1.3). Now, given $\varepsilon > 0$, we define the following four sets:

$$P = \{k : \| V_n^*(t^2; q_k; x) - x^2 \|_{x^2} \geq \varepsilon\},$$
$$P_1 = \left\{k : \alpha_k \geq \frac{\varepsilon}{3}\right\}, \qquad P_2 = \left\{k : \beta_k \geq \frac{\varepsilon}{3}\right\}, \qquad P_3 = \left\{k : \gamma_k \geq \frac{\varepsilon}{3}\right\}.$$

It is obvious that $P \subseteq P_1 \cup P_2 \cup P_3$. So, we get

$$\delta\{k \leq n : \| V_n^*(t; q_k; x) - x \|_{x^2} \geq \varepsilon\} \leq \delta\left\{k \leq n : \alpha_k \geq \frac{\varepsilon}{3}\right\} + \delta\left\{k \leq n : \beta_k \geq \frac{\varepsilon}{3}\right\} + \delta\left\{k \leq n : \gamma_k \geq \frac{\varepsilon}{3}\right\}.$$

So, the right-hand side of the inequalities is zero by (4.2), then

$$st\text{-}\lim_n \| V_n^*(t^2; q_n; x) - x^2 \|_{x^2} = 0.$$

So, the proof is completed. □

5 Rates of statistical convergence

In this part, rates of statistical convergence of operator (1.6) by means of modulus of continuity and Lipschitz functions are introduced.

Lemma 2 [15] *Let* $0 < q < 1$ *and* $a \in [0, bq]$, $b > 0$. *The inequality*

$$\int_a^b |t - x| \, d_q t \leq \left(\int_a^b |t - x|^2 \, d_q t\right)^{1/2} \left(\int_a^b d_q t\right)^{1/2}$$

is satisfied.

Let $C_B[0, \infty)$, the space of all bounded and continuous functions on $[0, \infty)$, and $x \geq 0$. Then, for $\delta > 0$, the modulus of continuity of f denoted by $\omega(f; \delta)$ is defined to be

$$\omega(f; \delta) = \sup_{x-\delta \leq t \leq x+\delta; t \in [0,\infty)} |f(t) - f(x)|.$$

Then it is known that $\lim_{\delta \to 0} \omega(f; \delta) = 0$ for $f \in C_B[0, \infty)$, and also, for any $\delta > 0$ and each $t, x \geq 0$, we have

$$|f(t) - f(x)| \leq \omega(f; \delta)\left(1 + \frac{|t - x|}{\delta}\right). \tag{5.1}$$

Theorem 4 *Let* (q_n) *be a sequence satisfying* (1.3). *For every non-decreasing* $f \in C_B[0, \infty)$, $x \geq 0$ *and* $n \in \mathbb{N}$, *we have*

$$|V_n^*(f; q_n; x) - f(x)| \leq 2\omega\left(f; \sqrt{\delta_n(x)}\right),$$

where

$$\delta_n(x) = \left(\frac{q_n^{n-2}[n+2]_{q_n}}{[n+1]_{q_n}} - 1\right)x^2 + \left(\frac{q_n^{n-1}}{[n+1]_{q_n}} + \frac{(2q_n+1)}{[n+1]_{q_n}[3]_{q_n}} - \frac{2q_n}{[n+1]_{q_n}[2]_{q_n}}\right)x + \frac{q_n}{[n+1]_{q_n}^2[3]_{q_n}}. \tag{5.2}$$

Proof Let non-decreasing $f \in C_B[0,\infty)$ and $x \geq 0$. Using linearity and positivity of the operators $V_n^*(f;q_n;x)$ and then applying (5.1), we get for $\delta > 0$ and $n \in \mathbb{N}$ that

$$\begin{aligned}\left|V_n^*(f;q_n;x) - f(x)\right| &\leq V_n^*\left(\left|f(t) - f(x)\right|;q_n;x\right) \\ &\leq \omega(f;\delta)\left\{V_n^*(1;q_n;x) + \frac{1}{\delta}V_n^*\left(|t-x|;q_n;x\right)\right\}.\end{aligned}$$

