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# Mild solutions for abstract fractional differential equations with almost sectorial operators and infinite delay

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**Abstract**

Of concern is the existence of mild solutions to delay fractional differential equations with almost sectorial operators. Combining the techniques of operator semigroup, noncompact measures and fixed point theory, we obtain a new existence theorem without the assumptions that the nonlinearity $f$ satisfies a Lipschitz-type condition, and the resolvent operator associated with $A$ is compact. An example is presented.

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**Keywords:** fractional differential equations; mild solution; infinite delay; measure of noncompactness; fixed point theorem

## 1 Introduction

Fractional differential equations have been increasingly used for many mathematical models in probability, engineering, physics, astrophysics, economics, *etc.*, so the theory of fractional differential equations has in recent years been an object of investigations with increasing interest [1–15].

Most of the previous research on the fractional differential equations was done provided that the operator in the linear part is the infinitesimal generator of a strongly continuous operator semigroup, a compact semigroup, or an analytic semigroup, or is a Hille-Yosida operator (see, *e.g.*, [2, 3, 7, 11, 12]). However, as presented in Example 1.1 and Example 1.2 in [15], the resolvent operators do not satisfy the required estimate to be a sectorial operator. In [16], W. von Wahl first introduced examples of almost sectorial operators which are not sectorial. To the author's knowledge, there are few papers about the fractional evolution equations with almost sectorial operators.

Moreover, equations with delay are often more useful to describe concrete systems than those without delay. So, the study of these equations has attracted so much attention (*cf.*, *e.g.*, [7, 11, 17–21] and references therein).

In this paper, we pay our attention to the investigation of the existence of mild solutions to the following fractional differential equations with almost sectorial operators and infinite delay on a separable complex Banach space $X$:

$$\begin{aligned} &{}^cD_t^q u(t) = Au(t) + f\big(t, u(t), u_t\big), \quad t \in (0, T], \\ &u_0 = \phi \in \mathcal{P}, \end{aligned} \tag{1.1}$$

where $T > 0$, $0 < q < 1$. The fractional derivative is understood here in the Caputo sense. $\mathcal{P}$ is a phase space that will be defined later (see Definition 2.1). $A$ is an almost sectorial operator to be introduced later. Here, $f : [0,T] \times X \times \mathcal{P} \to X$, and $u_t : (-\infty, 0] \to X$ is defined by $u_t(\tau) = u(t+\tau)$ for $\tau \in (-\infty, 0]$.

Let us recall the following definition of almost sectorial operator; for more details, we refer the readers to [22, 23].

**Definition 1.1** Let $-1 < \gamma < 0$ and $0 < \omega < \frac{\pi}{2}$. By $\Theta_\omega^\gamma(X)$ we denote the family of all linear closed operators $A : D(A) \subset X \to X$ which satisfy

(1) $\sigma(A) \subset S_\omega = \{z \in \mathbf{C} \setminus \{0\}; |\arg z| \leq \omega\} \cup \{0\}$ and

(2) for every $\omega < \zeta < \pi$, there exists a constant $C_\zeta$ such that

$$\|R(z;A)\|_{L(X)} \leq C_\zeta |z|^\gamma \quad \text{for all } z \in \mathbf{C} \setminus S_\zeta.$$

A linear operator $A$ will be called an almost sectorial operator on $X$ if $A \in \Theta_\omega^\gamma(X)$.

**Remark 1.2** Let $A \in \Theta_\omega^\gamma(X)$, then the definition implies that $0 \in \rho(A)$.

We denote the semigroup associated with $A$ by $T(t)$. For $t \in S^0_{\frac{\pi}{2}-\omega} = \{z \in \mathbf{C}\setminus\{0\}; |\arg z| < \frac{\pi}{2} - \omega\}$,

$$T(t) = \mathrm{e}^{-tz}(A) = \frac{1}{2\pi i} \int_{\Gamma_\theta} \mathrm{e}^{-tz} R(z;A)\, \mathrm{d}z,$$

forms an analytic semigroup of growth order $1+\gamma$, here $\omega < \theta < \mu < \frac{\pi}{2} - |\arg t|$, the integral contour $\Gamma_\theta := \{\mathbf{R}_+ e^{i\theta}\} \cup \{\mathbf{R}_+ e^{-i\theta}\}$ is oriented counter-clockwise [15, 17, 23]. Moreover, $T(t)$ satisfies the following properties.

(i) There exists a constant $C_0 = C_0(\gamma) > 0$ such that

$$\|T(t)\|_{L(X)} \leq C_0 t^{-\gamma-1} \quad \text{for all } t > 0;$$

(ii) If $\beta > 1 + \gamma$, then $D(A^\beta) \subset \Sigma_T = \{x \in X; \lim_{t\to 0; t>0} T(t)x = x\}$;

(iii) The functional equation $T(s+t) = T(s)T(t)$ for all $s, t \in S^0_{\frac{\pi}{2}-\omega}$ holds. However, it is not satisfied for $t = 0$ or $s = 0$.

