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Asymptotic behavior for third-order quasi-linear differential equations

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Abstract
In this paper, a class of third-order quasi-linear differential equations with continuously distributed delay is studied. Applying the generalized Riccati transformation, integral averaging technique of Philos type and Young's inequality, a set of new criteria for oscillation or certain asymptotic behavior of nonoscillatory solutions of this equations is given. Our results essentially improve and complement some earlier publications.

Keywords: third-order quasi-linear differential equations; oscillation; nonoscillation

1 Introduction

Consider the following third-order quasi-linear differential equation:

$$\left[a(t)\left(\left[x(t)+\int_a^b p(t,\mu)x\left[\tau(t,\mu)\right]d\mu\right]''\right)^{\gamma}\right]'+\int_c^d q(t,\xi)f\left(x\left[\sigma(t,\xi)\right]\right)d\xi=0. \tag{1}$$

We build up the following hypotheses firstly:

(H1) $a(t)\in C([t_0,\infty),(0,\infty))$ and $\int_{t_0}^{\infty}a(s)^{-\frac{1}{\gamma}}\,ds=\infty$;

(H2) $p(t,\mu)\in C([t_0,\infty)\times[a,b],[0,\infty))$ and $0\le p(t)\equiv\int_a^b p(t,\mu)\,d\mu\le p<1$;

(H3) $\tau(t,\mu)\in C([t_0,\infty)\times[a,b],R)$ is not a decreasing function for μ and such that

$$\tau(t,\mu)\le t \quad\text{and}\quad \lim_{t\to\infty}\min_{\mu\in[a,b]}\tau(t,\mu)=\infty; \tag{2}$$

(H4) $q(t,\xi)\in C([t_0,\infty),(0,\infty))$;

(H5) $\sigma(t,\xi)\in C([t_0,\infty)\times[a,b],R)$ is not a decreasing function for ξ and such that

$$\sigma(t,\xi)\le t \quad\text{and}\quad \lim_{t\to\infty}\min_{\xi\in[c,d]}\sigma(t,\xi)=\infty; \tag{3}$$

(H6) $f(x)\in C(R,R)$ and $\frac{f(x)}{x^{\gamma}}\ge\delta>0$;

(H7) γ is a quotient of odd positive integers.

Define the function by

$$z(t)=x(t)+\int_a^b p(t,\mu)x\left[\tau(t,\mu)\right]d\mu. \tag{4}$$

A function $x(t)$ is a solution of (1) means that $x(t) \in C^2[T_x,\infty)$, $T_x \geq t_0$, $a(t)(z''(t))^\gamma \in C^1[T_x,\infty)$ and satisfies (1) on $[T_x,\infty)$. In this paper, we restrict our attention to those solutions of Eq. (1) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. We assume that Eq. (1) possesses such a solution. A solution of Eq. (1) is called oscillatory on $[T_x,\infty)$ if it is eventually positive or eventually negative; otherwise, it is called nonoscillatory.

In recent years, there has been much research activity concerning the oscillation theory and applications of differential equations; see [1–4] and the reference contained therein. Especially, the study content of oscillatory criteria of second-order differential equations is very rich. In contrast, the study of oscillatory criteria of third-order differential equations is relatively less, but most of works are about delay equations. Some interesting results have been obtained concerning the asymptotic behavior of solutions of Eq. (1) in the particular case. For example, [5] consider the third-order functional differential equations of the form

$$\bigl[a(t)\bigl(x''(t)\bigr)^\gamma\bigr]' + q(t)f\bigl(x\bigl[\sigma(t)\bigr]\bigr) = 0. \tag{5}$$

Zhang *et al.* [6] focus on the following the third-order neutral differential equations with continuously distributed delay:

$$\left[a(t)\left[x(t) + \int_a^b p(t,\mu)x\bigl[\tau(t,\mu)\bigr]\,d\mu\right]''\right]' + \int_c^d q(t,\xi)f\bigl(x\bigl[\sigma(t,\xi)\bigr]\bigr)\,d\xi = 0. \tag{6}$$

