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Existence and exponential stability of an equilibrium point for fuzzy BAM neural networks with time-varying delays in leakage terms on time scales

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Abstract

In this paper, by using a fixed point theorem and differential inequality techniques, we consider the existence and global exponential stability of an equilibrium point for a class of fuzzy bidirectional associative memory neural networks with time-varying delays in leakage terms on time scales. We also present a numerical example to show the feasibility of obtained results. The results of this paper are completely new and complementary to the previously known results.

MSC: 34B37; 34N05

Keywords: fuzzy BAM neural networks; exponential stability; leakage terms; time scales

1 Introduction

The bidirectional associative memory (BAM) neural networks were first introduced by Kosto in 1988 [1]. These are special recurrent neural networks that can store bipolar vector pairs and are composed of neurons arranged in two layers. The neurons in one layer are fully interconnected to the neurons in the other layer, while there are no interconnections among neurons in the same layer.

In recent years, due to their wide range of applications, for example, pattern recognition, associative memory, and combinatorial optimization, BAM neural networks have received much attention. There are lots of results on the existence and stability of an equilibrium point, periodic solutions or almost periodic solutions of BAM neural networks [2–10].

Based on traditional cellular neural networks, Yang and Yang proposed a fuzzy cellular neural network, which integrates fuzzy logic into the structure of traditional cellular neural networks and maintains local connectedness among cells [11]. The fuzzy neural network has fuzzy logic between its template input and/or output besides the sum of product operation. Studies have revealed that the fuzzy neural network is very useful for image processing problems, which is a cornerstone in image processing and pattern recognition. Besides, in reality, time delays often occur due to finite switching speeds of the amplifiers and communication time and can destroy a stable network or cause sustained oscillations, bifurcation or chaos. Hence, it is important to consider both the fuzzy logic and delay effect on dynamical behaviors of neural networks. There have been many results on the fuzzy neural networks with time delays [12–17]. Moreover, time delay in the leakage term also has a great impact on the dynamics of neural networks. As pointed out by the author in [18], time delay in the stabilizing negative feedback term has a tendency to destabilize a system. Therefore, it is meaningful to consider fuzzy neural networks with time delays in the leakage terms [19–24].

In fact, both continuous and discrete systems are very important in implementation and applications. To avoid the trouble of studying the dynamical properties for continuous and discrete systems respectively, it is meaningful to study those on time scales, which was initiated by Stefan Hilger in his PhD thesis, in order to unify continuous and discrete analyses. Lots of scholars have studied neural networks on time scales and obtained many good results [25–34]. However, to the best of our knowledge, there is no paper published on the stability of fuzzy BAM neural networks with time delays in the leakage terms on time scales.

Motivated by the above, in this paper, we integrate fuzzy operations into BAM neural networks with time delays in the leakage terms and study the stability of considered neural networks on time scales. By using a fixed point theorem and differential inequality techniques, we consider the existence and global exponential stability of an equilibrium point for the following BAM neural network with time-varying delays in leakage terms on time scales:

$$\begin{cases} x_i^{\Delta}(t) = -a_i x_i(t-\delta_i(t)) + \sum_{j=1}^{m} c_{ji} f_j(y_j(t-\tau_{ji}(t))) + \bigwedge_{j=1}^{m} \alpha_{ji} f_j(y_j(t-\tau_{ji}(t))) \\ \qquad + \bigwedge_{j=1}^{m} T_{ji}\mu_j + \bigvee_{j=1}^{m} \beta_{ji} f_j(y_j(t-\tau_{ji}(t))) \\ \qquad + \bigvee_{j=1}^{m} H_{ji}\mu_j + I_i, \quad t \in \mathbb{T}, i = 1,2,\ldots,n, \\ y_j^{\Delta}(t) = -b_j y_j(t-\eta_j(t)) + \sum_{i=1}^{n} d_{ij} g_i(x_i(t-\sigma_{ij}(t))) + \bigwedge_{i=1}^{n} p_{ij} g_i(x_i(t-\sigma_{ij}(t))) \\ \qquad + \bigwedge_{i=1}^{n} F_{ij}\nu_i + \bigvee_{i=1}^{n} q_{ij} g_i(x_i(t-\sigma_{ij}(t))) \\ \qquad + \bigvee_{i=1}^{n} G_{ij}\nu_i + J_j, \quad t \in \mathbb{T}, j = 1,2,\ldots,m, \end{cases} \tag{1.1}$$

where $\mathbb{T}$ is a time scale; n, m are the number of neurons in layers; $x_i(t)$ and $y_j(t)$ denote the activations of the ith neuron and the jth neuron at time t; $a_i > 0$ and $b_j > 0$ represent the rate at which the ith neuron and the jth neuron will reset their potential to the resting state in isolation when they are disconnected from the network and the external inputs; $0 < \delta_i < \delta_i^+$ and $0 < \eta_j < \eta_j^+$ denote the leakage delays; f_j, g_i are the input-output functions (the activation functions); $0 \le \tau_{ji}(t) \le \tau_{ji}$ and $0 \le \sigma_{ij}(t) \le \sigma_{ij}$ are transmission delays; $t - \tau_{ji}(t) \in \mathbb{T}$, $t - \sigma_{ij}(t) \in \mathbb{T}$, $t - \delta_i(t) \in [0,\infty)_{\mathbb{T}}$ and $t - \eta_j(t) \in [0,\infty)_{\mathbb{T}}$; c_{ji}, d_{ij} are elements of feedback templates; α_{ji}, p_{ij} denote the elements of fuzzy feedback MIN templates and β_{ji}, q_{ij} are the elements of fuzzy feedback MAX templates; T_{ji}, F_{ij} are fuzzy feed-forward MIN templates and H_{ji}, G_{ij} are fuzzy feed-forward MAX templates; μ_j, ν_i denote the input of the ith neuron and the jth neuron; I_i, J_j denote biases of the ith neuron and the jth neuron, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$, $\bigwedge$ and $\bigvee$ denote the fuzzy AND and fuzzy OR operations, respectively.

The initial condition of (1.1) is of the form

$$\begin{cases} x_i(s) = \varphi_i(s), \quad s \in [-\sigma,0]_{\mathbb{T}}, \sigma = \max\{\max_{(i,j)}\sigma_{ij}, \max_i \delta_i^+\}, i = 1,2,\ldots,n, \\ y_j(s) = \psi_j(s), \quad s \in [-\tau,0]_{\mathbb{T}}, \tau = \max\{\max_{(i,j)}\tau_{ji}, \max_j \eta_j^+\}, j = 1,2,\ldots,m, \end{cases}$$

where $\varphi_i(\cdot)$, $\psi_j(\cdot)$ denote positive real-valued continuous functions on $[-\sigma,0]_{\mathbb{T}}$ and $[-\tau,0]_{\mathbb{T}}$, respectively.

For the sake of convenience, we introduce some notations. For matrix D, D^T denotes the transpose of D, $\rho(D)$ denotes the spectral radius of D. A matrix or a vector $D \geq 0$ means that all entries of D are greater than or equal to zero, $D > 0$ can be defined similarly. For matrices or vectors D and E, $D \geq E$ (respectively $D > E$) means that $D - E \geq 0$ (respectively $D - E > 0$).

Throughout this paper, we assume that the following condition holds:

(H) $f_j, g_i \in C(\mathbb{R}, \mathbb{R})$ and there exist positive constants L_j^f, L_i^g such that

$$\left|f_j(u) - f_j(v)\right| \leq L_j^f |u - v|, \qquad \left|g_i(u) - g_i(v)\right| \leq L_i^g |u - v|$$

for all $u, v \in \mathbb{R}$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

The organization of the rest of this paper is as follows. In Section 2, we introduce some preliminary results which are needed in the later sections. In Section 3, we establish some sufficient conditions for the existence and uniqueness of the equilibrium point of (1.1). In Section 4, we prove the equilibrium point of (1.1) is globally exponentially stable. In Section 5, we give an example to illustrate the feasibility and effectiveness of our results obtained in previous sections.