We refer the readers to [23] and references therein for more details on $T(t)$.

In this paper, we construct a pair of families of operators $\mathcal{S}_q(t)$ and $\mathcal{T}_q(t)$ ((2.3)-(2.4)) associated with $T(t)$ and use the fixed point theorem (Theorem 2.11) to study the existence of a mild solution of Equation (1.1). We obtain the existence theorem based on the theory on measures of noncompactness without the assumptions that the nonlinearity $f$ satisfies a Lipschitz-type condition, and the resolvent operator associated with $A$ is compact. An example is given to show the application of the abstract result.

## 2 Preliminaries

Throughout this paper, we set $J := [0,T]$ and denote by $X$ a separable complex Banach space with the norm $\|\cdot\|$, by $L(X)$ the Banach space of all linear and bounded operators on $X$, and by $C(J,X)$ the Banach space of all $X$-valued continuous functions on $J$ with the supremum norm. We abbreviate $\|\mu\|_{L^p(J,\mathbf{R}^+)}$ with $\|\mu\|_{L^p}$ for any $\mu \in L^p(J,\mathbf{R}^+)$.

We will employ an axiomatic definition of the phase space $\mathcal{P}$ from [18–21] which is a generalization of that given by Hale and Kato [24].

**Definition 2.1** A linear space $\mathcal{P}$ consisting of functions from $\mathbf{R}^-$ into $X$, with the seminorm $\|\cdot\|_{\mathcal{P}}$, is called an admissible phase space if $\mathcal{P}$ has the following properties.

(1) If $u:(-\infty,T]\to X$ is continuous on $J$ and $u_0\in\mathcal{P}$, then $u_t\in\mathcal{P}$ and $u_t$ is continuous in $t\in J$, and

$$\|\phi(0)\|\le M\|\phi\|_{\mathcal{P}},\quad \forall\phi\in\mathcal{P} \tag{2.1}$$

for a positive constant $M$.

(2) There exist a continuous function $C_1(t)>0$ and a locally bounded function $C_2(t)\ge 0$ in $t\ge 0$ such that

$$\|u_t\|_{\mathcal{P}}\le C_1(t)\sup_{s\in[0,t]}\|u(s)\|+C_2(t)\|u_0\|_{\mathcal{P}} \tag{2.2}$$

for $t\in[0,T]$ and $u$ as in (1).

(3) The space $\mathcal{P}$ is complete.

**Remark 2.2** Equation (2.1) in (1) is equivalent to $\|u(t)\|\le M\|u_t\|_{\mathcal{P}}$.

Based on the work in [15], we give the following definition.

**Definition 2.3** Let $\Psi_q(z)$ with $0<q<1$ be a function of Wright type (*cf.*, *e.g.*, [15])

$$\Psi_q(z):=\sum_{n=0}^{\infty}\frac{(-z)^n}{n!\Gamma(-qn+1-q)}=\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-z)^n}{(n-1)!}\Gamma(nq)\sin(n\pi q),\quad z\in\mathbf{C}.$$

For any $x\in X$, we define operator families $\{\mathcal{S}_q(t)\}|_{t\in S^0_{\frac{\pi}{2}-\omega}}$ and $\{\mathcal{T}_q(t)\}|_{t\in S^0_{\frac{\pi}{2}-\omega}}$ by the semigroup $T(t)$ associated with $A$ as follows:

$$\mathcal{S}_q(t)x=\int_0^\infty\Psi_q(\sigma)T(\sigma t^q)x\,\mathrm{d}\sigma,\quad t\in S^0_{\frac{\pi}{2}-\omega}, x\in X, \tag{2.3}$$

$$\mathcal{T}_q(t)x=\int_0^\infty q\sigma\Psi_q(\sigma)T(\sigma t^q)x\,\mathrm{d}\sigma,\quad t\in S^0_{\frac{\pi}{2}-\omega}, x\in X. \tag{2.4}$$

**Theorem 2.4** ([15]) *For each fixed $t\in S^0_{\frac{\pi}{2}-\omega}$, $\mathcal{S}_q(t)$ and $\mathcal{T}_q(t)$ are linear and bounded operators on $X$. Moreover, for all $t>0$, $-1<\gamma<0$, $0<q<1$,*

$$\begin{aligned}\|\mathcal{S}_q(t)x\|&\le M_1t^{-q(1+\gamma)}\|x\|,\quad x\in X,\\ \|\mathcal{T}_q(t)x\|&\le M_2t^{-q(1+\gamma)}\|x\|,\quad x\in X,\end{aligned} \tag{2.5}$$

