Wang *Advances in Difference Equations* 2013, **2013**:273
http://www.advancesindifferenceequations.com/content/2013/1/273

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

Anti-periodic solution for impulsive high-order Hopfield neural networks with time-varying delays in the leakage terms

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Abstract

This paper presents new results on anti-periodic solutions for impulsive high-order Hopfield neural networks with time-varying delays in the leakage terms. By employing a novel proof, some criteria are derived for guaranteeing the existence and exponential stability of the anti-periodic solution, which are new and complement previously known results. Moreover, a numerical simulation is given to show the effectiveness.

Keywords: impulsive high-order Hopfield neural networks; anti-periodic solution; exponential stability; time-varying delay; leakage term

1 Introduction

In this paper, we discuss anti-periodic solutions for impulsive high-order Hopfield neural networks (IHHNNs) with time-varying delays in the leakage terms

$$\begin{cases} x_i'(t) = -c_i(t)x_i(t-\eta_i(t)) + \sum_{j=1}^n a_{ij}(t)g_j(x_j(t-\tau_{ij}(t))) \\ \qquad + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}(t)g_j(x_j(t-\sigma_{ijl}(t)))g_l(x_l(t-\nu_{ijl}(t))) + I_i(t), \\ \qquad t>0, t\neq t_k, \\ \Delta x_i(t_k) = d_{ik}x_i(t_k), \\ x_i(t) = \varphi_i(t), \quad t\in[-\tau_i,0], \qquad k=1,2,\ldots, \end{cases} \tag{1.1}$$

where $i\in\mathcal{N} := \{1,2,\ldots,n\}$ and n is the number of units in a neural network, $x_i(t)$ corresponds to the state vector of the ith unit at time t, $c_i(t)>0$ represents the rate at which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs, $a_{ij}(t)$ and $b_{ijl}(t)$ are the first- and second-order connection weights of the neural network, $\eta_i(t)\geq 0$ denotes the leakage delay and $t-\eta_i(t)>0$ for all $t>0$. $\tau_{ij}(t)\geq 0$, $\sigma_{ijl}(t)\geq 0$, $\nu_{ijl}(t)\geq 0$ correspond to the transmission delays, $I_i(t)$ denotes the external inputs at time t, and g_j is the activation function of signal transmission. c_i, η_i, I_i, a_{ij}, b_{ijl}, g_j, τ_{ij}, σ_{ijl}, ν_{ijl} are continuous functions on R. $\tau_i = \max_{j,l\in\mathcal{N}}\max_{t\in[0,\omega]}\{\eta_i(t),\tau_{ij}(t),\sigma_{ijl}(t),\nu_{ijl}(t)\}$ is a positive constant. $\Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k)$, $x_i(t_k^+) = \lim_{\Delta t\to 0^+} x_i(t_k+\Delta t)$, $x_i(t_k) = \lim_{\Delta t\to 0^-} x_i(t_k+\Delta t)$, $i\in\mathcal{N}$, $k=1,2,\ldots$. $t_k>0$ are impulsive moments satisfying $t_k<t_{k+1}$ and $\lim_{k\to+\infty} t_k = +\infty$. $\varphi(t) = (\varphi_1,\varphi_2,\ldots,\varphi_n)^T$ is the

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initial condition and $\varphi_i(\cdot)$ denotes real-valued continuous functions defined on $[-\tau_i, 0]$, $i \in \mathcal{N}$.

The impulsive differential equations have been proposed in many fields such as control theory, physics, chemistry, population dynamics, biotechnologies, industrial robotics, economics, *etc.* [1–3]. High-order neural networks have been the object of intensive analysis by numerous authors since high-order neural networks have stronger approximation property, faster convergence rate, greater storage capacity, and higher fault tolerance than lower-order neural networks [4–8]. Thus, many high-order Hopfield neural networks with impulses have been studied extensively, and a great deal of literature focuses on the existence and stability of equilibrium points, periodic solutions, almost periodic solutions and anti-periodic solutions [9–16]. However, to the best of our knowledge, few authors have considered the existence and stability of an anti-periodic solution of system (1.1) with the leakage delay $\eta_i(t) \neq$ constant. We mention that arguments in [9–16] are not applicable to system (1.1).

The purpose of this paper is to discuss the existence and exponential stability of an anti-periodic solution for IHHNNs with time-varying delays in the leakage terms of system (1.1). The outline of the paper is as follows. In Section 2, some preliminaries and basic results are established. In Section 3, we give sufficient conditions for the existence and exponential stability of an anti-periodic solution for system (1.1). In Section 4, we give an example and numerical simulation to illustrate our results.

2 Preliminaries and basic results

Throughout this paper, we assume that the following conditions hold.

(H_1) For $i, j, l \in \mathcal{N}$ and $k \in Z^+$, where Z^+ denotes the set of all positive integers, there exists a constant $\omega > 0$ such that

$$\begin{cases} c_i(t+\omega) = c_i(t), \qquad \eta_i(t+\omega) = \eta_i(t), \\ a_{ij}(t+\omega)g_j(u) = -a_{ij}(t)g_j(-u), \\ \tau_{ij}(t+\omega) = \tau_{ij}(t), \qquad \sigma_{ijl}(t+\omega) = \sigma_{ijl}(t), \qquad \nu_{ijl}(t+\omega) = \nu_{ijl}(t), \\ b_{ijl}(t+\omega)g_j(u)g_l(u) = -b_{ijl}(t)g_j(-u)g_l(-u), \\ I_i(t+\omega) = -I_i(t), \quad t, u \in R. \end{cases} \tag{2.1}$$

(H_2) For $i, j, l \in \mathcal{N}$, there exist constants c_i^+, η_i^+, I_i^+, a_{ij}^+, τ_{ij}^+, b_{ijl}^+, σ_{ijl}^+, ν_{ijl}^+ such that

$$\begin{cases} c_i^+ = \max_{t\in[0,\omega]} c_i(t), \qquad \eta_i^+ = \max_{t\in[0,\omega]} \eta_i(t), \\ a_{ij}^+ = \max_{t\in[0,\omega]} |a_{ij}(t)|, \qquad \tau_{ij}^+ = \max_{t\in[0,\omega]} \tau_{ij}(t), \\ b_{ijl}^+ = \max_{t\in[0,\omega]} |b_{ijl}(t)|, \qquad \sigma_{ijl}^+ = \max_{t\in[0,\omega]} \sigma_{ijl}(t), \\ \nu_{ijl}^+ = \max_{t\in[0,\omega]} \nu_{ijl}(t), \qquad I_i^+ = \max_{t\in[0,\omega]} |I_i(t)|. \end{cases} \tag{2.2}$$

(H_3) $-2 \le d_{ik} \le 0$ for $i \in \mathcal{N}$ and $k \in Z^+$.