2 Preliminaries

In this section, we state some preliminary results.

Definition 2.1 [25] Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}_+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \qquad \rho(t) = \sup\{s \in \mathbb{T} : s < t\} \quad \text{and} \quad \mu(t) = \sigma(t) - t.$$

Lemma 2.1 [25] *Assume that $p, q : \mathbb{T} \to \mathbb{R}$ are two regressive functions, then*

(i) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$;

(ii) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;

(iii) $e_p(t,s)e_p(s,r) = e_p(t,r)$;

(iv) $(e_p(t,s))^\Delta = p(t)e_p(t,s)$.

Lemma 2.2 [25] *Let f, g be Δ-differentiable functions on T, then*

(i) $(\nu_1 f + \nu_2 g)^\Delta = \nu_1 f^\Delta + \nu_2 g^\Delta$ *for any constants* ν_1, ν_2;

(ii) $(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t))$.

Lemma 2.3 [35] *Assume that $p(t) \geq 0$ for $t \geq s$, then $e_p(t,s) \geq 1$.*

Definition 2.2 [35] A function $r : \mathbb{T} \to \mathbb{R}$ is called regressive if

$$1 + \mu(t)r(t) \neq 0$$

for all $t \in \mathbb{T}^k$. The set of all regressive and rd-continuous functions $r : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R}$. We define the set $\mathcal{R}^+ = \{r \in \mathcal{R} : 1 + \mu(t)r(t) > 0, \forall t \in \mathbb{T}\}$.

Lemma 2.4 [35] *Suppose that $p \in \mathcal{R}^+$, then*

(i) $e_p(t,s) > 0$ *for all* $t,s \in \mathbb{T}$;
(ii) *if* $p(t) \leq q(t)$ *for all* $t \geq s$, $t,s \in \mathbb{T}$, *then* $e_p(t,s) \leq e_q(t,s)$ *for all* $t \geq s$.

Lemma 2.5 [35] *If* $p \in \mathcal{R}$ *and* $a,b,c \in \mathbb{T}$, *then*

$$\left[e_p(c,\cdot)\right]^{\Delta} = -p\left[e_p(c,\cdot)\right]^{\sigma}$$

and

$$\int_a^b p(t)e_p\big(c,\sigma(t)\big)\Delta t = e_p(c,a) - e_p(c,b).$$

Lemma 2.6 [35] *Let* $a \in \mathbb{T}^k$, $b \in \mathbb{T}$ *and assume that* $f : \mathbb{T} \times \mathbb{T}^k \to \mathbb{R}$ *is continuous at* (t,t), *where* $t \in \mathbb{T}^k$ *with* $t > a$. *Also assume that* $f^{\Delta}(t,\cdot)$ *is rd-continuous on* $[a,\sigma(t)]$. *Suppose that for each* $\varepsilon > 0$, *there exists a neighborhood* U *of* $\tau \in [a,\sigma(t)]$ *such that*

$$\left|f\big(\sigma(t),\tau\big) - f(s,\tau) - f^{\Delta}(t,\tau)\big(\sigma(t) - s\big)\right| \leq \varepsilon\left|\sigma(t) - s\right|, \quad \forall s \in U,$$

where f^{Δ} *denotes the derivative of* f *with respect to the first variable. Then*
(i) $g(t) := \int_a^t f(t,\tau)\Delta\tau$ *implies* $g^{\Delta}(t) := \int_a^t f^{\Delta}(t,\tau)\Delta\tau + f(\sigma(t),t)$;
(ii) $h(t) := \int_t^b f(t,\tau)\Delta\tau$ *implies* $h^{\Delta}(t) := \int_t^b f^{\Delta}(t,\tau)\Delta\tau - f(\sigma(t),t)$.

Definition 2.3 A point $z^* = (x_1^*, x_2^*, \ldots, x_n^*, y_1^*, y_2^*, \ldots, y_m^*)^T \in R^{n+m}$ is said to be an equilibrium point of (1.1) if $z(t) = z^*$ is a solution of (1.1).

Lemma 2.7 [17] *Let* f_j *be defined on* R, $j = 1,2,\ldots,m$. *Then, for any* $a_{ij} \in R$, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$, *we have the following estimations*:

$$\left|\bigwedge_{j=1}^{m} a_{ij}f_j(u_j) - \bigwedge_{j=1}^{m} a_{ij}f_j(v_j)\right| \leq \sum_{j=1}^{m} |a_{ij}|\left|f_j(u_j) - f_j(v_j)\right|$$

and

$$\left|\bigvee_{j=1}^{m} a_{ij}f_j(u_j) - \bigvee_{j=1}^{m} a_{ij}f_j(v_j)\right| \leq \sum_{j=1}^{m} |a_{ij}|\left|f_j(u_j) - f_j(v_j)\right|,$$

where $u_j, v_j \in R$, $j = 1,2,\ldots,m$.

Definition 2.4 [36] A real matrix $A = (a_{ij})_{n\times n}$ is said to be an M-matrix if $a_{ij} \leq 0$, $i,j = 1,2,\ldots,n$, $i \neq j$ and all successive principal minors of A are positive.

Lemma 2.8 [36] *Let* $A = (a_{ij})_{n\times n}$ *be a matrix with nonpositive off-diagonal elements, then the following statements are equivalent*:
(i) A *is an* M*-matrix*;
(ii) *there exists a vector* $\eta > 0$ *such that* $A\eta > 0$;
(iii) *there exists a vector* $\xi > 0$ *such that* $\xi^T A > 0$;
(iv) *there exists a positive definite* $n \times n$ *diagonal matrix* D *such that* $AD + DA^T > 0$.

Lemma 2.9 [36] *Let $A \geq 0$ be an $l \times l$ matrix with $\rho(A) < 1$, then $(E_l - A)^{-1} \geq 0$, where $\rho(A)$ denotes the spectral radius of A and E_l is the identity matrix of size l.*

Definition 2.5 Let $z^* = (x_1^*, x_2^*, \ldots, x_n^*, y_1^*, y_2^*, \ldots, y_m^*)^T$ be an equilibrium point of (1.1). If there exists a positive constant λ with $-\lambda \in \mathcal{R}^+$ such that for $t_0 \in [-\vartheta, 0]_{\mathbb{T}}$, there exists $M > 1$ such that for an arbitrary solution $z(t) = (x_1(t), x_2(t), \ldots, x_n(t), y_1(t), y_2(t), \ldots, y_m(t))^T$ of (1.1) with initial value $\phi(s) = (\varphi_1(s), \varphi_2(s), \ldots, \varphi_n(s), \psi_1(s), \psi_2(s), \ldots, \psi_m(s))^T$ satisfies

$$\left|z(t) - z^*\right|_1 \leq M \left\|\phi - z^*\right\| e_{\ominus\lambda}(t, t_0), \quad t \in [t_0, \infty)_{\mathbb{T}}, t \geq t_0,$$

where $\vartheta = \max\{\sigma, \tau\}$, $|z(t) - z^*|_1 = \max\{\max_{1 \leq i \leq n}\{|x_i(t) - x_i^*|\}, \max_{1 \leq j \leq m}\{|y_j(t) - y_j^*|\}\}$, $\|\phi - z^*\| = \max\{\max_{1 \leq i \leq n} \max_{s \in [-\sigma, 0]_{\mathbb{T}}}\{|\varphi_i(s) - x_i^*|\}, \max_{1 \leq j \leq m} \max_{s \in [-\tau, 0]_{\mathbb{T}}}\{|\psi_j(s) - y_j^*|\}\}$. Then the equilibrium point z^* is said to be globally exponentially stable.

3 Existence and uniqueness of an equilibrium point

In this section, we study the existence and uniqueness of an equilibrium point of (1.1).