*where $M_1=\frac{C_0\Gamma(-\gamma)}{\Gamma(1-q(1+\gamma))}$ and $M_2=\frac{qC_0\Gamma(1-\gamma)}{\Gamma(1-q\gamma)}$.*

**Theorem 2.5** ([15], Theorem 3.2) *For $t>0$, $\mathcal{S}_q(t)$ and $\mathcal{T}_q(t)$ are continuous in the uniform operator topology. Moreover, for every $\widetilde{r}>0$, the continuity is uniform on $[\widetilde{r},\infty)$.*

**Remark 2.6** ([15], Theorem 3.4) Let $\beta > 1 + \gamma$. Then, for all $x \in D(A^{\beta})$,

$$\lim_{t\to 0; t>0} \mathcal{S}_q(t)x = x.$$

Let $\mathcal{B}_T$ be a set defined by

$$\mathcal{B}_T = \bigl\{u : (-\infty, T] \to X \text{ such that } u|_{(-\infty,0]} \in \mathcal{P} \text{ and } u|_J \in C(J, X)\bigr\}.$$

Motivated by [3, 15], when $\phi(0) \in D(A^{\beta})$ with $\beta > 1 + \gamma$, we give the following definition of a mild solution of Equation (1.1).

**Definition 2.7** A function $u \in \mathcal{B}_T$ satisfying the equation

$$u(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ \mathcal{S}_q(t)\phi(0) + \int_0^t (t-s)^{q-1}\mathcal{T}_q(t-s)f(s, u(s), u_s)\,\mathrm{d}s, & t \in J, \end{cases} \tag{2.6}$$

is called a mild solution of Equation (1.1).

**Remark 2.8** In general, since the operator $\mathcal{S}_q(t)$ is singular at $t = 0$, solutions to problem (1.1) are assumed to have the same kind of singularity at $t = 0$ as the operator $\mathcal{S}_q(t)$. When $\phi(0) \in D(A^{\beta})$ with $\beta > 1 + \gamma$, it follows from Remark 2.6 that the mild solution is continuous at $t = 0$.

Next, we recall that the Hausdorff measure of noncompactness $\chi(\cdot)$ on each bounded subset $\Omega$ of a Banach space $X$ is defined by

$$\chi(\Omega) = \inf\{\varepsilon > 0; \Omega \text{ has a finite } \varepsilon\text{-net in } X\}.$$

This measure of noncompactness satisfies some basic properties as follows.

**Lemma 2.9** ([25]) *Let $Y$ be a Banach space, and let $U, V \subseteq Y$ be bounded. Then*

(1) $\chi(U) = 0$ *if and only if $U$ is precompact*;
(2) $\chi(U) = \chi(\overline{U}) = \chi(\operatorname{conv} U)$, *where $\overline{U}$ and* $\operatorname{conv} U$ *mean the closure and convex hull of $U$, respectively*;
(3) $\chi(U) \le \chi(V)$ *if $U \subseteq V$*;
(4) $\chi(U \cup V) \le \max\{\chi(U), \chi(V)\}$;
(5) $\chi(U + V) \le \chi(U) + \chi(V)$, *where $U + V = \{x + y; x \in U, y \in V\}$*;
(6) $\chi(\lambda U) = |\lambda|\chi(U)$ *for any $\lambda \in \mathbf{R}$*.

**Definition 2.10** A continuous map $Q : W \subseteq Y \to Y$ is said to be a $\chi$-contraction if there exists a positive constant $\nu < 1$ such that $\chi(QU) \le \nu \cdot \chi(U)$ for any bounded closed subset $U \subseteq W$.

**Theorem 2.11** ([25]) (Darbo-Sadovskii) *If $U \subseteq X$ is bounded closed and convex, the continuous map $\mathcal{F} : U \to U$ is a $\chi$-contraction, then the map $\mathcal{F}$ has at least one fixed point in $U$.*

In Section 3, we use the above fixed point theorem to obtain main result. To this end, we present the following assertion about $\chi$-estimates for a multivalued integral (Theorem 4.2.3 of [26]).

Let $2^Y$ be the family of all nonempty subsets of $Y$, and let $G:[0,h]\to 2^Y$ be a multifunction. It is called:

(i) integrable if it admits a Bochner integrable selection $g:[0,h]\to Y$, $g(t)\in G(t)$ for a.e. $t\in[0,h]$;

(ii) integrably bounded if there exists a function $\vartheta\in L^1([0,h],Y)$ such that

$$\|G(t)\| := \sup\{\|g\|; g\in G(t)\}\le \vartheta(t) \quad \text{a.e. } t\in[0,h].$$

**Proposition 2.12** *For an integrable, integrably bounded multifunction* $G:[0,h]\to 2^X$, *where* $X$ *is a separable Banach space, let*

$$\chi\big(G(t)\big)\le m(t), \quad \textit{for a.e. } t\in[0,h],$$

*where* $m\in L^1_+([0,h])$. *Then* $\chi(\int_0^t G(s)\,ds)\le\int_0^t m(s)\,ds$ *for all* $t\in[0,h]$.