(H_4) There exists $q \in Z^+$ such that

$$d_{i(k+q)} = d_{ik}, \qquad t_{k+q} = t_k + \omega.$$

(H_5) For each $j \in \mathcal{N}$, the activation functions $g_j : R \to R$ are continuous and there exist nonnegative constants L_j and M such that, for all $u, v \in R$,

$$g_j(0) = 0, \qquad \left|g_j(u) - g_j(v)\right| \le L_j|u - v|, \qquad \left|g_j(u)\right| \le M.$$

(H_6) For all $t > 0$ and $i \in \mathcal{N}$, there exist positive constants ξ_i and η such that

$$\begin{aligned}-\eta > -\left[c_i(t) - c_i(t)\eta_i(t)c_i^+\right]\xi_i + \sum_{j=1}^{n}\left(\left|a_{ij}(t)\right| + c_i(t)\eta_i(t)a_{ij}^+\right)L_j\xi_j \\ + \sum_{j=1}^{n}\sum_{l=1}^{n}\left(\left|b_{ijl}(t)\right| + c_i(t)\eta_i(t)b_{ijl}^+\right)(L_j\xi_j + L_l\xi_l)M. \qquad (2.3)\end{aligned}$$

For convenience, let R^n be the set of all real vectors. We use $x = (x_1, x_2, \ldots, x_n)^T \in R^n$ to denote a column vector, in which the symbol (T) denotes the transpose of a vector. As usual in the theory of impulsive differential equations, at the points of discontinuity t_k of the solution $t \mapsto (x_1(t), x_2(t), \ldots, x_n(t))^T$, we assume that $(x_1(t), x_2(t), \ldots, x_n(t))^T = (x_1(t - 0), x_2(t - 0), \ldots, x_n(t - 0))^T$. It is clear that, in general, the derivative $x_i'(t_k)$ does not exist. On the other hand, according to system (1.1), there exists the limit $x_i'(t_k \mp 0)$. In view of the above convention, we assume that $x_i'(t_k) \equiv x_i'(t_k - 0)$.

Definition 2.1 A solution $x(t)$ of (1.1) is said to be ω-anti-periodic if

$$\begin{cases} x(t + \omega) = -x(t), & t \neq t_k, \\ x((t_k + \omega)^+) = -x(t_k^+), & k \in Z^+, \end{cases}$$

where the smallest positive number ω is called the anti-period of function $x(t)$.

In what follows, we shall prove the lemmas which will be used to prove our main results in Section 3.

Lemma 2.1 *Let* (H_1)-(H_6) *hold. Suppose that* $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ *is a solution of system* (1.1) *with initial conditions*

$$x_i(s) = \varphi_i(s), \qquad \left|\varphi_i(s)\right| < \xi_i\frac{\gamma}{\eta}, \quad s \in [-\tau_i, 0], \qquad (2.4)$$

where $\gamma = 1 + \max_{i\in\mathcal{N}}\{[c_i^+\eta_i^+ + 1]I_i^+\}$, $i \in \mathcal{N}$. *Then*

$$\left|x_i(s)\right| < \xi_i\frac{\gamma}{\eta} \quad \textit{for all } t > 0, i \in \mathcal{N}. \qquad (2.5)$$

Proof Assume that (2.5) does not hold. From (H_3), we have

$$\left|x_i\left(t_k^+\right)\right| = |1 + d_{ik}|\left|x_i(t_k)\right| \le \left|x_i(t_k)\right|.$$

So, if $|x_i(t_k^+)| > \xi_i\frac{\gamma}{\eta}$, then $|x_i(t_k)| > \xi_i\frac{\gamma}{\eta}$. Thus, we may assume that there exist $i \in \mathcal{N}$ and $t_* \in (t_k, t_{k+1})$ such that

$$\left|x_i(t_*)\right| = \xi_i\frac{\gamma}{\eta}, \quad \text{and} \quad \left|x_j(t)\right| < \xi_j\frac{\gamma}{\eta} \quad \text{for all } t \in [-\tau_j, t_*), j \in \mathcal{N}. \qquad (2.6)$$

In view of (1.1), for $i \in \mathcal{N}$, we obtain

$$\begin{aligned}
x_i'(t) &= -c_i(t)x_i\big(t-\eta_i(t)\big) + \sum_{j=1}^n a_{ij}(t)g_j\big(x_j\big(t-\tau_{ij}(t)\big)\big) \\
&\quad + \sum_{j=1}^n\sum_{l=1}^n b_{ijl}(t)g_j\big(x_j\big(t-\sigma_{ijl}(t)\big)\big)g_l\big(x_l\big(t-\nu_{ijl}(t)\big)\big) + I_i(t) \\
&= -c_i(t)x_i(t) + c_i(t)\big[x_i(t) - x_i\big(t-\eta_i(t)\big)\big] + \sum_{j=1}^n a_{ij}(t)g_j\big(x_j\big(t-\tau_{ij}(t)\big)\big) \\
&\quad + \sum_{j=1}^n\sum_{l=1}^n b_{ijl}(t)g_j\big(x_j\big(t-\sigma_{ijl}(t)\big)\big)g_l\big(x_l\big(t-\nu_{ijl}(t)\big)\big) + I_i(t) \\
&= -c_i(t)x_i(t) + c_i(t)\int_{t-\eta_i(t)}^t x_i'(s)\,ds + \sum_{j=1}^n a_{ij}(t)g_j\big(x_j\big(t-\tau_{ij}(t)\big)\big) \\
&\quad + \sum_{j=1}^n\sum_{l=1}^n b_{ijl}(t)g_j\big(x_j\big(t-\sigma_{ijl}(t)\big)\big)g_l\big(x_l\big(t-\nu_{ijl}(t)\big)\big) + I_i(t), \quad t > 0, t \neq t_k. \qquad (2.7)
\end{aligned}$$

Calculating the upper left derivative of $|x_i(t)|$, together with (2.6), (H_5), (H_6) and