Theorem 3.1 *Let* (H) *hold. Suppose further that $\rho(F) < 1$, where $F = A^{-1}BL$, $A = \operatorname{diag}(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m)$, $L = \operatorname{diag}(L_1^f, L_2^f, \ldots, L_m^f, L_1^g, L_2^g, \ldots, L_n^g)$ and*

$$B = \begin{pmatrix} 0_{n\times n} & P^T \\ Q^T & 0_{m\times m} \end{pmatrix},$$

$P = (|c_{ji}| + |\alpha_{ji}| + |\beta_{ji}|)_{m\times n}$, $Q = (|d_{ij}| + |p_{ij}| + |q_{ij}|)_{n\times m}$. Then (1.1) *has one unique equilibrium point.*

Proof Let $z^* = (x_1^*, x_2^*, \ldots, x_n^*, y_1^*, y_2^*, \ldots, y_m^*)^T$ be an equilibrium point of (1.1), then we have

$$\begin{cases} a_i x_i^* = \sum_{j=1}^m c_{ji} f_j(y_j^*) + \bigwedge_{j=1}^m \alpha_{ji} f_j(y_j^*) \\ \qquad + \bigwedge_{j=1}^m T_{ji}\mu_j + \bigvee_{j=1}^m \beta_{ji} f_j(y_j^*) + \bigvee_{j=1}^m H_{ji}\mu_j + I_i, \\ b_j y_j^* = \sum_{i=1}^n d_{ij} g_i(x_i^*) + \bigwedge_{i=1}^n p_{ij} g_i(x_i^*) + \bigwedge_{i=1}^n F_{ij}\nu_i \\ \qquad + \bigvee_{i=1}^n q_{ij} g_i(x_i^*) + \bigvee_{i=1}^n G_{ij}\nu_i + J_j, \end{cases}$$

where $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. Define a mapping $\Phi : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ as follows:

$$\Phi(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m) = (\Phi_1, \Phi_2, \ldots, \Phi_n, \Phi_{n+1}, \ldots, \Phi_{n+m})^T,$$

where

$$\begin{cases} \Phi_i = a_i^{-1}[\sum_{j=1}^m c_{ji} f_j(y_j) + \bigwedge_{j=1}^m \alpha_{ji} f_j(y_j) \\ \qquad + \bigwedge_{j=1}^m T_{ji}\mu_j + \bigvee_{j=1}^m \beta_{ji} f_j(y_j) + \bigvee_{j=1}^m H_{ji}\mu_j + I_i], \\ \Phi_{n+j} = b_j^{-1}[\sum_{i=1}^n d_{ij} g_i(x_i) + \bigwedge_{i=1}^n p_{ij} g_i(x_i) + \bigwedge_{i=1}^n F_{ij}\nu_i \\ \qquad + \bigvee_{i=1}^n q_{ij} g_i(x_i) + \bigvee_{i=1}^n G_{ij}\nu_i + J_j] \end{cases}$$

for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. Obviously, we need to show that $\Phi : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ is a contraction mapping on $\mathbb{R}^{n+m}$. In fact, for any $\Theta = (h_1, h_2, \ldots, h_n, v_1, v_2, \ldots, v_m)$ and $\bar{\Theta} =$

$(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_n, \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_m) \in \mathbb{R}^{n+m}$, we have

$$\begin{aligned} &\left|\Phi_i(h_i) - \Phi_i(\bar{h}_i)\right| \\ &\quad = \left|a_i^{-1}\sum_{j=1}^{m} c_{ji}\big(f_j(v_j) - f_j(\bar{v}_j)\big)\right| + \left|a_i^{-1}\bigwedge_{j=1}^{m} \alpha_{ji}\big(f_j(v_j) - f_j(\bar{v}_j)\big)\right| \\ &\qquad + \left|a_i^{-1}\bigvee_{j=1}^{m} \beta_{ji}\big(f_j(v_j) - f_j(\bar{v}_j)\big)\right| \\ &\quad \le a_i^{-1}\sum_{j=1}^{m}\big(|c_{ji}| + |\alpha_{ji}| + |\beta_{ji}|\big)L_j^f |v_j - \bar{v}_j|, \quad i = 1, 2, \ldots, n \end{aligned}$$

and

$$\begin{aligned} &\left|\Phi_{n+j}(v_j) - \Phi_{n+j}(\bar{v}_j)\right| \\ &\quad = \left|b_j^{-1}\sum_{i=1}^{n} d_{ij}\big(g_i(h_i) - g_i(\bar{h}_i)\big)\right| + \left|b_j^{-1}\bigwedge_{i=1}^{n} p_{ij}\big(g_i(h_i) - g_i(\bar{h}_i)\big)\right| \\ &\qquad + \left|b_j^{-1}\bigvee_{i=1}^{n} q_{ij}\big(g_i(h_i) - g_i(\bar{h}_i)\big)\right| \\ &\quad \le b_j^{-1}\sum_{i=1}^{n}\big(|d_{ij}| + |p_{ij}| + |q_{ij}|\big)L_i^g |h_i - \bar{h}_i|, \quad j = 1, 2, \ldots, m. \end{aligned}$$

It follows that

$$\begin{aligned} &\left|\Phi(h_1, h_2, \ldots, h_n, v_1, v_2, \ldots, v_m) - \Phi(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_n, \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_m)\right| \\ &\quad \le \begin{pmatrix} a_1^{-1}\sum_{j=1}^{m}(|c_{j1}| + |\alpha_{j1}| + |\beta_{j1}|)L_j^f |v_j - \bar{v}_j| \\ \vdots \\ a_n^{-1}\sum_{j=1}^{m}(|c_{jn}| + |\alpha_{jn}| + |\beta_{jn}|)L_j^f |v_j - \bar{v}_j| \\ b_1^{-1}\sum_{i=1}^{n}(|d_{i1}| + |p_{i1}| + |q_{i1}|)L_i^g |h_i - \bar{h}_i| \\ \vdots \\ b_m^{-1}\sum_{i=1}^{n}(|d_{im}| + |p_{im}| + |q_{im}|)L_i^g |h_i - \bar{h}_i| \end{pmatrix} = F \begin{pmatrix} |h_1 - \bar{h}_1| \\ \vdots \\ |h_n - \bar{h}_n| \\ |v_1 - \bar{v}_1| \\ \vdots \\ |v_m - \bar{v}_m| \end{pmatrix}. \end{aligned} \tag{3.1}$$

Let N be a positive integer. In view of (3.1), we have

$$\begin{aligned} &\left|\Phi^N(h_1, \ldots, h_n, v_1, \ldots, v_m) - \Phi^N(\bar{h}_1, \ldots, \bar{h}_n, \bar{v}_1, \ldots, \bar{v}_m)\right| \\ &\quad \le F^N\big(|h_1 - \bar{h}_1|, \ldots, |h_n - \bar{h}_n|, |v_1 - \bar{v}_1|, \ldots, |v_m - \bar{v}_m|\big)^T. \end{aligned}$$

Since $\rho(F) < 1$, we obtain $\lim_{N\to+\infty} F^N = 0$, which implies that there exist a positive integer M and a positive constant $r < 1$ such that

$$F^M = \big(A^{-1}BL\big)^M = (l_{ij})^{(n+m)\times(n+m)}, \quad \sum_{j=1}^{n+m} l_{ij} \le r, i = 1, 2, \ldots, n + m.$$