Taking into account $V_n^*(1;q_n;x) = 1$ and then applying Lemma 2 with $a = [k]_q/[n+1]_q$ and $b = [k+1]_q/[n+1]_q$, we may write

$$\begin{aligned}\left|V_n^*(f;q_n;x) - f(x)\right| \leq \omega(f;\delta)\Biggl\{1 + \frac{1}{\delta}\frac{[n+1]_q}{[n]_q}\sum_{k=0}^{\infty}\frac{p_{n,k}(q;x)}{q^{2k-1}}\left(\int_{[k]_q/[n+1]_q}^{[k+1]_q/[n+1]_q}(t-x)^2\,d_qt\right)^{1/2} \\ \times\left(\int_{[k]_q/[n+1]_q}^{[k+1]_q/[n+1]_q}d_qt\right)^{1/2}\Biggr\}.\end{aligned}$$

By using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\left|V_n^*(f;q_n;x) - f(x)\right| &= \omega(f;\delta)\Biggl\{1 + \frac{1}{\delta}\left(\frac{[n+1]_q}{[n]_q}\sum_{k=0}^{\infty}\frac{p_{n,k}(q;x)}{q^{2k-1}}\int_{[k]_q/[n+1]_q}^{[k+1]_q/[n+1]_q}(t-x)^2\,d_qt\right)^{1/2} \\ &\quad\times\left(\frac{[n+1]_q}{[n]_q}\sum_{k=0}^{\infty}\frac{p_{n,k}(q;x)}{q^{2k-1}}\int_{[k]_q/[n+1]_q}^{[k+1]_q/[n+1]_q}d_qt\right)^{1/2}\Biggr\} \\ &\leq \omega(f;\delta)\left\{1 + \frac{1}{\delta}\left(V_n^*\left((t-x)^2;q_n;x\right)\right)^{1/2}\left(V_n^*(1;q_n;x)\right)^{1/2}\right\}.\end{aligned}$$

Taking $q = q_n$, a sequence satisfying (1.3), and using $\delta_n(x) = V_n^*((t-x)^2;q_n;x)$ and then choosing $\delta = \delta_n(x)$ as in (5.2), the theorem is proved. □

Notice that by the conditions in (1.3), $st\text{-}\lim_n \delta_n = 0$. By (5.1), we have

$$st\text{-}\lim_n \omega(f;\delta_n) = 0.$$

This gives us the pointwise rate of statistical convergence of the operators $V_n^*(f;q_n;x)$ to $f(x)$.

Now we will study the rate of convergence of the operator $V_n^*(f;q_n;x)$ with the help of functions of the Lipschitz class $\mathrm{Lip}_M(\alpha)$, where $M > 0$ and $0 < \alpha \leq 1$. Recall that a function $f \in C_B[0,\infty)$ belongs to $\mathrm{Lip}_M(\alpha)$ if the inequality

$$\left|f(t) - f(x)\right| \leq M|t-x|^\alpha; \quad \forall t,x \in [0,\infty).$$

We have the following theorem.

Theorem 5 *Let the sequence $q = (q_n)$ satisfy the conditions given in* (1.3), *and let $f \in \mathrm{Lip}_M(\alpha)$, $x \geq 0$ with $0 < \alpha \leq 1$. Then*

$$\left|V_n^*(f;q_n;x) - f(x)\right| \leq M\delta_n^{\alpha/2}(x), \tag{5.3}$$

where $\delta_n(x)$ is given as in (5.2).