$$\gamma > \big[c_i^+\eta_i^+ + 1\big]I_i^+,$$

we obtain

$$\begin{aligned}
0 &\le D^-\big|x_i(t_*)\big| \\
&\le -c_i(t_*)\big|x_i(t_*)\big| + c_i(t_*)\int_{t_*-\eta_i(t_*)}^{t_*}\big|x_i'(s)\big|\,ds + \sum_{j=1}^n\big|a_{ij}(t_*)\big|\big|g_j\big(x_j\big(t_*-\tau_{ij}(t_*)\big)\big)\big| \\
&\quad + \sum_{j=1}^n\sum_{l=1}^n\big|b_{ijl}(t_*)\big|\big|g_j\big(x_j\big(t_*-\sigma_{ijl}(t_*)\big)\big)g_l\big(x_l\big(t_*-\nu_{ijl}(t_*)\big)\big)\big| + \big|I_i(t_*)\big| \\
&= -c_i(t_*)\big|x_i(t_*)\big| + c_i(t_*)\int_{t_*-\eta_i(t_*)}^{t_*}\bigg|-c_i(s)x_i\big(s-\eta_i(s)\big) \\
&\quad + \sum_{j=1}^n a_{ij}(s)\big[g_j\big(x_j\big(s-\tau_{ij}(s)\big)\big) - g_j(0)\big] \\
&\quad + \sum_{j=1}^n\sum_{l=1}^n b_{ijl}(s)\big(g_j\big(x_j\big(s-\sigma_{ijl}(s)\big)\big) - g_j(0)\big)g_l\big(x_l\big(s-\nu_{ijl}(s)\big)\big) + I_i(s)\bigg|\,ds \\
&\quad + \sum_{j=1}^n\big|a_{ij}(t_*)\big|\big|g_j\big(x_j\big(t_*-\tau_{ij}(t_*)\big)\big) - g_j(0)\big| \\
&\quad + \sum_{j=1}^n\sum_{l=1}^n\big|b_{ijl}(t_*)\big|\big|g_j\big(x_j\big(t_*-\sigma_{ijl}(t_*)\big)\big) - g_j(0)\big|\big|g_l\big(x_l\big(t_*-\nu_{ijl}(t_*)\big)\big)\big| + \big|I_i(t_*)\big| \\
&\le -\big[c_i(t_*) - c_i(t_*)\eta_i(t_*)c_i^+\big]\big|x_i(t_*)\big| + \sum_{j=1}^n\big(\big|a_{ij}(t_*)\big| + c_i(t_*)\eta_i(t_*)a_{ij}^+\big)L_j\xi_j\frac{\gamma}{\eta}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{n}\sum_{l=1}^{n}\bigl(\bigl|b_{ijl}(t_*)\bigr|+c_i(t_*)\eta_i(t_*)b_{ijl}^{+}\bigr)L_j\xi_j\frac{\gamma}{\eta}M+\bigl[c_i^{+}\eta_i^{+}+1\bigr]I_i^{+}\\
<&-\bigl[c_i(t_*)-c_i(t_*)\eta_i(t_*)c_i^{+}\bigr]\xi_i\frac{\gamma}{\eta}+\sum_{j=1}^{n}\bigl(\bigl|a_{ij}(t_*)\bigr|+c_i(t_*)\eta_i(t_*)a_{ij}^{+}\bigr)L_j\xi_j\frac{\gamma}{\eta}\\
&+\sum_{j=1}^{n}\sum_{l=1}^{n}\bigl(\bigl|b_{ijl}(t_*)\bigr|+c_i(t_*)\eta_i(t_*)b_{ijl}^{+}\bigr)(L_j\xi_j+L_l\xi_l)M\frac{\gamma}{\eta}+\bigl[c_i^{+}\eta_i^{+}+1\bigr]I_i^{+}\\
=&\Biggl\{-\bigl[c_i(t_*)-c_i(t_*)\eta_i(t_*)c_i^{+}\bigr]\xi_i+\sum_{j=1}^{n}\bigl(\bigl|a_{ij}(t_*)\bigr|+c_i(t_*)\eta_i(t_*)a_{ij}^{+}\bigr)L_j\xi_j\\
&+\sum_{j=1}^{n}\sum_{l=1}^{n}\bigl(\bigl|b_{ijl}(t_*)\bigr|+c_i(t_*)\eta_i(t_*)b_{ijl}^{+}\bigr)(L_j\xi_j+L_l\xi_l)M\Biggr\}\frac{\gamma}{\eta}+\bigl[c_i^{+}\eta_i^{+}+1\bigr]I_i^{+}\\
<&-\eta\frac{\gamma}{\eta}+\bigl[c_i^{+}\eta_i^{+}+1\bigr]I_i^{+}\\
<&\ 0.
\end{aligned}
$$

It is a contradiction and shows that (2.5) holds. The proof is now completed. □

Remark 2.1 After conditions (H_1)-(H_6), the solution of system (1.1) always exists (see [1, 2]). In view of the boundedness of this solution, from the theory of impulsive differential equations in [1], it follows that the solution of system (1.1) can be defined on $[0,+\infty)$.

Lemma 2.2 *Suppose that* (H_1)-(H_6) *are true. Let* $x^*(t)=(x_1^*(t),x_2^*(t),\dots,x_n^*(t))^T$ *be the solution of system* (1.1) *with initial value* $\varphi^*(t)=(\varphi_1^*(t),\varphi_2^*(t),\dots,\varphi_n^*(t))^T$, *and let* $x(t)=(x_1(t),x_2(t),\dots,x_n(t))^T$ *be the solution of system* (1.1) *with initial value* $\varphi(t)=(\varphi_1(t),\varphi_2(t),\dots,\varphi_n(t))^T$. *Then there exists a positive constant* λ *such that*

$$x_i(t)-x_i^*(t)=O\bigl(e^{-\lambda t}\bigr),\quad i\in\mathcal{N}.$$

Proof Let $y(t)=x(t)-x^*(t)$. Then, for $i\in\mathcal{N}$, it follows that

$$
\begin{cases}
y_i'(t)=-c_i(t)y_i(t-\eta_i(t))+\sum_{j=1}^{n}a_{ij}(t)[g_j(x_j(t-\tau_{ij}(t)))-g_j(x_j^*(t-\tau_{ij}(t)))]\\
\qquad+\sum_{j=1}^{n}\sum_{l=1}^{n}b_{ijl}(t)[g_j(x_j(t-\sigma_{ijl}(t)))g_l(x_l(t-\nu_{ijl}(t)))\\
\qquad-g_j(x_j^*(t-\sigma_{ijl}(t)))g_l(x_l^*(t-\nu_{ijl}(t)))],\quad t>0,t\neq t_k,\\
y_i^+(t_k^+)=(1+d_{ik})y_i(t_k),\quad k\in Z^+.
\end{cases}
\tag{2.8}
$$

Define continuous functions $\Gamma_i(r)$ by setting

$$
\begin{aligned}
\Gamma_i(r)={}&-\bigl[c_i(t)e^{r\eta_i(t)}-r-c_i(t)e^{r\eta_i(t)}\eta_i(t)\bigl(r+c_i^{+}e^{r\eta_i^{+}}\bigr)\bigr]\xi_i\\
&+\sum_{j=1}^{n}\bigl(\bigl|a_{ij}(t)\bigr|e^{r\tau_{ij}(t)}+a_{ij}^{+}c_i(t)e^{r\tau_{ij}^{+}}\eta_i(t)\bigr)L_j\xi_j\\
&+\sum_{j=1}^{n}\sum_{l=1}^{n}b_{ijl}^{+}c_i(t)e^{r\eta_i(t)}\eta_i(t)\bigl(e^{r\nu_{ijl}(t)}L_l\xi_l+e^{r\sigma_{ijl}(t)}L_j\xi_j\bigr)M\\
&+\sum_{j=1}^{n}\sum_{l=1}^{n}\bigl|b_{ijl}(t)\bigr|\bigl(e^{r\nu_{ijl}(t)}L_l\xi_l+e^{r\sigma_{ijl}(t)}L_j\xi_j\bigr)M,\quad r\geq 0,t\geq 0,i\in\mathcal{N}.
\end{aligned}
$$