Hence, we have

$$\begin{aligned}
&\left|\Phi^M(h_1,\dots,h_n,v_1,\dots,v_m)-\Phi^M(\bar{h}_1,\dots,\bar{h}_n,\bar{v}_1,\dots,\bar{v}_m)\right| \\
&\quad\le F^M\begin{pmatrix}|h_1-\bar{h}_1|\\ \vdots\\ |h_n-\bar{h}_n|\\ |v_1-\bar{v}_1|\\ \vdots\\ |v_m-\bar{v}_m|\end{pmatrix}\le F^M\begin{pmatrix}\|h_1-\bar{h}_1\|\\ \vdots\\ \|h_n-\bar{h}_n\|\\ \|v_1-\bar{v}_1\|\\ \vdots\\ \|v_m-\bar{v}_m\|\end{pmatrix}=\|\Theta-\bar{\Theta}\|\begin{pmatrix}\sum_{j=1}^{n+m}l_{1j}\\ \vdots\\ \sum_{j=1}^{n+m}l_{nj}\\ \sum_{j=1}^{n+m}l_{(n+1)j}\\ \vdots\\ \sum_{j=1}^{n+m}l_{(n+m)j}\end{pmatrix},
\end{aligned}$$

which implies that $\|\Phi^M(\Theta)-\Phi^M(\bar{\Theta})\|\le r\|\Theta-\bar{\Theta}\|$. Since $r<1$, it is obvious that the mapping $\Phi^M:\mathbb{R}^{n+m}\to\mathbb{R}^{n+m}$ is a contraction mapping. By the fixed point theorem of a Banach space, Φ possesses a unique fixed point in $\mathbb{R}^{n+m}$, that is, there exists a unique equilibrium point of (1.1). The proof of Theorem 3.1 is completed. □

4 Global exponential stability of an equilibrium point

In this section, we study the global exponential stability of the equilibrium point of (1.1).

Theorem 4.1 *Let* (H) *and* $\rho(F)<1$ *hold. Suppose further that*

(H_5) $\max\{\epsilon_1,\epsilon_2\}<1$, *where*

$$\epsilon_1=\max_{1\le i\le n}\left\{a_i\delta_i^+ +\left(\delta_i^+ +a_i^{-1}\right)\sum_{j=1}^{m}\left(|c_{ji}|+|\alpha_{ji}|+|\beta_{ji}|\right)L_j^f\right\}$$

and

$$\epsilon_2=\max_{1\le j\le m}\left\{b_j\eta_j^+ +\left(\eta_j^+ +b_j^{-1}\right)\sum_{i=1}^{n}\left(|d_{ij}|+|p_{ij}|+|q_{ij}|\right)L_i^g\right\}.$$

Then the equilibrium point of (1.1) *is globally exponentially stable.*

Proof By Theorem 3.1, (1.1) has a unique equilibrium point $z^*=(x_1^*,x_2^*,\dots,x_n^*,y_1^*,y_2^*,\dots,y_m^*)^T$. Suppose that $z(t)=(x_1(t),x_2(t),\dots,x_n(t),y_1(t),y_2(t),\dots,y_m(t))^T$ is an arbitrary solution of (1.1) with the initial condition $\phi(s)=(\varphi_1(s),\varphi_2(s),\dots,\varphi_n(s),\psi_1(s),\psi_2(s),\dots,\psi_m(s))^T$. Let $w(t)=(u_1(t),u_2(t),\dots,u_n(t),v_1(t),v_2(t),\dots,v_m(t))^T=z(t)-z^*$, where $u_i(t)=x_i(t)-x_i^*$, $v_j(t)=y_j(t)-y_j^*$, $i=1,2,\dots,n$, $j=1,2,\dots,m$. Then (1.1) can be rewritten as

$$\begin{cases}
u_i^\Delta(t)=-a_iu_i(t-\delta_i(t))+\sum_{j=1}^m c_{ji}\widetilde{f}_j(v_j(t-\tau_{ji}(t)))+\bigwedge_{j=1}^m\alpha_{ji}f_j(v_j(t-\tau_{ji}(t))+y_j^*)\\
\qquad -\bigwedge_{j=1}^m\alpha_{ji}f_j(y_j^*)+\bigvee_{j=1}^m\beta_{ji}f_j(v_j(t-\tau_{ji}(t))+y_j^*)\\
\qquad -\bigvee_{j=1}^m\beta_{ji}f_j(y_j^*),\quad i=1,2,\dots,n,\\
v_j^\Delta(t)=-b_jv_j(t-\eta_j(t))+\sum_{i=1}^n d_{ij}\widetilde{g}_i(u_i(t-\sigma_{ij}(t)))+\bigwedge_{i=1}^n p_{ij}g_i(u_i(t-\sigma_{ij}(t))+x_i^*)\\
\qquad -\bigwedge_{i=1}^n p_{ij}g_i(x_i^*)+\bigvee_{i=1}^n q_{ij}g_i(u_i(t-\sigma_{ij}(t))+x_i^*)\\
\qquad -\bigvee_{i=1}^n q_{ij}g_i(x_i^*),\quad j=1,2,\dots,m,
\end{cases}\tag{4.1}$$

where $t\in\mathbb{T}$ and

$$\widetilde{f}_j\big(v_j\big(t-\tau_{ji}(t)\big)\big)=f_j\big(y_j\big(t-\tau_{ji}(t)\big)\big)-f_j\big(y_j^*\big),\qquad \widetilde{g}_i\big(u_i\big(t-\sigma_{ij}(t)\big)\big)=g_i\big(x_i\big(t-\sigma_{ij}(t)\big)\big)-g_i\big(x_i^*\big).$$

The initial condition of (4.1) is the following:

$$\begin{cases} u_i(s) = \varphi_i(s) - x_i^*, & s \in [-\sigma, 0]_{\mathbb{T}}, i = 1, 2, \ldots, n, \\ v_j(s) = \psi_j(s) - y_j^*, & s \in [-\tau, 0]_{\mathbb{T}}, j = 1, 2, \ldots, m. \end{cases}$$

We rewrite (4.1) as follows:

$$\begin{aligned} u_i^{\Delta}(s) + a_i u_i(s) = {} & a_i \int_{s-\delta_i(s)}^{s} u_i^{\Delta}(\theta)\Delta\theta + \sum_{j=1}^{m} c_{ji}\widetilde{f}_j\bigl(v_j\bigl(s - \tau_{ji}(s)\bigr)\bigr) + \bigwedge_{j=1}^{m} \alpha_{ji} f_j\bigl(v_j\bigl(s - \tau_{ji}(s)\bigr) + y_j^*\bigr) \\ & - \bigwedge_{j=1}^{m} \alpha_{ji} f_j\bigl(y_j^*\bigr) + \bigvee_{j=1}^{m} \beta_{ji} f_j\bigl(v_j\bigl(s - \tau_{ji}(s)\bigr) + y_j^*\bigr) \\ & - \bigvee_{j=1}^{m} \beta_{ji} f_j\bigl(y_j^*\bigr), \quad i = 1, 2, \ldots, n \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} v_j^{\Delta}(s) + b_j v_j(s) = {} & b_j \int_{s-\eta_j(s)}^{s} v_j^{\Delta}(\theta)\Delta\theta + \sum_{i=1}^{n} d_{ij}\widetilde{g}_i\bigl(u_i\bigl(s - \sigma_{ij}(s)\bigr)\bigr) \\ & + \bigwedge_{i=1}^{n} p_{ij} g_i\bigl(u_i\bigl(s - \sigma_{ij}(s)\bigr) + x_i^*\bigr) \\ & - \bigwedge_{i=1}^{n} p_{ij} g_i\bigl(x_i^*\bigr) + \bigvee_{i=1}^{n} q_{ij} g_i\bigl(u_i\bigl(s - \sigma_{ij}(s)\bigr) + x_i^*\bigr) \\ & - \bigvee_{i=1}^{n} q_{ij} g_i\bigl(x_i^*\bigr), \quad j = 1, 2, \ldots, m. \end{aligned} \tag{4.3}$$