Baculíková and Džurina [7] are concerned with the couple of the third-order neutral differential equations of the form

$$\bigl[a(t)\bigl(\bigl[x(t) + p(t)x\bigl[\tau(t)\bigr]\bigr]''\bigr)^\gamma\bigr]' + q(t)x^\gamma\bigl[\sigma(t)\bigr] = 0. \tag{7}$$

However, as we know, oscillatory behaviors of solutions of Eq. (1) have not been considered up to now. In this paper, we try to discuss the problem of oscillatory criteria of Philos type of Eq. (1). Applying the generalized Riccati transformation, integral averaging technique of Philos type, Young's inequality, *etc.*, we obtain some new criteria for oscillation or certain asymptotic behavior of nonoscillatory solutions of this equations. We should point out that γ is any quotient of odd positive integers in this paper, but it is required that $\gamma = 1$ in [6].

2 Several lemmas

We start our work with the classification of possible nonoscillatory solutions of Eq. (1).

Lemma 2.1 *Let $x(t)$ be a positive solution of* (1), *and $z(t)$ is defined as in* (4). *Then $z(t)$ has only one of the following two properties eventually*:

(I) $z(t) > 0$, $z'(t) > 0$, $z''(t) > 0$;
(II) $z(t) > 0$, $z'(t) < 0$, $z''(t) > 0$.

Proof Let $x(t)$ be a positive solution of (1), eventually (if it is eventually negative, the proof is similar). Then $[a(t)(z''(t))^\gamma]' < 0$. Thus, $a(t)(z''(t))^\gamma$ is decreasing and of one sign and it follows hypotheses (H2)-(H7) that there exists $t_1 \geq t_0$ such that $z''(t)$ is of fixed sign for $t \geq t_1$. If we admit $z''(t) < 0$, then there exists a constant $M > 0$ such that

$$z''(t) \leq -\frac{M}{a(t)^{\frac{1}{\gamma}}}, \quad t \geq t_1. \tag{8}$$

Integrating from t_1 to t, we get

$$z'(t) \le z'(t_1) - M \int_{t_1}^{t} a(s)^{-\frac{1}{\gamma}}\, ds. \tag{9}$$

Let $t \to \infty$ and using (H1), we have $z'(t) \to -\infty$. Thus $z'(t) < 0$ eventually, which together with $z''(t) < 0$ implies $z(t) < 0$, which contradicts our assumption $z(t) > 0$. This contradiction shows that $z''(t) > 0$, eventually. Therefore $z'(t)$ is increasing and thus (I) or (II) holds for $z(t)$, eventually. □

Lemma 2.2 *Let $x(t)$ be a positive solution of* (1), *and correspondingly $z(t)$ has the property* (II). *Assume that*

$$\int_{t_0}^{\infty} \int_{v}^{\infty} \left[\frac{1}{a(u)} \int_{u}^{\infty} \int_{c}^{d} q(s,\xi)\, d\xi\, ds \right]^{1/\gamma} du\, dv = \infty. \tag{10}$$

Then

$$\lim_{t\to\infty} x(t) = 0. \tag{11}$$

Proof Let $x(t)$ be a positive solution of Eq. (1). Since $z(t)$ satisfies the property (II), it is obvious that there exists a finite limit

$$\lim_{t\to\infty} z(t) = l. \tag{12}$$

Next, we claim that $l = 0$. Assume that $l > 0$, then we have $l < z(t) < l + \varepsilon$ for all $\varepsilon > 0$ and t enough large. Choosing $\varepsilon < l(1-p)/p$, we obtain