Then

$$\begin{aligned}\Gamma_i(0) &= -\big[c_i(t) - c_i(t)\eta_i(t)c_i^+\big]\xi_i + \sum_{j=1}^n\big(\big|a_{ij}(t)\big| + c_i(t)\eta_i(t)a_{ij}^+\big)L_j\xi_j \\ &\quad + \sum_{j=1}^n\sum_{l=1}^n\big(\big|b_{ijl}(t)\big| + c_i(t)\eta_i(t)b_{ijl}^+\big)(L_l\xi_l + L_j\xi_j)M \\ &< 0, \quad t \ge 0, i \in \mathcal{N},\end{aligned}$$

which, for $i \in \mathcal{N}$, together with the continuity of $\Gamma_i(t)$, implies that we can choose sufficiently small λ satisfying $c_i(t) > \lambda > 0$ and $\overline{\eta} > 0$ such that

$$\begin{aligned}-\overline{\eta} &> \Gamma_i(\lambda) \\ &= -\big[c_i(t)e^{\lambda\eta_i(t)} - \lambda - c_i(t)e^{\lambda\eta_i(t)}\eta_i(t)\big(\lambda + c_i^+e^{\lambda\eta_i^+}\big)\big]\xi_i \\ &\quad + \sum_{j=1}^n\big(\big|a_{ij}(t)\big|e^{\lambda\tau_{ij}(t)} + a_{ij}^+c_i(t)e^{\lambda\tau_{ij}^+}\eta_i(t)\big)L_j\xi_j \\ &\quad + \sum_{j=1}^n\sum_{l=1}^n b_{ijl}^+c_i(t)e^{\lambda\eta_i(t)}\eta_i(t)\big(e^{\lambda\nu_{ijl}^+}L_l\xi_l + e^{\lambda\sigma_{ijl}^+}L_j\xi_j\big)M \\ &\quad + \sum_{j=1}^n\sum_{l=1}^n\big|b_{ijl}(t)\big|\big(e^{\lambda\nu_{ijl}(t)}L_l\xi_l + e^{\lambda\sigma_{ijl}(t)}L_j\xi_j\big)M, \quad t \ge 0.\end{aligned} \tag{2.9}$$

Let

$$Y_i(t) = y_i(t)e^{\lambda t}, \quad i \in \mathcal{N}.$$

Then, for $i \in \mathcal{N}$,

$$\begin{aligned}Y_i'(t) &= \lambda Y_i(t) - c_i(t)e^{\lambda t}y_i\big(t - \eta_i(t)\big) \\ &\quad + e^{\lambda t}\Bigg\{\sum_{j=1}^n a_{ij}(t)\big[g_j\big(x_j\big(t - \tau_{ij}(t)\big)\big) - g_j\big(x_j^*\big(t - \tau_{ij}(t)\big)\big)\big] \\ &\quad + \sum_{j=1}^n\sum_{l=1}^n b_{ijl}(t)\big[g_j\big(x_j\big(t - \sigma_{ijl}(t)\big)\big)g_l\big(x_l\big(t - \nu_{ijl}(t)\big)\big) \\ &\quad - g_j\big(x_j^*\big(t - \sigma_{ijl}(t)\big)\big)g_l\big(x_l^*\big(t - \nu_{ijl}(t)\big)\big)\big]\Bigg\} \\ &= \lambda Y_i(t) - c_i(t)e^{\lambda\eta_i(t)}Y_i(t) + c_i(t)e^{\lambda\eta_i(t)}\big[Y_i(t) - Y_i\big(t - \eta_i(t)\big)\big] \\ &\quad + e^{\lambda t}\Bigg\{\sum_{j=1}^n a_{ij}(t)\big[g_j\big(x_j\big(t - \tau_{ij}(t)\big)\big) - g_j\big(x_j^*\big(t - \tau_{ij}(t)\big)\big)\big] \\ &\quad + \sum_{j=1}^n\sum_{l=1}^n b_{ijl}(t)\big[g_j\big(x_j\big(t - \sigma_{ijl}(t)\big)\big)g_l\big(x_l\big(t - \nu_{ijl}(t)\big)\big) \\ &\quad - g_j\big(x_j^*\big(t - \sigma_{ijl}(t)\big)\big)g_l\big(x_l^*\big(t - \nu_{ijl}(t)\big)\big)\big]\Bigg\}\end{aligned}$$

$$\begin{aligned}
&= \lambda Y_i(t) - c_i(t)e^{\lambda\eta_i(t)}Y_i(t) + c_i(t)e^{\lambda\eta_i(t)}\int_{t-\eta_i(t)}^{t} Y_i'(s)\,ds \\
&\quad + e^{\lambda t}\Biggl\{\sum_{j=1}^{n} a_{ij}(t)\bigl[g_j\bigl(x_j\bigl(t-\tau_{ij}(t)\bigr)\bigr) - g_j\bigl(x_j^*\bigl(t-\tau_{ij}(t)\bigr)\bigr)\bigr] \\
&\quad + \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(t)\bigl[g_j\bigl(x_j\bigl(t-\sigma_{ijl}(t)\bigr)\bigr)g_l\bigl(x_l\bigl(t-\nu_{ijl}(t)\bigr)\bigr) \\
&\quad - g_j\bigl(x_j^*\bigl(t-\sigma_{ijl}(t)\bigr)\bigr)g_l\bigl(x_l^*\bigl(t-\nu_{ijl}(t)\bigr)\bigr)\bigr]\Biggr\} \\
&= \lambda Y_i(t) - c_i(t)e^{\lambda\eta_i(t)}Y_i(t) + c_i(t)e^{\lambda\eta_i(t)}\int_{t-\eta_i(t)}^{t}\Biggl\{\lambda Y_i(s) - c_i(s)e^{\lambda s}y_i\bigl(s-\eta_i(s)\bigr) \\
&\quad + e^{\lambda s}\sum_{j=1}^{n} a_{ij}(s)\bigl[g_j\bigl(x_j\bigl(s-\tau_{ij}(s)\bigr)\bigr) - g_j\bigl(x_j^*\bigl(s-\tau_{ij}(s)\bigr)\bigr)\bigr] \\
&\quad + e^{\lambda s}\sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(s)\bigl[g_j\bigl(x_j\bigl(s-\sigma_{ijl}(s)\bigr)\bigr)g_l\bigl(x_l\bigl(s-\nu_{ijl}(s)\bigr)\bigr) \\
&\quad - g_j\bigl(x_j^*\bigl(s-\sigma_{ijl}(s)\bigr)\bigr)g_l\bigl(x_l^*\bigl(s-\nu_{ijl}(s)\bigr)\bigr)\bigr]\Biggr\}\,ds \\
&\quad + e^{\lambda t}\Biggl\{\sum_{j=1}^{n} a_{ij}(t)\bigl[g_j\bigl(x_j\bigl(t-\tau_{ij}(t)\bigr)\bigr) - g_j\bigl(x_j^*\bigl(t-\tau_{ij}(t)\bigr)\bigr)\bigr] \\
&\quad + \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(t)\bigl[g_j\bigl(x_j\bigl(t-\sigma_{ijl}(t)\bigr)\bigr)g_l\bigl(x_l\bigl(t-\nu_{ijl}(t)\bigr)\bigr) \\
&\quad - g_j\bigl(x_j^*\bigl(t-\sigma_{ijl}(t)\bigr)\bigr)g_l\bigl(x_l^*\bigl(t-\nu_{ijl}(t)\bigr)\bigr)\bigr]\Biggr\}, \quad t > 0, t \neq t_k, \qquad (2.10)
\end{aligned}$$

and

$$\bigl|Y_i\bigl(t_k^+\bigr)\bigr| = \bigl|(1+d_{ik})Y_i(t_k)\bigr|. \qquad (2.11)$$

We define a positive constant $\overline{M}$ as follows:

$$\overline{M} = \max_{i\in\mathcal{N}}\Bigl\{\sup_{s\in[-\tau_i,0]}\bigl|Y_i(s)\bigr|\Bigr\}.$$