Multiplying both sides of (4.2) by $e_{-a_i}(t, \sigma(s))$ and integrating on $[t_0, t]_{\mathbb{T}}$, where $t_0 \in [-\vartheta, 0]_{\mathbb{T}}$, we get

$$\begin{aligned} u_i(t) = {} & u_i(t_0)e_{-a_i}(t, t_0) \\ & + \int_{t_0}^{t} e_{-a_i}\bigl(t, \sigma(s)\bigr) \Biggl\{ a_i \int_{s-\delta_i(s)}^{s} u_i^{\Delta}(\theta)\Delta\theta + \sum_{j=1}^{m} c_{ji}\widetilde{f}_j\bigl(v_j\bigl(s - \tau_{ji}(s)\bigr)\bigr) \\ & + \bigwedge_{j=1}^{m} \alpha_{ji} f_j\bigl(v_j\bigl(s - \tau_{ji}(s)\bigr) + y_j^*\bigr) - \bigwedge_{j=1}^{m} \alpha_{ji} f_j\bigl(y_j^*\bigr) + \bigvee_{j=1}^{m} \beta_{ji} f_j\bigl(v_j\bigl(s - \tau_{ji}(s)\bigr) + y_j^*\bigr) \\ & - \bigvee_{j=1}^{m} \beta_{ji} f_j\bigl(y_j^*\bigr) \Biggr\} \Delta s, \quad i = 1, 2, \ldots, n. \end{aligned} \tag{4.4}$$

Similarly, multiplying both sides of (4.3) by $e_{-b_j}(t, \sigma(s))$ and integrating on $[t_0, t]_{\mathbb{T}}$, we get

$$\begin{aligned} v_j(t) = {} & v_j(t_0)e_{-b_j}(t, t_0) \\ & + \int_{t_0}^{t} e_{-b_j}\bigl(t, \sigma(s)\bigr) \Biggl\{ b_j \int_{s-\eta_j(s)}^{s} v_j^{\Delta}(\theta)\Delta\theta + \sum_{i=1}^{n} d_{ij}\widetilde{g}_i\bigl(u_i\bigl(s - \sigma_{ij}(s)\bigr)\bigr) \end{aligned}$$

$$+\bigwedge_{i=1}^{n} p_{ij}g_i\big(u_i\big(s-\sigma_{ij}(s)\big)+x_i^*\big)-\bigwedge_{i=1}^{n} p_{ij}g_i\big(x_i^*\big)+\bigvee_{i=1}^{n} q_{ij}g_i\big(u_i\big(s-\sigma_{ij}(s)\big)+x_i^*\big)$$
$$\left.-\bigvee_{i=1}^{n} q_{ij}g_i\big(x_i^*\big)\right\}\Delta s, \quad j=1,2,\ldots,m. \tag{4.5}$$

For a positive constant $\alpha<\min\{\min_{1\le i\le n} a_i^-,\min_{1\le j\le m} b_j^-\}$ with $-\alpha\in\mathcal{R}^+$, we have $e_{\ominus\alpha}(t,t_0)>1$ for $t<t_0$. Take $M>\max\{\frac{1}{\epsilon_1},\frac{1}{\epsilon_2}\}$. In view of ($H_5$), it is obvious that $M>1$. Hence, we have

$$\big|w(t)\big|_1\le Me_{\ominus\alpha}(t,t_0)\big\|\phi-z^*\big\|,\quad \forall t\in[-\vartheta,t_0]_{\mathbb{T}}.$$

We claim that

$$\big|w(t)\big|_1\le Me_{\ominus\alpha}(t,t_0)\big\|\phi-z^*\big\|,\quad \forall t\in(t_0,+\infty)_{\mathbb{T}}. \tag{4.6}$$

To prove this claim, we show that for any $p>1$, the following inequality holds:

$$\big|w(t)\big|_1< pMe_{\ominus\alpha}(t,t_0)\big\|\phi-z^*\big\|,\quad \forall t\in(t_0,+\infty)_{\mathbb{T}}, \tag{4.7}$$

which means that for $i=1,2,\ldots,n$, we have

$$\big|u_i(t)\big|< pMe_{\ominus\alpha}(t,t_0)\big\|\phi-z^*\big\|,\quad \forall t\in(t_0,+\infty)_{\mathbb{T}} \tag{4.8}$$

and for $j=1,2,\ldots,m$, we have

$$\big|v_j(t)\big|< pMe_{\ominus\alpha}(t,t_0)\big\|\phi-z^*\big\|,\quad \forall t\in(t_0,+\infty)_{\mathbb{T}}. \tag{4.9}$$

By way of contradiction, assume that (4.7) does not hold. Firstly, we consider the following two cases.

Case One: (4.8) is not true and (4.9) is true. Then there exist $t_1\in(t_0,+\infty)_{\mathbb{T}}$ and $i_0\in\{1,2,\ldots,n\}$ such that

$$\big|u_{i_0}(t_1)\big|\ge pMe_{\ominus\alpha}(t_1,t_0)\big\|\phi-z^*\big\|,\qquad \big|u_{i_0}(t)\big|<pMe_{\ominus\alpha}(t,t_0)\big\|\phi-z^*\big\|,\quad t\in(t_0,t_1)_{\mathbb{T}},$$
$$\big|u_l(t)\big|<pMe_{\ominus\alpha}(t,t_0)\big\|\phi-z^*\big\|\quad \text{for } l\ne i_0, t\in(t_0,t_1]_{\mathbb{T}}, l=1,2,\ldots,n.$$

Hence, there must be a constant $c>1$ such that

$$\big|u_{i_0}(t_1)\big|=cpMe_{\ominus\alpha}(t_1,t_0)\big\|\phi-z^*\big\|,\qquad \big|u_{i_0}(t)\big|<cpMe_{\ominus\alpha}(t,t_0)\big\|\phi-z^*\big\|,\quad t\in(t_0,t_1)_{\mathbb{T}},$$
$$\big|u_l(t)\big|<cpMe_{\ominus\alpha}(t,t_0)\big\|\phi-z^*\big\|\quad \text{for } l\ne i_0, t\in(t_0,t_1]_{\mathbb{T}}, l=1,2,\ldots,n.$$

Note that in view of (4.4), we have

$$\big|u_{i_0}(t_1)\big|=\bigg|u_{i_0}(t_0)e_{-a_{i_0}}(t_1,t_0)+\int_{t_0}^{t_1}e_{-a_{i_0}}\big(t_1,\sigma(s)\big)$$
$$\times\left\{a_{i_0}\int_{s-\delta_{i_0}(s)}^{s}u_{i_0}^{\Delta}(\theta)\Delta\theta+\sum_{j=1}^{m}c_{ji_0}\widetilde{f}_j\big(v_j\big(s-\tau_{ji_0}(s)\big)\big)\right.$$