$$\begin{aligned} x(t) &= z(t) - \int_a^b p(t,\mu) x\big[\tau(t,\mu)\big]\, d\mu \ge l - \int_a^b p(t,\mu) z\big[\tau(t,\mu)\big]\, d\mu \\ &\ge l - p(t) z\big[\tau(t,a)\big] \ge l - p(l+\varepsilon) \\ &= K(l+\varepsilon) > K z(t), \end{aligned} \tag{13}$$

where $K = \frac{l-p(l+\varepsilon)}{l+\varepsilon} > 0$. □

Combining (H6), (13) with (1), one can get

$$\begin{aligned} \big(a(t)\big[z''(t)\big]^{\gamma}\big)' &\le -\delta K^{\gamma} \int_c^d q(t,\xi) \big(z\big[\sigma(t,\xi)\big]\big)^{\gamma}\, d\xi \\ &\le -\delta K^{\gamma} \big(z\big[\sigma(t,d)\big]\big)^{\gamma} \int_c^d q(t,\xi)\, d\xi \\ &\le -\delta K^{\gamma} \big(z\big[\sigma_0(t)\big]\big)^{\gamma} q_1(t), \end{aligned} \tag{14}$$

where $q_1(t) = \int_c^d q(t,\xi)\, d\xi$ and $\sigma_0(t) = \sigma(t,d)$. Integrating inequality (14) from t to ∞, we get immediately

$$a(t)\big[z''(t)\big]^{\gamma} \ge \delta K^{\gamma} \int_t^{\infty} q_1(s) \big(z\big[\sigma_0(s)\big]\big)^{\gamma}\, ds. \tag{15}$$

Using $z(\sigma_0(s)) > l$, we have

$$
\begin{aligned}
&z''(t) \geq \delta^{1/\gamma} Kl\left(\frac{1}{a(t)}\int_t^{\infty} q_1(s)\,ds\right)^{\frac{1}{\gamma}} \geq \delta^{1/\gamma} Kl\left(\frac{1}{a(t)}\int_t^{\infty}\int_c^d q(s,\xi)\,d\xi\,ds\right)^{\frac{1}{\gamma}}; \\
&-z'(t) \geq \delta^{1/\gamma} Kl\int_t^{\infty}\left(\frac{1}{a(u)}\int_u^{\infty}\int_c^d q(s,\xi)\,d\xi\,ds\right)^{\frac{1}{\gamma}} du; \\
&z(t_1) \geq \delta^{1/\gamma} Kl\int_{t_1}^{\infty}\int_v^{\infty}\left(\frac{1}{a(u)}\int_u^{\infty}\int_c^d q(s,\xi)\,d\xi\,ds\right)^{\frac{1}{\gamma}} du\,dv.
\end{aligned}
\tag{16}
$$

We have a contradiction with (10) and so it follows that $\lim_{t\to\infty} z(t) = 0$, which implies that

$$\lim_{t\to\infty} x(t) = 0. \tag{17}$$

Lemma 2.3 [7] *Assume that $u(t) > 0$, $u'(t) > 0$, $u''(t) < 0$ on $[t_0,\infty)$. Then, for each $\alpha \in (0,1)$, there exists $T_\alpha \geq t_0$ such that*

$$\frac{u(\sigma(t))}{\sigma(t)} \geq \alpha\frac{u(t)}{t} \quad \textit{for all } t \geq T_\alpha. \tag{18}$$

Lemma 2.4 [8] *Let $z(t) > 0$, $z'(t) > 0$, $z''(t) > 0$, $z'''(t) < 0$ on $[T_\alpha,\infty)$. Then there exist $\beta \in (0,1)$ and $T_\beta \geq T_\alpha$ such that*

$$z(t) \geq \beta t z'(t) \quad \textit{for all } t \geq T_\beta. \tag{19}$$

3 Main results

For simplicity, we introduce the following notations:

$$D = \{(t,s) : t \geq s \geq t_0\}; \qquad D_0 = \{(t,s) : t > s \geq t_0\}. \tag{20}$$

A function $H \in C^1(D,R)$ is said to belong to X class ($H \in X$) if it satisfies