## 3 Main result

Throughout this section, let $A\in\Theta^{\gamma}_{\omega}(X)$ with $-1<\gamma<0$, $0<\omega<\frac{\pi}{2}$. We will use fixed point techniques to establish a result on the existence of mild solutions for Equation (1.1). For this purpose, we consider the following hypotheses.

(H1) $f:J\times X\times\mathcal{P}\to X$ satisfies $f(\cdot,v,w):J\to X$ is measurable for all $(v,w)\in X\times\mathcal{P}$ and $f(t,\cdot,\cdot):X\times\mathcal{P}\to X$ is continuous for a.e. $t\in J$, and there exists a function $\mu(\cdot)\in L^p(J,\mathbf{R}^+)$ $(p>\frac{-1}{q\gamma}>\frac{1}{q}>1)$ such that

$$\big\|f(t,v,w)\big\|\le\mu(t)\big(1+\|w\|_{\mathcal{P}}\big)$$

for almost all $t\in J$;

(H2) For any bounded sets $D_1\subset X$, $D_2\subset\mathcal{P}$, there exists a nondecreasing function $\eta(\cdot)\in L^p(J,\mathbf{R}^+)$ such that

$$\chi\big(f(t,D_1,D_2)\big)\le\eta(t)\Big(\chi(D_1)+\sup_{-\infty<\tau\le 0}\chi\big(D_2(\tau)\big)\Big).$$

**Theorem 3.1** *Suppose that hypotheses* (H1) *and* (H2) *hold. Then, for every* $\phi(0)\in D(A^{\beta})$ *with* $\beta>1+\gamma$, *there exists a mild solution of* (1.1) *on* $(-\infty,T]$.

*Proof* Define the map $\mathcal{F}$ on the space $\mathcal{B}_T$ by $(\mathcal{F}u)|_0=\phi(t)$ and

$$(\mathcal{F}u)(t)=\mathcal{S}_q(t)\phi(0)+\int_0^t(t-s)^{q-1}\mathcal{T}_q(t-s)f\big(s,u(s),u_s\big)\,ds, \quad t\in J.$$

From Theorems 2.4-2.5 and (H1), we infer that $\mathcal{F}u\in\mathcal{B}_T$.

Let $\overline{x}(\cdot):(-\infty,T]\to X$ be the function defined by

$$\overline{x}(t)=\begin{cases}\phi(t), & t\in(-\infty,0],\\ \mathcal{S}_q(t)\phi(0), & t\in J.\end{cases}$$

Write $u(t) = \overline{x}(t) + y(t)$, $t \in (-\infty, T]$. It is clear that $u$ satisfies (2.6) if and only if $y$ satisfies $y_0 = 0$ and for $t \in J$,

$$y(t) = \int_0^t (t-s)^{q-1}\mathcal{T}_q(t-s)f\big(s, \overline{x}(s) + y(s), \overline{x}_s + y_s\big)\,\mathrm{d}s.$$

Set $Y_0 = \{y \in \mathcal{B}_T; y_0 = 0\}$. For any $y \in Y_0$,

$$\|y\|_{Y_0} = \sup_{t\in J}\big\|y(t)\big\| + \|y_0\|_{\mathcal{P}} = \sup_{t\in J}\big\|y(t)\big\|,$$

thus, $(Y_0, \|\cdot\|_{Y_0})$ is a Banach space.

In order to apply Theorem 2.11 to show that $\mathcal{F}$ has a fixed point, we let $\widetilde{\mathcal{F}} : Y_0 \to Y_0$ be an operator defined by $(\widetilde{\mathcal{F}}y)(t) = 0$, $t \in (-\infty, 0]$ and for $t \in J$,

$$(\widetilde{\mathcal{F}}y)(t) = \int_0^t (t-s)^{q-1}\mathcal{T}_q(t-s)f\big(s, \overline{x}(s) + y(s), \overline{x}_s + y_s\big)\,\mathrm{d}s.$$

Clearly, the operator $\widetilde{\mathcal{F}}$ has a fixed point is equivalent to $\mathcal{F}$ has one. So, it turns out to prove that $\widetilde{\mathcal{F}}$ has a fixed point.