$$
\begin{aligned}
&+ \bigwedge_{j=1}^{m} \alpha_{ji_0} f_j\big(\nu_j\big(s-\tau_{ji_0}(s)\big)+y_j^*\big) - \bigwedge_{j=1}^{m} \alpha_{ji_0} f_j\big(y_j^*\big) \\
&+ \bigvee_{j=1}^{m} \beta_{ji_0} f_j\big(\nu_j\big(s-\tau_{ji_0}(s)\big)+y_j^*\big) - \bigvee_{j=1}^{m} \beta_{ji_0} f_j\big(y_j^*\big) \Bigg\} \Delta s \Bigg| \\
={}& \Bigg| u_{i_0}(t_0) e_{-a_{i_0}}(t_1,t_0) + \int_{t_0}^{t_1} e_{-a_{i_0}}\big(t_1,\sigma(s)\big) \\
&\times \Bigg\{ a_{i_0} \int_{s-\delta_{i_0}(s)}^{s} \Bigg[-a_{i_0} u_{i_0}\big(\theta-\delta_{i_0}(\theta)\big) + \sum_{j=1}^{m} c_{ji_0} \widetilde{f}_j\big(\nu_j\big(\theta-\tau_{ji_0}(\theta)\big)\big) \\
&+ \bigwedge_{j=1}^{m} \alpha_{ji_0} f_j\big(\nu_j\big(\theta-\tau_{ji_0}(\theta)\big)+y_j^*\big) - \bigwedge_{j=1}^{m} \alpha_{ji_0} f_j\big(y_j^*\big) \\
&+ \bigvee_{j=1}^{m} \beta_{ji_0} f_j\big(\nu_j\big(\theta-\tau_{ji_0}(\theta)\big)+y_j^*\big) \\
&- \bigvee_{j=1}^{m} \beta_{ji_0} f_j\big(y_j^*\big) \Bigg] \Delta\theta + \sum_{j=1}^{m} c_{ji_0} \widetilde{f}_j\big(\nu_j\big(s-\tau_{ji_0}(s)\big)\big) \\
&+ \bigwedge_{j=1}^{m} \alpha_{ji_0} f_j\big(\nu_j\big(s-\tau_{ji_0}(s)\big)+y_j^*\big) - \bigwedge_{j=1}^{m} \alpha_{ji_0} f_j\big(y_j^*\big) \\
&+ \bigvee_{j=1}^{m} \beta_{ji_0} f_j\big(\nu_j\big(s-\tau_{ji_0}(s)\big)+y_j^*\big) - \bigvee_{j=1}^{m} \beta_{ji_0} f_j\big(y_j^*\big) \Bigg\} \Delta s \Bigg| \\
\le{}& e_{-a_{i_0}}(t_1,t_0)\|\phi - z^*\| + \int_{t_0}^{t_1} e_{-a_{i_0}}\big(t_1,\sigma(s)\big) \\
&\times \Bigg\{ a_{i_0} \int_{s-\delta_{i_0}(s)}^{s} \Bigg[a_{i_0} \big|u_{i_0}\big(\theta-\delta_{i_0}(\theta)\big)\big| + \Bigg| \sum_{j=1}^{m} c_{ji_0} \widetilde{f}_j\big(\nu_j\big(\theta-\tau_{ji_0}(\theta)\big)\big) \\
&+ \bigwedge_{j=1}^{m} \alpha_{ji_0} f_j\big(\nu_j\big(\theta-\tau_{ji_0}(\theta)\big)+y_j^*\big) - \bigwedge_{j=1}^{m} \alpha_{ji_0} f_j\big(y_j^*\big) + \bigvee_{j=1}^{m} \beta_{ji_0} f_j\big(\nu_j\big(\theta-\tau_{ji_0}(\theta)\big)+y_j^*\big) \\
&- \bigvee_{j=1}^{m} \beta_{ji_0} f_j\big(y_j^*\big) \Bigg| \Bigg] \Delta\theta + \Bigg| \sum_{j=1}^{m} c_{ji_0} \widetilde{f}_j\big(\nu_j\big(s-\tau_{ji_0}(s)\big)\big) \\
&+ \bigwedge_{j=1}^{m} \alpha_{ji_0} f_j\big(\nu_j\big(s-\tau_{ji_0}(s)\big)+y_j^*\big) - \bigwedge_{j=1}^{m} \alpha_{ji_0} f_j\big(y_j^*\big) \\
&+ \bigvee_{j=1}^{m} \beta_{ji_0} f_j\big(\nu_j\big(s-\tau_{ji_0}(s)\big)+y_j^*\big) - \bigvee_{j=1}^{m} \beta_{ji_0} f_j\big(y_j^*\big) \Bigg| \Bigg\} \Delta s \\
\le{}& \|\phi - z^*\| + \int_{t_0}^{t_1} e_{-a_{i_0}}\big(t_1,\sigma(s)\big) \Bigg\{ a_{i_0} \delta_{i_0}^{+} \Bigg[a_{i_0} cpMe_{\ominus\alpha}(t_1,t_0)\|\phi - z^*\| \\
&+ \sum_{j=1}^{m} \big(|c_{ji_0}| + |\alpha_{ji_0}| + |\beta_{ji_0}|\big) L_j^f cpMe_{\ominus\alpha}(t_1,t_0)\|\phi - z^*\| \Bigg] \\
&+ \sum_{j=1}^{m} \big(|c_{ji_0}| + |\alpha_{ji_0}| + |\beta_{ji_0}|\big) L_j^f cpMe_{\ominus\alpha}(t_1,t_0)\|\phi - z^*\| \Bigg\} \Delta s
\end{aligned}
$$

$$
\begin{aligned}
&\leq \|\phi - z^*\| e_{\ominus\alpha}(t_1, t_0) + \int_{t_0}^{t_1} e_{-a_{i_0}}\bigl(t_1, \sigma(s)\bigr) \\
&\quad \times \Biggl\{ a_{i_0}\delta_{i_0}^{+}\Biggl[a_{i_0} cpMe_{\ominus\alpha}(t_1, t_0)\|\phi - z^*\| \\
&\quad + \sum_{j=1}^{m}\bigl(|c_{ji_0}| + |\alpha_{ji_0}| + |\beta_{ji_0}|\bigr)L_j^f cpMe_{\ominus\alpha}(t_1, t_0)\|\phi - z^*\| \Biggr] \\
&\quad + \sum_{j=1}^{m}\bigl(|c_{ji_0}| + |\alpha_{ji_0}| + |\beta_{ji_0}|\bigr)L_j^f cpMe_{\ominus\alpha}(t_1, t_0)\|\phi - z^*\| \Biggr\} \Delta s \\
&= \|\phi - z^*\| e_{\ominus\alpha}(t_1, t_0) + cpMe_{\ominus\alpha}(t_1, t_0)\|\phi - z^*\| \\
&\quad \times \int_{t_0}^{t_1} e_{-a_{i_0}}\bigl(t_1, \sigma(s)\bigr) \Biggl\{ a_{i_0}\delta_{i_0}^{+}\Biggl[a_{i_0} + \sum_{j=1}^{m}\bigl(|c_{ji_0}| + |\alpha_{ji_0}| + |\beta_{ji_0}|\bigr)L_j^f \Biggr] \\
&\quad + \sum_{j=1}^{m}\bigl(|c_{ji_0}| + |\alpha_{ji_0}| + |\beta_{ji_0}|\bigr)L_j^f \Biggr\} \Delta s \\
&= \|\phi - z^*\| e_{\ominus\alpha}(t_1, t_0) - \frac{1}{a_{i_0}} cpMe_{\ominus\alpha}(t_1, t_0)\|\phi - z^*\| \\
&\quad \times \Biggl\{ a_{i_0}\delta_{i_0}^{+}\Biggl[a_{i_0} + \sum_{j=1}^{m}\bigl(|c_{ji_0}| + |\alpha_{ji_0}| + |\beta_{ji_0}|\bigr)L_j^f \Biggr] \\
&\quad + \sum_{j=1}^{m}\bigl(|c_{ji_0}| + |\alpha_{ji_0}| + |\beta_{ji_0}|\bigr)L_j^f \Biggr\} \int_{t_0}^{t_1} (-a_{i_0}) e_{-a_{i_0}}\bigl(t_1, \sigma(s)\bigr) \Delta s \\
&= \|\phi - z^*\| e_{\ominus\alpha}(t_1, t_0) - \frac{1}{a_{i_0}}\bigl(e_{a_{i_0}}(t_1, t_0) - 1\bigr) cpMe_{\ominus\alpha}(t_1, t_0)\|\phi - z^*\| \\
&\quad \times \Biggl\{ a_{i_0}\delta_{i_0}^{+}\Biggl[a_{i_0} + \sum_{j=1}^{m}\bigl(|c_{ji_0}| + |\alpha_{ji_0}| + |\beta_{ji_0}|\bigr)L_j^f \Biggr] \\
&\quad + \sum_{j=1}^{m}\bigl(|c_{ji_0}| + |\alpha_{ji_0}| + |\beta_{ji_0}|\bigr)L_j^f \Biggr\} \\
&< cpM\|\phi - z^*\| e_{\ominus\alpha}(t_1, t_0) \\
&\quad \times \Biggl\{ \frac{1}{cpM} + \frac{1}{a_{i_0}}\Biggl[a_{i_0}\delta_{i_0}^{+}\Biggl(a_{i_0} + \sum_{j=1}^{m}\bigl(|c_{ji_0}| + |\alpha_{ji_0}| + |\beta_{ji_0}|\bigr)L_j^f \Biggr) \\
&\quad + \sum_{j=1}^{m}\bigl(|c_{ji_0}| + |\alpha_{ji_0}| + |\beta_{ji_0}|\bigr)L_j^f \Biggr]\Biggr\} \\
&< cpM\|\phi - z^*\| e_{\ominus\alpha}(t_1, t_0) \\
&\quad \times \Biggl\{ \frac{1}{M} + \frac{1}{a_{i_0}}\Biggl[a_{i_0}\delta_{i_0}^{+}\Biggl(a_{i_0} + \sum_{j=1}^{m}\bigl(|c_{ji_0}| + |\alpha_{ji_0}| + |\beta_{ji_0}|\bigr)L_j^f \Biggr) \\
&\quad + \sum_{j=1}^{m}\bigl(|c_{ji_0}| + |\alpha_{ji_0}| + |\beta_{ji_0}|\bigr)L_j^f \Biggr]\Biggr\}
\end{aligned}
$$