Let K be a positive number such that

$$\bigl|Y_i(t)\bigr| \leq \overline{M} < K\xi_i \quad \text{for all } t \in [-\tau_i, 0], i \in \mathcal{N}. \qquad (2.12)$$

We claim that

$$\bigl|Y_i(t)\bigr| < K\xi_i \quad \text{for all } t > 0, i \in \mathcal{N}. \qquad (2.13)$$

Obviously, (2.13) holds for $t = 0$. We first prove that (2.13) is true for $0 < t \leq t_1$. Otherwise, there exist $i \in \mathcal{N}$ and $\rho \in (0, t_1]$ such that one of the following two cases must occur.

$$(1) \quad Y_i(\rho) = K\xi_i, \qquad \left|Y_j(t)\right| < K\xi_j \quad \text{for all } t \in [0,\rho), j \in \mathcal{N}. \tag{2.14}$$

$$(2) \quad Y_i(\rho) = -K\xi_i, \qquad \left|Y_j(t)\right| < K\xi_j \quad \text{for all } t \in [0,\rho), j \in \mathcal{N}. \tag{2.15}$$

Now, we consider two cases.

Case (i). If (2.14) holds. Then, from (2.9), (2.10) and (H_1)-(H_6), we have

$$\begin{aligned}
0 &\le Y_i'(\rho) \\
&= \lambda Y_i(\rho) - c_i(\rho)e^{\lambda\eta_i(\rho)}Y_i(\rho) + c_i(\rho)e^{\lambda\eta_i(\rho)}\int_{\rho-\eta_i(\rho)}^{\rho}\Bigg\{\lambda Y_i(s) - c_i(s)e^{\lambda s}y_i\big(s-\eta_i(s)\big) \\
&\quad + e^{\lambda s}\sum_{j=1}^{n} a_{ij}(s)\big[g_j\big(x_j\big(s-\tau_{ij}(s)\big)\big) - g_j\big(x_j^*\big(s-\tau_{ij}(s)\big)\big)\big] \\
&\quad + e^{\lambda s}\sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(s)\big[g_j\big(x_j\big(s-\sigma_{ijl}(s)\big)\big)g_l\big(x_l\big(s-\nu_{ijl}(s)\big)\big) \\
&\quad - g_j\big(x_j^*\big(s-\sigma_{ijl}(s)\big)\big)g_l\big(x_l^*\big(s-\nu_{ijl}(s)\big)\big)\big]\Bigg\}\,ds \\
&\quad + e^{\lambda\rho}\Bigg\{\sum_{j=1}^{n} a_{ij}(\rho)\big[g_j\big(x_j\big(\rho-\tau_{ij}(\rho)\big)\big) - g_j\big(x_j^*\big(\rho-\tau_{ij}(\rho)\big)\big)\big] \\
&\quad + \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(\rho)\big[g_j\big(x_j\big(\rho-\sigma_{ijl}(\rho)\big)\big)g_l\big(x_l\big(\rho-\nu_{ijl}(\rho)\big)\big) \\
&\quad - g_j\big(x_j^*\big(\rho-\sigma_{ijl}(\rho)\big)\big)g_l\big(x_l^*\big(\rho-\nu_{ijl}(\rho)\big)\big)\big]\Bigg\} \\
&\le \lambda Y_i(\rho) - c_i(\rho)e^{\lambda\eta_i(\rho)}Y_i(\rho) + c_i(\rho)e^{\lambda\eta_i(\rho)}\int_{\rho-\eta_i(\rho)}^{\rho}\Bigg\{\lambda Y_i(\rho) + c_i^+e^{\lambda\eta_i(s)}\big|Y_i\big(s-\eta_i(s)\big)\big| \\
&\quad + \sum_{j=1}^{n} a_{ij}^+L_je^{\lambda\tau_{ij}(s)}\big|Y_j\big(s-\tau_{ij}(s)\big)\big| + \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}^+ \\
&\quad \times\big[e^{\lambda s}\big|g_j\big(x_j\big(s-\sigma_{ijl}(s)\big)\big)\big|\big|g_l\big(x_l\big(s-\nu_{ijl}(s)\big)\big) - g_l\big(x_l^*\big(s-\nu_{ijl}(s)\big)\big)\big| \\
&\quad + e^{\lambda s}\big|g_l\big(x_l^*\big(s-\nu_{ijl}(s)\big)\big)\big|\big|g_j\big(x_j\big(s-\sigma_{ijl}(s)\big)\big) - g_j\big(x_j^*\big(s-\sigma_{ijl}(s)\big)\big)\big|\big]\Bigg\}\,ds \\
&\quad + \sum_{j=1}^{n}\big|a_{ij}(\rho)\big|L_je^{\lambda\tau_{ij}(\rho)}\big|Y_j\big(\rho-\tau_{ij}(\rho)\big)\big| + \sum_{j=1}^{n}\sum_{l=1}^{n}\big|b_{ijl}(\rho)\big| \\
&\quad \times\big[e^{\lambda\rho}\big|g_j\big(x_j\big(\rho-\sigma_{ijl}(\rho)\big)\big)\big|\big|g_l\big(x_l\big(\rho-\nu_{ijl}(\rho)\big)\big) - g_l\big(x_l^*\big(\rho-\nu_{ijl}(\rho)\big)\big)\big| \\
&\quad + e^{\lambda\rho}\big|g_l\big(x_l^*\big(\rho-\nu_{ijl}(\rho)\big)\big)\big|\big|g_j\big(x_j\big(\rho-\sigma_{ijl}(\rho)\big)\big) - g_j\big(x_j^*\big(\rho-\sigma_{ijl}(\rho)\big)\big)\big|\big] \\
&\le -\big[c_i(\rho)e^{\lambda\eta_i(\rho)} - \lambda - c_i(\rho)e^{\lambda\eta_i(\rho)}\eta_i(\rho)\big(\lambda + c_i^+e^{\lambda\eta_i^+}\big)\big]K\xi_i \\
&\quad + \sum_{j=1}^{n}\big(\big|a_{ij}(\rho)\big|e^{\lambda\tau_{ij}(\rho)} + a_{ij}^+c_i(\rho)e^{\lambda\tau_{ij}^+}\eta_i(\rho)\big)L_jK\xi_j
\end{aligned}$$

$$\begin{aligned}
&+\sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}^{+} c_i(\rho) e^{\lambda\eta_i(\rho)}\eta_i(\rho)\bigl(Me^{\lambda\nu_{ijl}^{+}}L_lK\xi_l + Me^{\lambda\sigma_{ijl}^{+}}L_jK\xi_j\bigr) \\
&+\sum_{j=1}^{n}\sum_{l=1}^{n} \bigl|b_{ijl}(\rho)\bigr|\bigl(Me^{\lambda\nu_{ijl}(\rho)}L_lK\xi_l + Me^{\lambda\sigma_{ijl}(\rho)}L_jK\xi_j\bigr) \\
= &\Biggl\{-\bigl[c_i(\rho)e^{\lambda\eta_i(\rho)} - \lambda - c_i(\rho)e^{\lambda\eta_i(\rho)}\eta_i(\rho)\bigl(\lambda + c_i^{+}e^{\lambda\eta_i^{+}}\bigr)\bigr]\xi_i \\
&+\sum_{j=1}^{n}\bigl(\bigl|a_{ij}(\rho)\bigr|e^{\lambda\tau_{ij}(\rho)} + a_{ij}^{+}c_i(\rho)e^{\lambda\tau_{ij}^{+}}\eta_i(\rho)\bigr)L_j\xi_j \\
&+\sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}^{+} c_i(\rho) e^{\lambda\eta_i(\rho)}\eta_i(\rho)\bigl(e^{\lambda\nu_{ijl}^{+}}L_l\xi_l + e^{\lambda\sigma_{ijl}^{+}}L_j\xi_j\bigr)M \\
&+\sum_{j=1}^{n}\sum_{l=1}^{n} \bigl|b_{ijl}(\rho)\bigr|\bigl(e^{\lambda\nu_{ijl}(\rho)}L_l\xi_l + e^{\lambda\sigma_{ijl}(\rho)}L_j\xi_j\bigr)M\Biggr\}K \\
< &-\overline{\eta}K \\
< &\ 0.
\end{aligned}$$