Proof Since $V_n^*(f;q_n;x)$ are linear positive operators and $f \in \mathrm{Lip}_M(\alpha)$, on $x \geq 0$ with $0 < \alpha \leq 1$, we can write

$$\begin{aligned}\left|V_n^*(f;q_n;x) - f(x)\right| &\leq V_n^*\left(\left|f(t) - f(x)\right|;q_n;x\right)\\ &\leq MV_n^*\left(|t-x|^\alpha;q_n;x\right).\end{aligned}$$

If we take $p = \frac{2}{\alpha}$, $q = \frac{2}{2-\alpha}$, applying Lemma 2 and Hölder's inequality, we obtain

$$\begin{aligned}\left|V_n^*(f;q_n;x) - f(x)\right| &\leq M\frac{[n+1]_q}{[n]_q}\sum_{k=0}^{\infty}\frac{p_{n,k}(q;x)}{q^{2k-1}}\left(\int_{[k]_q/[n+1]_q}^{[k+1]_q/[n+1]_q}(t-x)^2\,d_qt\right)^{\alpha/2}\\ &\quad\times\left(\int_{[k]_q/[n+1]_q}^{[k+1]_q/[n+1]_q}d_qt\right)^{(2-\alpha)/2}\Big\},\\ \left|V_n^*(f;q_n;x) - f(x)\right| &= M\left(\frac{[n+1]_q}{[n]_q}\sum_{k=0}^{\infty}\frac{p_{n,k}(q;x)}{q^{2k-1}}\int_{[k]_q/[n+1]_q}^{[k+1]_q/[n+1]_q}(t-x)^2\,d_qt\right)^{\alpha/2}\\ &\quad\times\left(\frac{[n+1]_q}{[n]_q}\sum_{k=0}^{\infty}\frac{p_{n,k}(q;x)}{q^{2k-1}}\int_{[k]_q/[n+1]_q}^{[k+1]_q/[n+1]_q}d_qt\right)^{(2-\alpha)/2}\\ &\leq M\left(V_n^*\left((t-x)^2;q_n;x\right)\right)^{\alpha/2}\left(V_n^*(1;q_n;x)\right)^{(2-\alpha)/2}\\ &\leq M\left(V_n^*\left((t-x)^2;q_n;x\right)\right)^{\alpha/2}.\end{aligned}$$

Taking $\delta_n(x) = (V_n^*((t-x)^2;q_n;x))$, as in (5.2), we get the desired result. □

6 The construct of the bivariate operators of Kantorovich type

The purpose of this part is to give a representation for the bivariate operators of Kantorovich type (1.6), introduce the statistical convergence of the operators to the function f and show the rate of statistical convergence of these operators.

For $f : C([0,\infty) \times [0,\infty)) \to C([0,\infty) \times [0,\infty))$ and $0 < q_{n_1}, q_{n_2} \leq 1$, let us define the bivariate case of operators (1.6) as follows:

$$\begin{aligned}&V_{n_1,n_2}^*(f;q_{n_1},q_{n_2},x,y)\\ &\quad= \frac{[n_1+1]_{q_{n_1}}[n_2+1]_{q_{n_2}}}{[n_1]_{q_{n_1}}[n_2]_{q_{n_2}}}\\ &\qquad\times\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\left(\int_{[k_1]_{q_{n_1}}/[n_1+1]_{q_{n_1}}}^{[k_1+1]_{q_{n_1}}/[n_1+1]_{q_{n_1}}}\int_{[k_2]_{q_{n_2}}/[n_2+1]_{q_{n_2}}}^{[k_2+1]_{q_{n_2}}/[n_2+1]_{q_{n_2}}}f(s,t)\,d_{q_{n_1}}s\,d_{q_{n_2}}t\right)\\ &\qquad\times\frac{p_{n_1,n_2}(q_{n_1},q_{n_2},x,y)}{q_{n_1}^{2k_1-1}q_{n_2}^{2k_2-1}},\end{aligned}\tag{6.1}$$

where

$$p_{n_1,n_2,k_1,k_2}(q_{n_1},q_{n_2},x,y) = \frac{q_{n_1}^{k_1(k_1-1)/2} q_{n_2}^{k_2(k_2-1)/2}}{B_{q_{n_1}}(k_1+1,n_1)B_{q_{n_2}}(k_2+1,n_2)} \frac{x^{k_1}y^{k_2}}{(1+x)_{q_{n_1}}^{n_1+k_1+1}(1+y)_{q_{n_2}}^{n_2+k_2+1}}.$$