For $L > 0$, let us introduce in the space $Y_0$ the equivalent norm defined as

$$\|y\|_* = \sup_{t\in J}\big(\mathrm{e}^{-Lt}\big\|y(t)\big\|\big),$$

since for any $\psi \in L^1(J, X)$,

$$\lim_{L\to+\infty}\sup_{t\in J}\int_0^t \mathrm{e}^{-L(t-s)}\psi(s)\,\mathrm{d}s = 0,$$

we can take the appropriate $L$ to satisfy

$$M_2 C_1^* \sup_{t\in J}\int_0^t \mathrm{e}^{-L(t-s)}(t-s)^{-1-q\gamma}\mu(s)\,\mathrm{d}s \le \frac{1}{2}, \tag{3.1}$$

where $C_1^* = \sup_{t\in J} C_1(t)$.

Consider the set

$$B_\rho = \big\{y \in Y_0; \|y\|_* \le \rho\big\},$$

here $\rho$ is a constant chosen so that

$$\frac{\rho}{2} \ge \ell := M_2 T^{\frac{-(1+pq\gamma)}{p}} l_{p,q}\|\mu\|_{L^p}\cdot(1+\alpha),$$

where $l_{p,q} := (\frac{p-1}{-pq\gamma-1})^{\frac{p-1}{p}}$, $\alpha = C_1^*\sup_{t\in J}\|\mathcal{S}_q(t)\phi(0)\| + C_2^*\|\phi\|_{\mathcal{P}}$, $C_2^* = \sup_{t\in J} C_2(t)$.

Let $\{v^n\}_{n\in\mathbf{N}} \subset B_\rho$ be a sequence such that $v^n \to v$ as $n \to \infty$. Obviously, the Lebesgue dominated convergence theorem enables us to prove that $\widetilde{\mathcal{F}}$ is continuous.

In what follows, we prove that $\widetilde{\mathcal{F}}B_\rho \subset B_\rho$. From (2.2), it follows that

$$\begin{aligned}\|\overline{x}_t + y_t\|_{\mathcal{P}} &\le \|\overline{x}_t\|_{\mathcal{P}} + \|y_t\|_{\mathcal{P}} \\ &\le C_1(t) \sup_{0\le s\le t} \|\overline{x}(s)\| + C_2(t)\|\overline{x}_0\|_{\mathcal{P}} + C_1(t) \sup_{0\le s\le t} \|y(s)\| + C_2(t)\|y_0\|_{\mathcal{P}} \\ &\le C_1^* \sup_{t\in J} \|\mathcal{S}_q(t)\phi(0)\| + C_2^*\|\phi\|_{\mathcal{P}} + C_1(t) \sup_{0\le s\le t} \|y(s)\| \\ &\le \alpha + C_1^* \sup_{0\le s\le t} \|y(s)\|. \end{aligned} \tag{3.2}$$

Moreover, we see from the Hölder inequality that

$$\int_0^t (t-s)^{-1-q\gamma} \mu(s)\,\mathrm{d}s \le t^{\frac{-(1+pq\gamma)}{p}} l_{p,q}\|\mu\|_{L^p} \le T^{\frac{-(1+pq\gamma)}{p}} l_{p,q}\|\mu\|_{L^p}. \tag{3.3}$$

For $t \in J$, $y \in B_\rho$, by (2.5), (H1) and (3.2)-(3.3), we have

$$\begin{aligned}\|(\widetilde{\mathcal{F}}y)(t)\| &\le \int_0^t (t-s)^{q-1} \|\mathcal{T}_q(t-s)f(s, \overline{x}(s)+y(s), \overline{x}_s + y_s)\|\,\mathrm{d}s \\ &\le M_2 \int_0^t (t-s)^{-1-q\gamma}\mu(s)\Big(1 + \alpha + C_1^* \sup_{0\le\sigma\le s} \|y(\sigma)\|\Big)\,\mathrm{d}s \\ &\le \ell + M_2C_1^* \int_0^t (t-s)^{-1-q\gamma}\mu(s) \sup_{0\le\sigma\le s} \|y(\sigma)\|\,\mathrm{d}s. \end{aligned}$$

Then

$$\begin{aligned}\mathrm{e}^{-Lt}\|(\widetilde{\mathcal{F}}y)(t)\| &\le \ell + M_2C_1^* \int_0^t \mathrm{e}^{-L(t-s)}(t-s)^{-1-q\gamma}\mu(s) \sup_{0\le\sigma\le s} \big(\mathrm{e}^{-L\sigma}\|y(\sigma)\|\big)\,\mathrm{d}s \\ &\le \ell + \rho M_2C_1^* \int_0^t \mathrm{e}^{-L(t-s)}(t-s)^{-1-q\gamma}\mu(s)\,\mathrm{d}s. \end{aligned}$$

It results that $\|\widetilde{\mathcal{F}}y\|_* \le \rho$ by (3.1). Hence, for some positive number $\rho$, $\widetilde{\mathcal{F}}B_\rho \subset B_\rho$.