Case (ii). If (2.15) holds. From (2.9), (2.10) and (H_1)-(H_6), using a similar method, we can obtain the contradiction. Therefore, (2.13) holds for $t \in [0, t_1]$. From (2.11) and (2.13), we know that

$$\bigl|Y_i(t_1)\bigr| = \bigl|y_i(t_1)\bigr|e^{\lambda t_1} < K\xi_i, \quad i \in \mathcal{N},$$

and

$$\bigl|Y_i\bigl(t_1^{+}\bigr)\bigr| = |1 + d_{i1}|\bigl|Y_i(t_1)\bigr| \le \bigl|Y_i(t_1)\bigr| < K\xi_i, \quad i \in \mathcal{N}.$$

Thus, for $t \in [t_1, t_2]$, we may repeat the above procedure and obtain

$$\bigl|Y_i(t)\bigr| = \bigl|y_i(t)\bigr|e^{\lambda t} < K\xi_i \quad \text{for all } t \in [t_1, t_2], i \in \mathcal{N}.$$

Further, we have

$$\bigl|Y_i(t)\bigr| = \bigl|y_i(t)\bigr|e^{\lambda t} < K\xi_i \quad \text{for all } t > 0, i \in \mathcal{N}.$$

That is,

$$\bigl|x_i(t) - x_i^*(t)\bigr| \le K\xi_i e^{-\lambda t}, \quad \forall t > 0, \text{ and } i \in \mathcal{N}. \qquad \square$$

Remark 2.2 If $x^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t))^T$ is an ω-anti-periodic solution of system (1.1), it follows from Lemma 2.2 that $x^*(t)$ is globally exponentially stable.

3 Main results

In this section, we study the existence and exponential stability for an anti-periodic solution of system (1.1).

Theorem 3.1 *Suppose that all conditions in Lemma* 2.2 *are satisfied. Then system* (1.1) *has exactly one ω-anti-periodic solution $x^*(t)$. Moreover, $x^*(t)$ is globally exponentially stable.*

Proof Let $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ be a solution of system (1.1). By Remark 2.1, the solution $x(t)$ can be defined for all $t \in [0, +\infty)$. By hypothesis (H_1), we have, for any natural number h and $i \in \mathcal{N}$,

$$\begin{aligned}
&\bigl((-1)^{h+1}x_i\bigl(t+(h+1)\omega\bigr)\bigr)' \\
&\quad = (-1)^{h+1}x_i\bigl(t+(h+1)\omega\bigr)' \\
&\quad = (-1)^{h+1}\Biggl\{-c_i\bigl(t+(h+1)\omega\bigr)x_i\bigl(t+(h+1)\omega-\eta_i\bigl(t+(h+1)\omega\bigr)\bigr) \\
&\qquad + \sum_{j=1}^{n} a_{ij}\bigl(t+(h+1)\omega\bigr)g_j\bigl(x_j\bigl(t+(h+1)\omega-\tau_{ij}\bigl(t+(h+1)\omega\bigr)\bigr)\bigr) \\
&\qquad + \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}\bigl(t+(h+1)\omega\bigr)g_j\bigl(x_j\bigl(t+(h+1)\omega-\sigma_{ijl}\bigl(t+(h+1)\omega\bigr)\bigr)\bigr) \\
&\qquad \times g_l\bigl(x_l\bigl(t+(h+1)\omega-\nu_{ijl}\bigl(t+(h+1)\omega\bigr)\bigr)\bigr) + I_i\bigl(t+(h+1)\omega\bigr)\Biggr\} \\
&\quad = (-1)^{h+1}\Biggl\{-c_i(t)x_i\bigl(t+(h+1)\omega-\eta_i(t)\bigr) \\
&\qquad + \sum_{j=1}^{n} a_{ij}(t)(-1)^{h+1}g_j\bigl((-1)^{h+1}x_j\bigl(t+(h+1)\omega-\tau_{ij}(t)\bigr)\bigr) \\
&\qquad + \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(t)(-1)^{h+1}g_j\bigl((-1)^{h+1}x_j\bigl(t+(h+1)\omega-\sigma_{ijl}(t)\bigr)\bigr) \\
&\qquad \times g_l\bigl((-1)^{h+1}x_l\bigl(t+(h+1)\omega-\nu_{ijl}(t)\bigr)\bigr) + (-1)^{h+1}I_i(t)\Biggr\} \\
&\quad = -c_i(t)(-1)^{h+1}x_i\bigl(t+(h+1)\omega-\eta_i(t)\bigr) \\
&\qquad + \sum_{j=1}^{n} a_{ij}(t)g_j\bigl((-1)^{h+1}x_j\bigl(t+(h+1)\omega-\tau_{ij}(t)\bigr)\bigr) \\
&\qquad + \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(t)g_j\bigl((-1)^{h+1}x_j\bigl(t+(h+1)\omega-\sigma_{ijl}(t)\bigr)\bigr) \\
&\qquad \times g_l\bigl((-1)^{h+1}x_l\bigl(t+(h+1)\omega-\nu_{ijl}(t)\bigr)\bigr) + I_i(t), \quad t \neq t_k.
\end{aligned} \tag{3.1}$$

Further, by hypothesis (H_4), we obtain

$$\begin{aligned}
&(-1)^{h+1}x_i\bigl(\bigl(t_k+(h+1)\omega\bigr)^+\bigr) \\
&\quad = (-1)^{h+1}x_i\bigl(t^+_{k+(h+1)q}\bigr) \\
&\quad = (-1)^{h+1}(1+d_{i(k+(h+1)q)})x_i(t_{k+(h+1)q}) \\
&\quad = (1+d_{ik})(-1)^{h+1}x_i\bigl(t_k+(h+1)\omega\bigr), \quad k = 1, 2, \dots.
\end{aligned} \tag{3.2}$$