$$\begin{aligned}
&= cpM\|\phi - z^*\| e_{\ominus\alpha}(t_1,t_0) \\
&\quad\times \left(\frac{1}{M} + a_{i_0}\delta_{i_0}^+ + \left(\delta_{i_0}^+ + a_{i_0}^{-1}\right)\sum_{j=1}^{m}\left(|c_{ji_0}| + |\alpha_{ji_0}| + |\beta_{ji_0}|\right)L_j^f\right) \\
&< cpMe_{\ominus\alpha}(t_1,t_0)\|\phi - z^*\|,
\end{aligned}$$

which is a contradiction.

Case Two: (4.8) is true and (4.9) is not true. Then there exist $t_2 \in (t_0,+\infty)_{\mathbb{T}}$ and $j_0 \in \{1,2,\dots,n\}$ such that

$$\begin{aligned}
&|v_{j_0}(t_2)| \ge pMe_{\ominus\alpha}(t_2,t_0)\|\phi - z^*\|, \qquad |v_{j_0}(t)| < pMe_{\ominus\alpha}(t,t_0)\|\phi - z^*\|, \quad t \in (t_0,t_2)_{\mathbb{T}}, \\
&|v_k(t)| < pMe_{\ominus\alpha}(t,t_0)\|\phi - z^*\| \quad \text{for } k \ne j_0, t \in (t_0,t_2]_{\mathbb{T}}, k = 1,2,\dots,m.
\end{aligned}$$

Hence, there must be a constant $c_1 > 1$ such that

$$\begin{aligned}
&|v_{j_0}(t_2)| = c_1pMe_{\ominus\alpha}(t_2,t_0)\|\phi - z^*\|, \\
&|v_{j_0}(t)| < c_1pMe_{\ominus\alpha}(t,t_0)\|\phi - z^*\|, \quad t \in (t_0,t_2)_{\mathbb{T}}, \\
&|v_k(t)| < c_1pMe_{\ominus\alpha}(t,t_0)\|\phi - z^*\| \quad \text{for } k \ne j_0, t \in (t_0,t_2]_{\mathbb{T}}, k = 1,2,\dots,m.
\end{aligned}$$

Note that in view of (4.5), we have

$$\begin{aligned}
|v_{j_0}(t_2)| &= \Bigg| v_{j_0}(t_0)e_{-b_{j_0}}(t_2,t_0) + \int_{t_0}^{t_2} e_{-b_{j_0}}\big(t_2,\sigma(s)\big) \\
&\quad\times \Bigg\{ b_{j_0}\int_{s-\eta_{j_0}(s)}^{s}\Bigg[-b_{j_0}v_{j_0}\big(\theta - \eta_{j_0}(\theta)\big) + \sum_{i=1}^{n} d_{ij_0}\widetilde{g}_i\big(u_i\big(\theta - \sigma_{ij_0}(\theta)\big)\big) \\
&\quad + \bigwedge_{i=1}^{n} p_{ij_0}g_i\big(u_i\big(\theta - \sigma_{ij_0}(\theta)\big) + x_i^*\big) - \bigwedge_{i=1}^{n} p_{ij_0}g_i\big(x_i^*\big) \\
&\quad + \bigvee_{i=1}^{n} q_{ij_0}g_i\big(u_i\big(\theta - \sigma_{ij_0}(\theta)\big) + x_i^*\big) \\
&\quad - \bigvee_{i=1}^{n} q_{ij_0}g_i\big(x_i^*\big)\Bigg]\Delta\theta + \sum_{i=1}^{n} d_{ij_0}\widetilde{g}_i\big(u_i\big(s - \sigma_{ij_0}(s)\big)\big) \\
&\quad + \bigwedge_{i=1}^{n} p_{ij_0}g_i\big(u_i\big(s - \sigma_{ij_0}(s)\big) + x_i^*\big) - \bigwedge_{i=1}^{n} p_{ij_0}g_i\big(x_i^*\big) \\
&\quad + \bigvee_{i=1}^{n} q_{ij_0}g_i\big(u_i\big(s - \sigma_{ij_0}(s)\big) + x_i^*\big) - \bigvee_{i=1}^{n} q_{ij_0}g_i\big(x_i^*\big)\Bigg\}\Delta s\Bigg| \\
&\le e_{-b_{j_0}}(t_2,t_0)\|\phi - z^*\| + \Bigg|\int_{t_0}^{t_2} e_{-b_{j_0}}\big(t_2,\sigma(s)\big) \\
&\quad\times \Bigg\{ b_{j_0}\int_{s-\eta_{j_0}(s)}^{s}\Bigg[-b_{j_0}v_{j_0}\big(\theta - \eta_{j_0}(\theta)\big) + \sum_{i=1}^{n} d_{ij_0}\widetilde{g}_i\big(u_i\big(\theta - \sigma_{ij_0}(\theta)\big)\big) \\
&\quad + \bigwedge_{i=1}^{n} p_{ij_0}g_i\big(u_i\big(\theta - \sigma_{ij_0}(\theta)\big) + x_i^*\big) - \bigwedge_{i=1}^{n} p_{ij_0}g_i\big(x_i^*\big)
\end{aligned}$$