(i) $H(t,t) = 0$, $t \geq t_0$; $H(t,s) > 0$, $(t,s) \in D_0$;

(ii) $\frac{\partial H(t,s)}{\partial s} < 0$, there exist $\rho \in C^1([t_0,\infty),(0,\infty))$ and $h \in C(D_0,R)$ such that

$$\frac{\partial H(t,s)}{\partial s} + \frac{\rho'(t)}{\rho(t)}H(t,s) = -h(t,s)\bigl(H(t,s)\bigr)^{\frac{\gamma}{1+\gamma}}. \tag{21}$$

Theorem 3.1 *Assume that* (10) *holds, there exist $\rho \in C^1([t_0,\infty),(0,\infty))$ and $H \in X$ such that*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^t\left[H(t,s)Q(s) - \frac{a(s)\rho(s)h^{\gamma+1}(t,s)}{(\gamma+1)^{\gamma+1}}\right]ds = \infty, \tag{22}$$

$$Q(s) = \delta(1-p)^{\gamma}\rho(s)\left(\frac{\alpha\beta\sigma^2(s,c)}{s}\right)^{\gamma}\int_c^d q(t,\xi)\,d\xi. \tag{23}$$

Suppose, further, that $a'(t) > 0$. Then every solution $x(t)$ of Eq. (1) *is either oscillatory or converges to zero.*

Proof Assume that Eq. (1) has a nonoscillatory solution $x(t)$. Without loss of generality, we may assume that $x(t) > 0$, $t \geq t_1$, $x(\tau(t,\mu)) > 0$, $(t,\mu) \in [t_0,\infty) \times [a,b]$, $x(\sigma(t,\xi)) > 0$, $(t,\xi) \in [t_0,\infty) \times [c,d]$, $z(t)$ is defined as in (4). By Lemma 2.1, we have that $z(t)$ has the property (I) or the property (II). If $z(t)$ has the property (II). Since (10) holds, then the conditions in Lemma 2.2 are satisfied. Hence $\lim_{t\to\infty} x(t) = 0$.

When $z(t)$ has the property (I), we obtain

$$\begin{aligned} x(t) &= z(t) - \int_a^b p(t,\mu)x\bigl[\tau(t,\mu)\bigr]\,d\mu \geq z(t) - \int_a^b p(t,\mu)z\bigl[\tau(t,\mu)\bigr]\,d\mu \\ &\geq z(t) - z\bigl[\tau(t,b)\bigr]\int_a^b p(t,\mu)\,d\mu \geq (1-p)z(t). \end{aligned} \tag{24}$$

Using (H5) and (H6), we have

$$\bigl(a(t)\bigl[z''(t)\bigr]^\gamma\bigr)' \leq -\delta(1-p)^\gamma\bigl(z\bigl[\sigma_1(t)\bigr]\bigr)^\gamma q_1(t), \tag{25}$$

where $q_1(t) = \int_c^d q(t,\xi)\,d\xi$ and $\sigma_1(t) = \sigma(t,c)$. Let

$$w(t) = \rho(t)a(t)\left(\frac{z''(t)}{z'(t)}\right)^\gamma, \quad t \geq t_1. \tag{26}$$

Then

$$\begin{aligned} &w'(t) - \frac{\rho'(t)}{\rho(t)}w(t) \\ &\quad\leq -\delta(1-p)^\gamma q_1(t)\left(\frac{z[\sigma_1(t)]}{z'(t)}\right)^\gamma - \gamma\left(\frac{1}{a(t)\rho(t)}\right)^{1/\gamma} w^{\frac{\gamma+1}{\gamma}}(t). \end{aligned} \tag{27}$$

Choosing $u(t) = z'(t)$ in Lemma 2.2, we obtain

$$\frac{1}{z'(t)} \geq \frac{\alpha\sigma_1(t)}{tz'(\sigma_1(t))}, \quad t \geq T_\alpha \geq t_1. \tag{28}$$