Thus, for any natural number h, we obtain that $(-1)^{h+1}x(t+(h+1)\omega)$ is a solution of system (1.1) for all $t+(h+1)\omega \geq 0$. Hence, $-x(t+\omega)$ is also a solution of (1.1) with initial values

$$-x_i(s+\omega), \quad s \in [-\tau_i, 0], i \in \mathcal{N}.$$

Then, by the proof of Lemma 2.2, for $i \in \mathcal{N}$, there exists a constant $K > 0$ such that for any natural number h, we have

$$\begin{aligned}
&\bigl|(-1)^{h+1}x_i\bigl(t+(h+1)\omega\bigr) - (-1)^h x_i(t+h\omega)\bigr| \\
&\quad = \bigl|x_i(t+h\omega) - \bigl(-x_i(t+h\omega+\omega)\bigr)\bigr| \\
&\quad \leq K\xi_i e^{-\lambda(t+h\omega)} \\
&\quad = K\xi_i e^{-\lambda t}\left(\frac{1}{e^{\lambda\omega}}\right)^h, \quad t+h\omega \geq 0, t \neq t_k,
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
&\bigl|(-1)^{h+1}x_i\bigl(\bigl(t_k+(h+1)\omega\bigr)^+\bigr) - (-1)^h x_i\bigl((t_k+h\omega)^+\bigr)\bigr| \\
&\quad = |1+d_{ik}|\bigl|x_i(t_k+h\omega) - \bigl(-x_i(t_k+h\omega+\omega)\bigr)\bigr| \\
&\quad \leq K\xi_i e^{-\lambda(t_k+h\omega)} \\
&\quad = K\xi_i e^{-\lambda t_k}\left(\frac{1}{e^{\lambda\omega}}\right)^h, \quad k \in Z^+.
\end{aligned} \tag{3.4}$$

Moreover, for any natural number m, and $i \in \mathcal{N}$, we can obtain

$$\begin{aligned}
&(-1)^{m+1}x_i\bigl(t+(m+1)\omega\bigr) \\
&\quad = x_i(t) + \sum_{h=0}^{m}\bigl[(-1)^{h+1}x_i\bigl(t+(h+1)\omega\bigr) - (-1)^h x_i(t+h\omega)\bigr], \quad t+h\omega \geq 0, t \neq t_k,
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
&(-1)^{m+1}x_i\bigl(\bigl(t_k+(m+1)\omega\bigr)^+\bigr) \\
&\quad = x_i\bigl(t_k^+\bigr) + \sum_{h=0}^{m}\bigl[(-1)^{h+1}x_i\bigl(\bigl(t_k+(h+1)\omega\bigr)^+\bigr) - (-1)^h x_i\bigl((t_k+h\omega)^+\bigr)\bigr], \quad k \in Z^+.
\end{aligned} \tag{3.6}$$

Combining (3.3)-(3.4) with (3.5)-(3.6), we know that $(-1)^m x(t+m\omega)$ converges uniformly to a piecewise continuous function $x^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t))^T$ on any compact set of R.

Now we are in a position to prove that $x^*(t)$ is an ω-anti-periodic solution of system (1.1). It is easily known that $x^*(t)$ is ω-anti-periodic since

$$\begin{aligned}
x_i^*(t+\omega) &= \lim_{m\to+\infty}(-1)^m x_i(t+\omega+m\omega) = -\lim_{m+1\to+\infty}(-1)^{m+1}x_i\bigl(t+(m+1)\omega\bigr) \\
&= -x_i^*(t), \quad t \neq t_k,
\end{aligned}$$

and

$$x_i^*\big((t_k+\omega)^+\big) = -\lim_{m+1\to+\infty}(-1)^{m+1}x_i\big(\big(t_k+(m+1)\omega\big)^+\big) = -x_i^*\big(t_k^+\big), \quad k\in Z^+,$$

where $i \in \mathcal{N}$. Noting that the right-hand side of (1.1) is piecewise continuous, together with (3.1) and (3.2), we know that $(-1)^{m+1}\{x_i'(t+(m+1)\omega)\}$ converges uniformly to a piecewise continuous function on any compact set of $R\setminus\{t_1,t_2,\ldots\}$. Therefore, letting $m\to+\infty$ on both sides of (3.1) and (3.2), we get

$$\left.\begin{array}{l} x_i^{*\prime}(t) = -c_i(t)x_i^*(t-\eta_i(t)) + \sum_{j=1}^n a_{ij}(t)g_j(x_j^*(t-\tau_{ij}(t))) \\ \qquad + \sum_{j=1}^n\sum_{l=1}^n b_{ijl}(t)g_j(x_j^*(t-\sigma_{ijl}(t)))g_l(x_l^*(t-\nu_{ijl}(t))) + I_i(t), \\ \qquad t>0, t\neq t_k, \\ x_i^*(t_k^+) = (1+d_{ik})x_i^*(t_k), \quad k\in Z^+, \end{array}\right\}\quad i\in\mathcal{N}.$$

Thus, $x^*(t) = \big(x_1^*(t),x_2^*(t),\ldots,x_n^*(t)\big)^T$ is an ω-anti-periodic solution of system (1.1).

Finally, by Lemma 2.2, we can prove that $x^*(t)$ is globally exponentially stable. This completes the proof. □

4 Example

In this section, we give an example to demonstrate the results obtained in previous sections.

Example 4.1 Consider the following IHHNNs consisting of two neurons with time-varying delays in the leakage terms:

$$\begin{cases} \left.\begin{array}{l} x_1'(t) = -1.5x_1(t-\frac{|\sin\pi t|}{1{,}000}) + \frac{|\sin\pi t|}{32}g_1(x_1(t-|\sin\pi t|)) \\ \qquad + \frac{|\cos\pi t|}{32}g_2(x_2(t-|\cos\pi t|)) \\ \qquad + \frac{\cos\pi t}{32}[g_1(x_1(t-2|\cos\pi t|))g_1(x_1(t-2|\sin\pi t|)) \\ \qquad + g_1(x_1(t-2|\sin\pi t|))g_2(x_2(t-2|\cos\pi t|)) \\ \qquad + g_2(x_2(t-2|\sin\pi t|))g_1(x_1(t-2|\cos\pi t|)) \\ \qquad + g_2(x_2(t-2|\sin\pi t|))g_2(x_2(t-2|\cos\pi t|))] \\ \qquad + 10\sin\pi t, \\ x_2'(t) = -1.5x_2(t-\frac{|\sin\pi t|}{1{,}000}) + \frac{|\cos\pi t|}{32}g_1(x_1(t-|\cos\pi t|)) \\ \qquad + \frac{|\sin\pi t|}{32}g_2(x_2(t-|\sin\pi t|)) \\ \qquad + \frac{\sin\pi t}{32}[g_1(x_1(t-2|\sin\pi t|))g_1(x_1(t-2|\cos\pi t|)) \\ \qquad + g_1(x_1(t-2|\cos\pi t|))g_2(x_2(t-2|\sin\pi t|)) \\ \qquad + g_2(x_2(t-2|\cos\pi t|))g_1(x_1(t-2|\sin\pi t|)) \\ \qquad + g_2(x_2(t-2|\cos\pi t|))g_2(x_2(t-2|\sin\pi t|))] \\ \qquad + 10\cos\pi t, \end{array}\right\} \quad t\neq k-0.5, \\ \left.\begin{array}{l} x_i(t_k^+) = (1+d_{ik})x_i(t_k), \\ d_{i(2s)} = -2, \qquad d_{i(2s-1)} = -1, \end{array}\right\} \quad t_k = k-0.5, i=1,2, k,s=1,2,\ldots. \end{cases} \tag{4.1}$$