For $y \in B_\rho$, let $\delta > 0$, $t_1, t_2 \in (0, T]$ such that $0 < t_2 - t_1 \le \delta$, we get

$$\begin{aligned}&\bigg\|\int_0^{t_1} (t_1-s)^{q-1}\mathcal{T}_q(t_1-s)f(s, \overline{x}(s)+y(s), \overline{x}_s+y_s)\,\mathrm{d}s \\ &\quad - \int_0^{t_2} (t_2-s)^{q-1}\mathcal{T}_q(t_2-s)f(s, \overline{x}(s)+y(s), \overline{x}_s+y_s)\,\mathrm{d}s\bigg\| \\ &\le \bigg\|\int_0^{t_1} \big[(t_1-s)^{q-1} - (t_2-s)^{q-1}\big]\mathcal{T}_q(t_1-s)f(s, \overline{x}(s)+y(s), \overline{x}_s+y_s)\,\mathrm{d}s\bigg\| \\ &\quad + \bigg\|\int_0^{t_1} (t_2-s)^{q-1}\big[\mathcal{T}_q(t_1-s) - \mathcal{T}_q(t_2-s)\big]f(s, \overline{x}(s)+y(s), \overline{x}_s+y_s)\,\mathrm{d}s\bigg\| \\ &\quad + \bigg\|\int_{t_1}^{t_2} (t_2-s)^{q-1}\mathcal{T}_q(t_2-s)f(s, \overline{x}(s)+y(s), \overline{x}_s+y_s)\,\mathrm{d}s\bigg\|. \end{aligned} \tag{3.4}$$

We will show that each term on the right-hand side of (3.4) uniformly converges to zero.

Combining with (3.2), we have

$$\left\| f\left(t, \overline{x}(t) + y(t), \overline{x}_t + y_t\right) \right\| \le \mu(t) K(t), \tag{3.5}$$

where $K(t) = 1 + \alpha + C_1^* \sup_{0 \le s \le t} \|y(s)\|$.

Taking $t_2 \to t_1$ and using (3.5), we conclude

$$\begin{aligned}
&\left\| \int_0^{t_1} \left[(t_1 - s)^{q-1} - (t_2 - s)^{q-1}\right] \mathcal{T}_q(t_1 - s) f\left(s, \overline{x}(s) + y(s), \overline{x}_s + y_s\right) \mathrm{d}s \right\| \\
&\quad \le M_2 \int_0^{t_1} \left|(t_1 - s)^{q-1} - (t_2 - s)^{q-1}\right| (t_1 - s)^{-q(1+\gamma)} \mu(s) K(s) \,\mathrm{d}s \\
&\quad \to 0
\end{aligned}$$

and

$$\begin{aligned}
&\left\| \int_{t_1}^{t_2} (t_2 - s)^{q-1} \mathcal{T}_q(t_2 - s) f\left(s, \overline{x}(s) + y(s), \overline{x}_s + y_s\right) \mathrm{d}s \right\| \\
&\quad \le M_2 \int_{t_1}^{t_2} (t_2 - s)^{-q\gamma - 1} \mu(s) K(s) \,\mathrm{d}s \\
&\quad \to 0.
\end{aligned}$$

For $\varepsilon > 0$ small enough, noting that (2.5) and (3.5), we obtain

$$\begin{aligned}
&\left\| \int_0^{t_1} (t_2 - s)^{q-1} \left[\mathcal{T}_q(t_1 - s) - \mathcal{T}_q(t_2 - s)\right] f\left(s, \overline{x}(s) + y(s), \overline{x}_s + y_s\right) \mathrm{d}s \right\| \\
&\quad \le \int_0^{t_1 - \varepsilon} (t_2 - s)^{q-1} \left\| \mathcal{T}_q(t_2 - s) - \mathcal{T}_q(t_1 - s) \right\|_{L(X)} \mu(s) K(s) \,\mathrm{d}s \\
&\qquad + \int_{t_1 - \varepsilon}^{t_1} (t_2 - s)^{q-1} \left\| \mathcal{T}_q(t_2 - s) - \mathcal{T}_q(t_1 - s) \right\|_{L(X)} \mu(s) K(s) \,\mathrm{d}s \\
&\quad \le \sup_{s \in [0, t_1 - \varepsilon]} \left\| \mathcal{T}_q(t_2 - s) - \mathcal{T}_q(t_1 - s) \right\|_{L(X)} \cdot \int_0^{t_1 - \varepsilon} (t_2 - s)^{q-1} \mu(s) K(s) \,\mathrm{d}s \\
&\qquad + M_2 \int_{t_1 - \varepsilon}^{t_1} \left( \frac{(t_2 - s)^{q-1}}{(t_2 - s)^{q(\gamma+1)}} + \frac{(t_2 - s)^{q-1}}{(t_1 - s)^{q(\gamma+1)}} \right) \mu(s) K(s) \,\mathrm{d}s.
\end{aligned}$$

This together with Theorem 2.5 shows that the right-hand side tends to zero as $t_2 \to t_1$ and $\varepsilon \to 0$.