Using Lemma 2.3, we get

$$z\bigl(\sigma_1(t)\bigr) \geq \beta\sigma_1(t)z'\bigl(\sigma_1(t)\bigr)t \geq T_\beta \geq T_\alpha. \tag{29}$$

Combining with (27)-(29), we have

$$w'(t) \leq -Q(t) + \frac{\rho'(t)}{\rho(t)}w(t) - \gamma\left(\frac{1}{a(t)\rho(t)}\right)^{1/\gamma} w^{\frac{\gamma+1}{\gamma}}(t), \tag{30}$$

where $Q(t)$ is defined by (23). Let

$$A(t) = \frac{\rho'(t)}{\rho(t)}, \qquad B(t) = \gamma\left(\frac{1}{a(t)\rho(t)}\right)^{1/\gamma}.$$

For $t \geq t_2 \geq T_\beta$, we have

$$\begin{aligned}\int_{t_2}^{t} H(t,s)Q(s)\,ds &\leq \int_{t_2}^{t} H(t,s)\bigl[-w'(s) + A(s)w(s) - B(s)w^{\frac{\gamma+1}{\gamma}}(s)\bigr]\,ds \\ &= H(t,t_2)w(t_2) - \int_{t_2}^{t}\bigl[h(t,s)F(t,s) + B(s)\bigl(F(t,s)\bigr)^{\frac{\gamma+1}{\gamma}}\bigr]\,ds,\end{aligned} \tag{31}$$

where $F(t,s) = w(s)H^{\frac{\gamma}{\gamma+1}}(t,s)$. By Young's inequality

$$\frac{(B^{\frac{\gamma}{\gamma+1}}(s)F(t,s))^{\frac{\gamma+1}{\gamma}}}{\frac{\gamma+1}{\gamma}} + \frac{(\gamma B^{-\frac{\gamma}{\gamma+1}}(s)\frac{h(t,s)}{\gamma+1})^{\gamma+1}}{\gamma+1} \geq \frac{\gamma}{\gamma+1}\bigl|h(t,s)\bigr|F(t,s), \tag{32}$$

we obtain

$$B(s)F^{\frac{\gamma+1}{\gamma}}(t,s) \geq \bigl|h(t,s)\bigr|F(t,s) - \frac{a(s)\rho(s)h^{\gamma+1}(t,s)}{(\gamma+1)^{\gamma+1}}. \tag{33}$$

Applying (33) to inequality (31), we obtain

$$\begin{aligned}\int_{t_2}^{t} H(t,s)Q(s)\,ds \leq H(t,t_2)w(t_2) + \int_{t_2}^{t} \frac{a(s)\rho(s)h^{\gamma+1}(t,s)}{(\gamma+1)^{\gamma+1}}\,ds \\ - \int_{t_2}^{t}\bigl[h(t,s) + \bigl|h(t,s)\bigr|\bigr]F(t,s)\,ds.\end{aligned} \tag{34}$$

Therefore, we have

$$w(t_2) \geq \frac{1}{H(t,t_2)}\int_{t_2}^{t}\biggl[H(t,s)Q(s)\,ds - \frac{a(s)\rho(s)h^{\gamma+1}(t,s)}{(\gamma+1)^{\gamma+1}}\biggr]\,ds. \tag{35}$$

The last inequality contradicts (22). □

Theorem 3.2 *Assume that other conditions of Theorem* 3.1 *are satisfied except condition* (22). *Further, for every T, the following inequalities hold*:

$$0 < \inf_{s\geq T}\liminf_{t\to\infty}\frac{H(t,s)}{H(t,T)} \leq \infty \tag{36}$$

and

$$\limsup_{t\to\infty}\int_{T}^{t}\frac{a(s)\rho(s)h^{\gamma+1}(t,s)}{H(t,T)}\,ds < \infty. \tag{37}$$