Here, it is assumed that the activation functions

$$g_1(x) = g_2(x) = |x+1| - |x-1|.$$

Note that

$$\begin{aligned}
&c_1(t) = c_2(t) = 1.5, \qquad L_1 = L_2 = 2, \qquad M = 2,\\
&a_{11}(t) = \frac{|\sin \pi t|}{32}, \qquad a_{12}(t) = \frac{|\cos \pi t|}{32},\\
&a_{21}(t) = \frac{|\cos \pi t|}{32}, \qquad a_{22}(t) = \frac{|\sin \pi t|}{32},\\
&b_{111}(t) = b_{112}(t) = b_{121}(t) = b_{122}(t) = \frac{\cos \pi t}{32},\\
&b_{211}(t) = b_{212}(t) = b_{221}(t) = b_{222}(t) = \frac{\sin \pi t}{32},\\
&\eta_1(t) = \eta_2(t) = \frac{|\sin \pi t|}{1{,}000}, \qquad I_1(t) = 10 \sin \pi t, \qquad I_2(t) = 10 \cos \pi t,\\
&\tau_{11}(t) = |\sin \pi t|, \qquad \tau_{12}(t) = |\cos \pi t|, \qquad \tau_{21}(t) = |\cos \pi t|, \qquad \tau_{22}(t) = |\sin \pi t|,\\
&\sigma_{111}(t) = 2|\cos \pi t|, \qquad \sigma_{112}(t) = \sigma_{121}(t) = \sigma_{122}(t) = 2|\sin \pi t|,\\
&\sigma_{211}(t) = 2|\sin \pi t|, \qquad \sigma_{212}(t) = \sigma_{221}(t) = \sigma_{222}(t) = 2|\cos \pi t|,\\
&\nu_{111}(t) = 2|\sin \pi t|, \qquad \nu_{112}(t) = \nu_{121}(t) = \nu_{122}(t) = 2|\cos \pi t|,\\
&\nu_{211}(t) = 2|\cos \pi t|, \qquad \nu_{212}(t) = \nu_{221}(t) = \nu_{222}(t) = 2|\sin \pi t|.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
&-\bigl[c_i(t) - c_i(t)\eta_i(t)c_i^+\bigr]\xi_i + \sum_{j=1}^{n}\bigl(\bigl|a_{ij}(t)\bigr| + c_i(t)\eta_i(t)a_{ij}^+\bigr)L_j\xi_j\\
&\quad + \sum_{j=1}^{n}\sum_{l=1}^{n}\bigl(\bigl|b_{ijl}(t)\bigr| + c_i(t)\eta_i(t)b_{ijl}^+\bigr)(L_j\xi_j + L_l\xi_l)M\\
&< -1.5 + 1.5 \times \frac{1}{1{,}000} \times 1.5 + \left(\frac{1}{32} + 3 \times \frac{1}{1{,}000} \times \frac{1}{32}\right) \times 2 \times 2\\
&\quad + \left(\frac{1}{32} + 3 \times \frac{1}{1{,}000} \times \frac{1}{32}\right) \times (2+2) \times 2 \times 4\\
&= -0.369375 < -0.3, \quad \xi_i = 1, i = 1, 2.
\end{aligned} \tag{4.2}$$

It follows that system (4.1) satisfies all the conditions in Theorem 3.1. Hence, system (4.1) has exactly one 1-anti-periodic solution. Moreover, the 1-anti-periodic solution is globally exponentially stable. The fact is verified by the numerical simulation in Figure 1.

Remark 4.1 Since [9–16] only dealt with IHHNNs without leakage delays, one can observe that all the results in these works and the references therein cannot be applicable to prove the existence and exponential stability of 1-anti-periodic solution for IHHNNs (4.1). This implies that the results of this paper are essentially new.

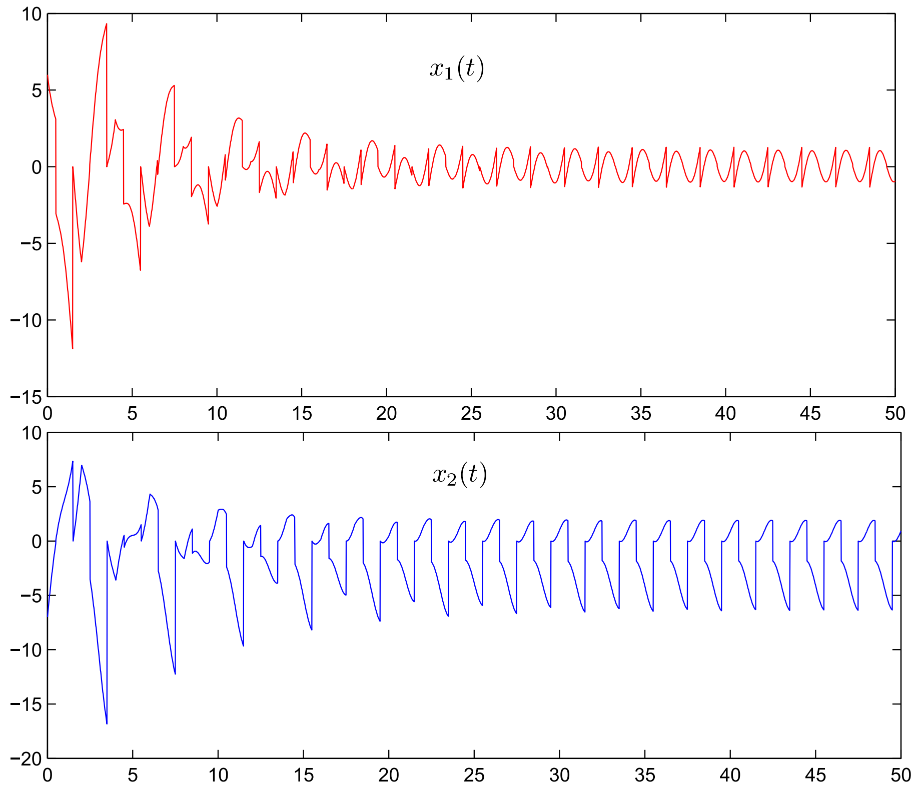

Figure 1 Numerical solution $x(t) = (x_1(t), x_2(t))^T$ of systems (4.1) for initial value $\varphi(s) \equiv (6, -7)^T$, $s \in [-2, 0]$.

Competing interests
The author declares that they have no competing interests.

Author's contributions
The author has made this manuscript independently. The author read and approved the final version.

Acknowledgements
I am grateful to the referees for their suggestions that improved the writing of the paper. This work was supported by the National Natural Science Foundation of China (grant no. 11201184), the Natural Scientific Research Fund of Zhejiang Provincial of P.R. China (grant no. LY12A01018), and the Natural Scientific Research Fund of Zhejiang Provincial Education Department of P.R. China (grant no. Z201122436).

Received: 3 May 2013 Accepted: 21 August 2013 Published: 11 September 2013

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doi:10.1186/1687-1847-2013-273
Cite this article as: Wang: **Anti-periodic solution for impulsive high-order Hopfield neural networks with time-varying delays in the leakage terms.** *Advances in Difference Equations* 2013 **2013**:273.