In [18], Erkuş and Duman proved the statistical Korovkin-type approximation theorem for the bivariate linear positive operators to the functions in space H_{ω_2}.

In 2006, Doğru and Gupta [19] introduced a bivariate generalization of the q-MKZ operators and investigated its Korovkin-type approximation properties.

Recently, Ersan and Doğru [8] obtained the statistical Korovkin-type theorem and lemma for the bivariate linear positive operators defined in the space H_{ω_2} as follows.

Theorem 6 [8] *Let L_{n_1,n_2} be the sequence of linear positive operators acting from $H_{\omega_2}(R_+^2)$ into $C_B(R_+)$, where $R_+ = [0,\infty)$. Then, for any $f \in H_{\omega_2}$,*

$$st\text{-}\lim_{n_1,n_2} \| L_{n_1,n_2}(f) - f \| = 0.$$

Lemma 3 [8] *The bivariate operators defined in* [8] *satisfy the following items*:

(i) $L_{n_1,n_2}(f_0;q_{n_1},q_{n_2},x,y) = q_{n_1}q_{n_2}$,

(ii) $L_{n_1,n_2}(f_1;q_{n_1},q_{n_2},x,y) = q_{n_1}q_{n_2}\frac{[n_1]_{q_{n_1}}}{[n_1+1]_{q_{n_1}}}\frac{x}{1+x}$,

(iii) $L_{n_1,n_2}(f_2;q_{n_1},q_{n_2},x,y) = q_{n_1}q_{n_2}\frac{[n_2]_{q_{n_2}}}{[n_2+1]_{q_{n_2}}}\frac{y}{1+y}$,

(iv) $L_{n_1,n_2}(f_3;q_{n_1},q_{n_2},x,y) = q_{n_1}^3 q_{n_2}\frac{[n_1]_{q_{n_1}}[n_1-1]_{q_{n_1}}}{[n_1+1]_{q_{n_1}}^2}\frac{x^2}{(1+x)(1+q_{n_1}x)} + q_{n_1}q_{n_2}\frac{[n_1]_{q_{n_1}}}{[n_1+1]_{q_{n_1}}^2}\frac{x}{1+x} + q_{n_1}q_{n_2}^3\frac{[n_2]_{q_{n_2}}[n_2-1]_{q_{n_2}}}{[n_2+1]_{q_{n_2}}^2}\frac{y^2}{(1+y)(1+q_{n_2}y)} + q_{n_1}q_{n_2}\frac{[n_2]_{q_{n_2}}}{[n_2+1]_{q_{n_2}}^2}\frac{y}{1+y}$.

In order to obtain the statistical convergence of operator (6.1), we need the following lemma.

Lemma 4 *The bivariate operators defined in* (6.1) *satisfy the following equalities*:

(i) $V_{n_1,n_2}^*(f_0;q_{n_1},q_{n_2},x,y) = 1$,

(ii) $V_{n_1,n_2}^*(f_1;q_{n_1},q_{n_2},x,y) = x + \frac{q_{n_1}}{[2]_{q_{n_1}}[n_1+1]_{q_{n_1}}}$,

(iii) $V_{n_1,n_2}^*(f_2;q_{n_1},q_{n_2},x,y) = y + \frac{q_{n_2}}{[2]_{q_{n_2}}[n_2+1]_{q_{n_2}}}$,