Therefore, the set $\{(\widetilde{\mathcal{F}}y)(\cdot); y \in B_\rho\}$ is equicontinuous.

For a bounded set $\Omega \subset Y_0$, we define the Hausdorff measure of noncompactness $\chi_1$ on $Y_0$ as follows:

$$\chi_1(\Omega) = \sup_{t \in J} \left( \mathrm{e}^{-rt} \sup_{s \in [0,t]} \chi\left(\Omega(s)\right) \right),$$

where $r > 0$ is a constant chosen so that

$$\widetilde{L} := 2M_2 \sup_{t \in J} \int_0^t \mathrm{e}^{-r(t-s)} (t - s)^{-q\gamma - 1} \eta(s) \,\mathrm{d}s < 1. \tag{3.6}$$

For any $t \in J$, we set

$$\widetilde{\mathcal{F}}(\Omega)(t) = \left\{ \int_0^t (t-s)^{q-1}\mathcal{T}_q(t-s)f\left(s, \overline{x}(s) + y(s), \overline{x}_s + y_s\right) \mathrm{d}s; y \in \Omega \right\}.$$

We consider the multifunction $s \in [0,t] \multimap H(s)$,

$$H(s) = \left\{(t-s)^{q-1}\mathcal{T}_q(t-s)f\left(s, \overline{x}(s) + y(s), \overline{x}_s + y_s\right); y \in \Omega\right\}.$$

Obviously, $H$ is integrable, and from (2.5), (H1) and (3.5) it follows that $H$ is integrably bounded. Moreover, noting that (H2), we have the following estimate for a.e. $s \in [0,t]$:

$$\begin{aligned}
\chi\left(H(s)\right) &= \chi\left(\left\{(t-s)^{q-1}\mathcal{T}_q(t-s)f\left(s, \overline{x}(s) + y(s), \overline{x}_s + y_s\right); y \in \Omega\right\}\right) \\
&= \chi\left((t-s)^{q-1}\mathcal{T}_q(t-s)f\left(s, \overline{x}(s) + \Omega(s), \overline{x}_s + \Omega_s\right)\right) \\
&\le M_2(t-s)^{-q\gamma-1}\eta(s)\Big(\chi\left(\Omega(s)\right) + \sup_{\sigma\in[0,s]} \chi\left(\Omega(\sigma)\right)\Big) \\
&\le 2M_2(t-s)^{-q\gamma-1}\eta(s) \sup_{\sigma\in[0,s]} \chi\left(\Omega(\sigma)\right).
\end{aligned}$$

Applying Proposition 2.12, we obtain

$$\chi\left(\widetilde{\mathcal{F}}(\Omega)(t)\right) = \chi\left(\int_0^t H(s)\,\mathrm{d}s\right) \le 2M_2 \int_0^t (t-s)^{-q\gamma-1}\eta(s) \sup_{\sigma\in[0,s]} \chi\left(\Omega(\sigma)\right) \mathrm{d}s,$$

which implies

$$\begin{aligned}
\chi_1\left(\widetilde{\mathcal{F}}(\Omega)\right) &= \sup_{t\in J}\Big(\mathrm{e}^{-rt} \sup_{s\in[0,t]} \chi\left(\widetilde{\mathcal{F}}(\Omega)(s)\right)\Big) \\
&\le 2M_2 \sup_{t\in J} \int_0^t \mathrm{e}^{-r(t-s)}(t-s)^{-q\gamma-1}\eta(s)\Big(\mathrm{e}^{-rs} \sup_{\sigma\in[0,s]} \chi\left(\Omega(\sigma)\right)\Big)\,\mathrm{d}s \\
&\le \widetilde{L}\chi_1(\Omega).
\end{aligned} \tag{3.7}$$

Hence $\widetilde{\mathcal{F}}$ is a $\chi_1$ -contraction on $Y_0$ by Definition 2.10. According to Theorem 2.11, the operator $\widetilde{\mathcal{F}}$ has at least one fixed point $y$ in $B_\rho$. Let $u(t) = \overline{x}(t) + y(t)$, $t \in (-\infty, T]$, then $u(t)$ is a fixed point of the operator $\mathcal{F}$ which is a mild solution of Equation (1.1). This ends the proof. □