If there exists $\psi \in C([t_0,\infty),R)$ such that

$$\limsup_{t\to\infty}\int_{T}^{t}\biggl[\frac{\psi_+^{\gamma+1}(s)}{\rho(s)a(s)}\biggr]^{1/\gamma}\,ds = \infty, \tag{38}$$

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}\biggl[H(t,s)Q(s) - \frac{a(s)\rho(s)h^{\gamma+1}(t,s)}{(\gamma+1)^{\gamma+1}}\biggr]\,ds \geq \psi(T), \tag{39}$$

where $\psi_+(s) = \max\{\psi(s), 0\}$, *then every solution* $x(t)$ *of Eq.* (1) *is either oscillatory or converges to zero.*

Proof As the proof of Theorem 3.1, we can see that (31) holds. It follows that

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{H(t,t_2)} \int_{t_2}^{t} \bigl(H(t,s)Q(s) - G(t,s)\bigr)\,ds \\
&\quad\le w(t_2) - \liminf_{t\to\infty} \frac{1}{H(t,t_2)} \int_{t_2}^{t} \bigl[h(t,s)F(t,s) + B(s)\bigl(F(t,s)\bigr)^{\frac{\gamma+1}{\gamma}} + G(t,s)\bigr]\,ds,
\end{aligned} \tag{40}$$

where $G(t,s) = \frac{a(s)\rho(s)h^{\gamma+1}(t,s)}{(\gamma+1)^{\gamma+1}}$.

By (45), we get

$$\psi(t_2) \le w(t_2) - \liminf_{t\to\infty} \frac{1}{H(t,t_2)} \int_{t_2}^{t} \bigl[h(t,s)F(t,s) + B(s)\bigl(F(t,s)\bigr)^{\frac{\gamma+1}{\gamma}} + G(t,s)\bigr]\,ds, \tag{41}$$

and hence

$$\begin{aligned}
0 &\le \liminf_{t\to\infty} \frac{1}{H(t,t_2)} \int_{t_2}^{t} \bigl[h(t,s)F(t,s) + B(s)\bigl(F(t,s)\bigr)^{\frac{\gamma+1}{\gamma}} + G(t,s)\bigr]\,ds \\
&\le w(t_2) - \psi(t_2) < \infty.
\end{aligned} \tag{42}$$

Define the functions $\alpha(t)$ and $\beta(t)$ as follows:

$$\begin{aligned}
\alpha(t) &= \frac{1}{H(t,t_2)} \int_{t_2}^{t} h(t,s)F(t,s)\,ds, \\
\beta(t) &= \frac{1}{H(t,t_2)} \int_{t_2}^{t} B(s)\bigl(F(t,s)\bigr)^{\frac{\gamma+1}{\gamma}}\,ds.
\end{aligned} \tag{43}$$

From (37) and (42), we obtain

$$\liminf_{t\to\infty} \bigl[\alpha(t) + \beta(t)\bigr] < \infty. \tag{44}$$

The remainder of the proof is similar to the theorem given in [9–11] and hence is omitted. If $z(t)$ has the property (II), since (10) holds, by Lemma 2.2, we have $\lim_{t\to\infty} x(t) = 0$. □

Theorem 3.3 *If we replace* (37) *by*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)Q(s)\,ds < \infty, \tag{45}$$

and assume that the other assumptions of Theorem 3.2 *hold, then every solution of Eq.* (1) *is either oscillatory or converges to zero.*

Proof The proof is similar to Theorem 3.2 and hence is omitted. □

Remark 3.4 When $\gamma = 1$, Theorems 3.1-3.3 with condition (37) reduce to Theorems 3.1-3.3 of Zhang [6], respectively.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The work presented here was carried out in collaboration between all authors. GQ carried out the design of the study and drafted the manuscript. CH and YX conceived, instructed the design of the study and polished the manuscript. FW participated in discussion and completed the revision of the manuscript. All authors read and approved the final manuscript.

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