(iv) $V_{n_1,n_2}^*(f_3;q_{n_1},q_{n_2},x,y) = \frac{q_{n_1}^{n_1-2}[n_1+2]_{q_{n_1}}}{[n_1+1]_{q_{n_1}}}x^2 + (\frac{q_{n_1}^{n_1-1}}{[n_1+1]_{q_{n_1}}} + \frac{(2q_{n_1}+1)}{[n_1+1]_{q_{n_1}}[3]_{q_{n_1}}})x + \frac{q_{n_1}}{[n_1+1]_{q_{n_1}}^2[3]_{q_{n_1}}} + \frac{q_{n_2}^{n_2-2}[n_2+1]_{q_{n_2}}}{[n_2+1]_{q_{n_2}}}y^2 + (\frac{q_{n_2}^{n_2-1}}{[n_2+1]_{q_{n_2}}} + \frac{(2q_{n_2}+1)}{[n_2+1]_{q_{n_2}}[3]_{q_{n_2}}})y + \frac{q_{n_2}}{[n_2+1]_{q_{n_2}}^2[3]_{q_{n_2}}}$.

Proof By the help of the proofs for the bivariate operator in [20], the conditions may be easily obtained. So, the proof can be omitted.

Let $q = (q_{n_1})$ and $q = (q_{n_2})$ be the sequence that converges statistically to 1 but does not converge in ordinary sense, so for $0 < q_{n_1}, q_{n_2} \leq 1$, it can be written as

$$st\text{-}\lim_{n_1} q_{n_1} = st\text{-}\lim_{n_2} q_{n_2} = 1. \tag{6.2}$$

Now, under the condition in (6.2), let us show the statistical convergence of bivariate operator (6.1) with the help of the proof of Theorem 2. □

Theorem 7 *Let $q = (q_{n_1})$ and $q = (q_{n_2})$ be a sequence satisfying* (6.2) *for $0 < q_{n_1}, q_{n_2} \leq 1$, and let $V^*_{n_1,n_2}$ be a sequence of linear positive operators from $C(K)$ into $C(K)$ given by* (1.6). *Then, for any function $f \in C(K_1 \times K_1) \subset C(K \times K)$ and $x \in K_1 \times K_1 \subset K \times K$, where $K = [0,\infty) \times [0,\infty)$, $K_1 = [0,\mu] \times [0,\mu]$, we have*

$$st\text{-}\lim_{n_1,n_2} \| V^*_{n_1,n_2}(f) - f \|_{C(K_1 \times K_1)} = 0.$$

Proof Using Lemma 4, the proof can be obtained similar to the proof of Theorem 2. So, we shall omit this proof. □

7 Rates of convergence of bivariate operators

Let $K = [0,\infty) \times [0,\infty)$. Then the sup norm on $C_B(K)$ is given by

$$\|f\| = \sup_{(x,y)\in K} |f(x,y)|, \quad f \in C_B(K).$$

We consider the modulus of continuity $\omega_2(f;\delta_1,\delta_2)$ for bivariate case given by $\delta_1, \delta_2 > 0$,

$$\omega_2(f;\delta_1,\delta_2) = \{\sup|f(x',y') - f(x,y)| : (x',y'), (x,y) \in K \text{ and } |x'-x| \leq \delta_1, |y'-y| \leq \delta_2\}.$$

It is clear that a necessary and sufficient condition for a function $f \in C_B(K)$ is

$$\lim_{\delta_1,\delta_2 \to 0} \omega(f;\delta_1,\delta_2) = 0,$$

and $\omega(f;\delta_1,\delta_2)$ satisfy the following condition:

$$|f(x',y') - f(x,y)| \leq \omega(f;\delta_1,\delta_2)\left(1 + \frac{|x'-x|}{\delta_1}\right)\left(1 + \frac{|y'-y|}{\delta_2}\right) \tag{7.1}$$

for each $f \in C_B(K)$. Then observe that any function in $C_B(K)$ is continuous and bounded on K. Details of the modulus of continuity for bivariate case can be found in [21].