## 4 Application

Let $X = L^3(\mathbf{R}^2)$, we consider the following integrodifferential problem:

$$\begin{cases}
\partial_t^q v(t,x) = \widehat{A}v(t,x) + a(t)\cos(|v(t,x)|) + \int_{-\infty}^0 c(\tau)\sin(t^b|v(\tau,x)|)\,\mathrm{d}\tau, \\
\quad t \in (0,1], x \in [0,1], \\
v(\tau,x) = v_0(\tau,x), \quad -\infty < \tau \le 0,
\end{cases} \tag{4.1}$$

where

$$\widehat{A} = (-i\Delta + \sigma)^{\frac{1}{2}}, \qquad D(\widehat{A}) = W^{1,3}\big(\mathbf{R}^2\big) \quad \text{(a Sobolev space, see Example 6.3 in [15])}$$

$a(t) \in L^p([0,1], \mathbf{R}^+)$ $(p > \frac{6}{q})$ and $a(t)$ is nondecreasing, $b > 0$, $c : (-\infty, 0] \to \mathbf{R}$, $v_0 : (-\infty, 0] \times [0,1] \to \mathbf{R}$ are continuous functions, and $\int_{-\infty}^0 |c(\tau)|\,d\tau < \infty$.

In Example 6.3 of [15], the authors demonstrate that $\widehat{A} \in \Theta_\omega^\gamma(L^3(\mathbf{R}^2))$ for some $0 < \omega < \frac{\pi}{2}$ and $\gamma = -\frac{1}{6}$. We denote the semigroup associated with $\widehat{A}$ by $T(t)$ and $\|T(t)\|_{L(X)} \leq C_0 t^{-\frac{5}{6}}$ ($C_0$ is a constant).

Let the phase space $\mathcal{P}$ be $\mathrm{BUC}(\mathbf{R}^-, X)$, the space of bounded uniformly continuous functions endowed with the following norm:

$$\|\varphi\|_{\mathcal{P}} = \sup_{-\infty < \tau \leq 0} \|\varphi(\tau)\| \quad \text{for all } \varphi \in \mathcal{P},$$

then we can see that $C_1(t) = 1$ in (2.2).

For $t \in [0,1]$, $x \in [0,1]$ and $\varphi \in \mathrm{BUC}(\mathbf{R}^-, X)$, we set

$$u(t)(x) = v(t,x),$$

$$\phi(\tau)(x) = v_0(\tau, x), \quad \tau \in (-\infty, 0],$$

$$f\bigl(t, u(t), \varphi\bigr)(x) = a(t)\cos\bigl(\bigl|u(t)(x)\bigr|\bigr) + \int_{-\infty}^0 c(\tau)\sin\bigl(t^b\bigl|\varphi(\tau)(x)\bigr|\bigr)\,d\tau.$$

Then we can rewrite Equation (4.1) above as abstract Equation (1.1).

Moreover, we have

$$\begin{aligned}\bigl\|f\bigl(t, u(t), \varphi\bigr)(x)\bigr\| &\leq a(t) + t^b\|\varphi\|_{\mathcal{P}} \int_{-\infty}^0 \bigl|c(\tau)\bigr|\,d\tau \\ &= \mu(t)\bigl(1 + \|\varphi\|_{\mathcal{P}}\bigr), \quad \text{for } t \in [0,1],\end{aligned}$$

where $\mu(t) := \max\{a(t), t^b \int_{-\infty}^0 |c(\tau)|\,d\tau\}$.

For any $u_1, u_2 \in X$, $\varphi, \widetilde{\varphi} \in \mathcal{P}$,

$$\begin{aligned}&\bigl\|f\bigl(t, u_1(t), \varphi\bigr)(x) - f\bigl(t, u_2(t), \widetilde{\varphi}\bigr)(x)\bigr\| \\ &\quad\leq a(t)\bigl\|u_1(t) - u_2(t)\bigr\| + t^b \int_{-\infty}^0 \bigl|c(\tau)\bigr| \cdot \bigl\|\varphi(\tau) - \widetilde{\varphi}(\tau)\bigr\|\,d\tau,\end{aligned}$$

which implies that for any bounded sets $D_1 \subset X$, $D_2 \subset \mathcal{P}$,

$$\chi\bigl(f(t, D_1, D_2)\bigr) \leq \eta(t)\Bigl(\chi(D_1) + \sup_{-\infty < \tau \leq 0} \chi\bigl(D_2(\tau)\bigr)\Bigr), \quad t \in [0,1],$$

where $\eta(t) = \mu(t)$.

Thus, problem (4.1) has at least a mild solution by Theorem 3.1 for every $\phi(0) \in D(\widehat{A}^\beta)$ $(\beta > \frac{5}{6})$.

**Competing interests**

The author has no competing interests.

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