Now, the rate of statistical convergence of bivariate operator (6.1) by means of modulus of continuity in $f \in C_B(K)$ will be given in the following theorem.

Theorem 8 *Let $q = (q_{n_1})$ and $q = (q_{n_2})$ be a sequence satisfying the condition in* (6.2). *So, we have*

$$|V^*_{n_1,n_2}(f;q_{n_1},q_{n_2},x,y) - f(x,y)| \leq 4\omega(f;\sqrt{\delta_{n_1(x)}},\sqrt{\delta_{n_2(x)}}),$$

where

$$\delta_{n_1}(x) = \left(\frac{q_{n_1}^{n_1-2}[n_1+2]_{q_{n_1}}}{[n_1+1]_{q_{n_1}}} - 1\right)x^2 + \left(\frac{q_{n_1}^{n_1-1}}{[n_1+1]_{q_{n_1}}} + \frac{(2q_{n_1}+1)}{[n_1+1]_{q_{n_1}}[3]_{q_{n_1}}} - \frac{2q_{n_1}}{[n_1+1]_{q_{n_1}}[2]_{q_{n_1}}}\right)x + \frac{q_{n_1}}{[n_1+1]^2_{q_{n_1}}[3]_{q_{n_1}}}, \tag{7.2}$$

$$\begin{aligned}\delta_{n_2}(y) &= \left(\frac{q_{n_2}^{n_2-2}[n_2+2]_{q_{n_2}}}{[n_2+1]_{q_{n_2}}}-1\right)y^2+\left(\frac{q_{n_2}^{n_2-1}}{[n_2+1]_{q_{n_2}}}+\frac{(2q_{n_2}+1)}{[n_2+1]_{q_{n_2}}[3]_{q_{n_2}}}\right.\\&\quad\left.-\frac{2q_{n_2}}{[n_2+1]_{q_{n_2}}[2]_{q_{n_2}}}\right)y+\frac{q_{n_2}}{[n_2+1]_{q_{n_2}}^2[3]_{q_{n_2}}}.\end{aligned}\tag{7.3}$$

Proof By using the condition in (7.1), we get for $\delta_{n_1},\delta_{n_2}>0$ and $n\in\mathbb{N}$ that

$$\begin{aligned}&\left|V^*_{n_1,n_2}(f;q_{n_1},q_{n_2},x,y)-f(x,y)\right|\\&\quad\le V^*_{n_1,n_2}\left(\left|f(x',y')-f(x,y)\right|;q_{n_1},q_{n_2},x,y\right)\\&\quad\le\omega(f;\delta_{n_1(x)},\delta_{n_2(x)})\left\{V^*_{n_1,n_2}(f_0;q_{n_1},q_{n_2},x,y)+\frac{1}{\delta_{n_1}}V^*_n\left(\left|x'-x\right|;q_{n_1},q_{n_2},x,y\right)\right\}\\&\qquad\times\left\{V^*_{n_1,n_2}(f_0;q_{n_1},q_{n_2},x,y)+\frac{1}{\delta_{n_2}}V^*_n\left(\left|y'-y\right|;q_{n_1},q_{n_2},x,y\right)\right\}.\end{aligned}$$

If the Cauchy-Schwarz inequality is applied, we have

$$V^*_n\left(\left|x'-x\right|;q_{n_1},q_{n_2},x,y\right)\le\left(V^*_n\left((x'-x)^2;q_{n_1},q_{n_2},x,y\right)\right)^{1/2}\left(V^*_{n_1,n_2}(f_0;q_{n_1},q_{n_2},x,y)\right)^{1/2}.$$

So, if it is substituted in the above equation, the proof is completed. □

At last, the following theorem represents the rate of statistical convergence of bivariate operator (6.1) by means of Lipschitz $\mathrm{Lip}_M(\alpha_1,\alpha_2)$ functions for the bivariate case, where $f\in C_B[0,\infty)$ and $M>0$ and $0<\alpha_1\le1$, $0<\alpha_2\le1$, then let us define $\mathrm{Lip}_M(\alpha_1,\alpha_2)$ as

$$\left|f(x',y')-f(x,y)\right|\le M\left|x'-x\right|^{\alpha_1}\left|y'-y\right|^{\alpha_2};\quad\forall x,x',y,y'\in[0,\infty).$$

We have the following theorem.

Theorem 9 *Let the sequence $q=(q_{n_1})$ and $q=(q_{n_2})$ satisfy the conditions given in* (6.2), *and let $f\in\mathrm{Lip}_M(\alpha_1,\alpha_2)$, $x\ge0$ and $0<\alpha_1\le1$, $0<\alpha_2\le1$. Then*

$$\left|V^*_{n_1,n_2}(f;q_{n_1},q_{n_2},x,y)-f(x,y)\right|\le M{\delta_{n_1}}^{\alpha_1/2}(x){\delta_{n_2}}^{\alpha_2/2}(x),$$

where $\delta_{n_1}(x)$ and $\delta_{n_2}(x)$ are defined in (7.2), (7.3).

Proof Since $V^*_{n_1,n_2}(f;q_{n_1},q_{n_2},x,y)$ are linear positive operators and $f\in\mathrm{Lip}_M(\alpha_1,\alpha_2)$, $x\ge0$ and $0<\alpha_1\le1$, $0<\alpha_2\le1$, we can write

$$\begin{aligned}\left|V^*_{n_1,n_2}(f;q_{n_1},q_{n_2},x,y)-f(x,y)\right|&\le V^*_{n_1,n_2}\left(\left|f(x',y')-f(x,y)\right|;q_{n_1},q_{n_2},x,y\right)\\&\le MV^*_{n_1,n_2}\left(\left|x'-x\right|^{\alpha_1}\left|y'-y\right|^{\alpha_2};q_{n_1},q_{n_2},x,y\right)\\&=MV^*_{n_1,n_2}\left(\left|x'-x\right|^{\alpha_1};q_{n_1},q_{n_2},x,y\right)\\&\quad\times V^*_{n_1,n_2}\left(\left|y'-y\right|^{\alpha_2};q_{n_1},q_{n_2},x,y\right).\end{aligned}$$

If we take $p_1 = \frac{2}{\alpha_1}$, $p_2 = \frac{2}{\alpha_2}$, $q_1 = \frac{2}{2-\alpha_1}$, $q_2 = \frac{2}{2-\alpha_2}$, applying Hölder's inequality, we obtain

$$\begin{aligned}\left|V^*_{n_1,n_2}(f;q_{n_1},q_{n_2},x,y) - f(x,y)\right| &\le M\left(V^*_{n_1,n_2}\left(x'-x\right)^2;q_{n_1},q_{n_2},x,y\right)^{\alpha_1/2} \\ &\quad \times \left(V^*_{n_1,n_2}(f_0;q_{n_1},q_{n_2},x,y)\right)^{2-\alpha_1/2} \\ &\quad \times \left(V^*_{n_1,n_2}\left(\left(y'-y\right)^{\alpha_2};q_{n_1},q_{n_2},x,y\right)^{\alpha_2/2}\right. \\ &\quad \times \left(V^*_{n_1,n_2}(f_0;q_{n_1},q_{n_2},x,y)\right)^{2-\alpha_2/2} \\ &= M{\delta_{n_1}}^{\alpha_1/2}(x){\delta_{n_2}}^{\alpha_2/2}(x).\end{aligned}$$

So, the proof is completed. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
VNM, KK and LNM computed the moments of modified operators and established the asymptotic formula. VNM conceived of the study and participated in its design and coordination. VNM, KK and LNM contributed equally and significantly in writing this paper. All the authors drafted the manuscript, read and approved the final manuscript.

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