$$
\begin{aligned}
&\quad + \bigvee_{i=1}^{n} q_{ij_0} g_i\big(u_i\big(\theta - \sigma_{ij_0}(\theta)\big) + x_i^*\big) \\
&\quad - \bigvee_{i=1}^{n} q_{ij_0} g_i\big(x_i^*\big)\Bigg]\Delta\theta + \sum_{i=1}^{n} d_{ij_0}\widetilde{g}_i\big(u_i\big(s - \sigma_{ij_0}(s)\big)\big) \\
&\quad + \bigwedge_{i=1}^{n} p_{ij_0} g_i\big(u_i\big(s - \sigma_{ij_0}(s)\big) + x_i^*\big) - \bigwedge_{i=1}^{n} p_{ij_0} g_i\big(x_i^*\big) \\
&\quad + \bigvee_{i=1}^{n} q_{ij_0} g_i\big(u_i\big(s - \sigma_{ij_0}(s)\big) + x_i^*\big) - \bigvee_{i=1}^{n} q_{ij_0} g_i\big(x_i^*\big)\Bigg\}\Delta s\Bigg| \\
&\le \big\|\phi - z^*\big\| + \int_{t_0}^{t_2} e_{-b_{j_0}}\big(t_2, \sigma(s)\big)\Bigg\{b_{j_0}\eta_{j_0}^+\Bigg[b_{j_0} c_1 pMe_{\ominus\alpha}(t_2, t_0)\big\|\phi - z^*\big\| \\
&\quad + \sum_{i=1}^{n}\big(|d_{ij_0}| + |p_{ij_0}| + |q_{ij_0}|\big)L_i^g c_1 pMe_{\ominus\alpha}(t_2, t_0)\big\|\phi - z^*\big\|\Bigg] \\
&\quad + \sum_{i=1}^{n}\big(|d_{ij_0}| + |p_{ij_0}| + |q_{ij_0}|\big)L_i^g c_1 pMe_{\ominus\alpha}(t_2, t_0)\big\|\phi - z^*\big\|\Bigg\}\Delta s \\
&\le \big\|\phi - z^*\big\| e_{\ominus\alpha}(t_2, t_0) + \int_{t_0}^{t_2} e_{-b_{j_0}}\big(t_2, \sigma(s)\big) \\
&\quad \times \Bigg\{b_{j_0}\eta_{j_0}^+\Bigg[b_{j_0} c_1 pMe_{\ominus\alpha}(t_2, t_0)\big\|\phi - z^*\big\| \\
&\quad + \sum_{i=1}^{n}\big(|d_{ij_0}| + |p_{ij_0}| + |q_{ij_0}|\big)L_i^g c_1 pMe_{\ominus\alpha}(t_2, t_0)\big\|\phi - z^*\big\|\Bigg] \\
&\quad + \sum_{i=1}^{n}\big(|d_{ij_0}| + |p_{ij_0}| + |q_{ij_0}|\big)L_i^g c_1 pMe_{\ominus\alpha}(t_2, t_0)\big\|\phi - z^*\big\|\Bigg\}\Delta s \\
&= \big\|\phi - z^*\big\| e_{\ominus\alpha}(t_2, t_0) + c_1 pMe_{\ominus\alpha}(t_2, t_0)\big\|\phi - z^*\big\| \\
&\quad \times \int_{t_0}^{t_2} e_{-b_{j_0}}\big(t_2, \sigma(s)\big)\Bigg\{b_{j_0}\eta_{j_0}^+\Bigg[b_{j_0} + \sum_{i=1}^{n}\big(|d_{ij_0}| + |p_{ij_0}| + |q_{ij_0}|\big)L_i^g\Bigg] \\
&\quad + \sum_{i=1}^{n}\big(|d_{ij_0}| + |p_{ij_0}| + |q_{ij_0}|\big)L_i^g\Bigg\}\Delta s \\
&= \big\|\phi - z^*\big\| e_{\ominus\alpha}(t_2, t_0) - \frac{1}{b_{j_0}} c_1 pMe_{\ominus\alpha}(t_2, t_0)\big\|\phi - z^*\big\| \\
&\quad \times \Bigg\{b_{j_0}\eta_{j_0}^+\Bigg[b_{j_0} + \sum_{i=1}^{n}\big(|d_{ij_0}| + |p_{ij_0}| + |q_{ij_0}|\big)L_i^g\Bigg] \\
&\quad + \sum_{i=1}^{n}\big(|d_{ij_0}| + |p_{ij_0}| + |q_{ij_0}|\big)L_i^g\Bigg\}\int_{t_0}^{t_2} e_{-b_{j_0}}\big(t_2, \sigma(s)\big)\Delta s \\
&= \big\|\phi - z^*\big\| e_{\ominus\alpha}(t_2, t_0) - \frac{1}{b_{j_0}}\big(e_{b_{j_0}}(t_2, t_0) - 1\big) c_1 pMe_{\ominus\alpha}(t_2, t_0)\big\|\phi - z^*\big\| \\
&\quad \times \Bigg\{b_{j_0}\eta_{j_0}^+\Bigg[b_{j_0} + \sum_{i=1}^{n}\big(|d_{ij_0}| + |p_{ij_0}| + |q_{ij_0}|\big)L_i^g\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \sum_{i=1}^{n}\bigl(|d_{ij_0}| + |p_{ij_0}| + |q_{ij_0}|\bigr)L_i^g\Biggr\} \\
&< c_1pM\|\phi - z^*\|e_{\ominus\alpha}(t_2,t_0) \\
&\quad\times\Biggl\{\frac{1}{c_1pM} + \frac{1}{b_{j_0}}\Biggl[b_{j_0}\eta_{j_0}^+\Biggl(b_{j_0} + \sum_{i=1}^{n}\bigl(|d_{ij_0}| + |p_{ij_0}| + |q_{ij_0}|\bigr)L_i^g\Biggr) \\
&\quad + \sum_{i=1}^{n}\bigl(|d_{ij_0}| + |p_{ij_0}| + |q_{ij_0}|\bigr)L_i^g\Biggr]\Biggr\} \\
&< c_1pM\|\phi - z^*\|e_{\ominus\alpha}(t_2,t_0) \\
&\quad\times\Biggl\{\frac{1}{M} + \frac{1}{b_{j_0}}\Biggl[b_{j_0}\eta_{j_0}^+\Biggl(b_{j_0} + \sum_{i=1}^{n}\bigl(|d_{ij_0}| + |p_{ij_0}| + |q_{ij_0}|\bigr)L_i^g\Biggr) \\
&\quad + \sum_{i=1}^{n}\bigl(|d_{ij_0}| + |p_{ij_0}| + |q_{ij_0}|\bigr)L_i^g\Biggr]\Biggr\} \\
&= c_1pM\|\phi - z^*\|e_{\ominus\alpha}(t_2,t_0)\Biggl(\frac{1}{M} + b_j\eta_j^+ + \bigl(\eta_j^+ + b_j^{-1}\bigr)\sum_{i=1}^{n}\bigl(|d_{ij}| + |p_{ij}| + |q_{ij}|\bigr)L_i^g\Biggr) \\
&< c_1pMe_{\ominus\alpha}(t_2,t_0)\|\phi - z^*\|,
\end{aligned}
$$

which is also a contradiction.

By the above two cases, for other cases of negative proposition of (4.7), we can obtain a contradiction. Therefore, (4.7) holds. Let $p \to 1$, then (4.6) holds. Hence, we have that

$$
|w(t)|_1 \le M\|\phi - z^*\|e_{\ominus\alpha}(t,t_0), \quad t \in [-\vartheta,\infty)_{\mathbb{T}}, t \ge t_0,
$$

which means that the equilibrium point z^* of (1.1) is globally exponentially stable. The proof of Theorem 4.1 is completed. □

5 Example

In this section, we present an example to illustrate the feasibility of our results.

Example 5.1 Let $n = m = 2$. Consider the following fuzzy BAM system with delays in leakage terms on a time scale $\mathbb{T}$:

$$
\begin{cases}
x_i^\Delta(t) = -a_ix_i(t-\delta_i(t)) + \sum_{j=1}^{2}c_{ji}f_j(y_j(t-\tau_{ji}(t))) + \bigwedge_{j=1}^{2}\alpha_{ji}f_j(y_j(t-\tau_{ji}(t))) \\
\qquad + \bigwedge_{j=1}^{2}T_{ji}\mu_j + \bigvee_{j=1}^{2}\beta_{ji}f_j(y_j(t-\tau_{ji}(t))) + \bigvee_{j=1}^{2}H_{ji}\mu_j + I_i, \quad t \in \mathbb{T}, i = 1,2, \\
y_j^\Delta(t) = -b_jy_j(t-\eta_j(t)) + \sum_{i=1}^{2}d_{ij}g_i(x_i(t-\sigma_{ij}(t))) + \bigwedge_{i=1}^{2}p_{ij}g_i(x_i(t-\sigma_{ij}(t))) \\
\qquad + \bigwedge_{i=1}^{2}F_{ij}\nu_i + \bigvee_{i=1}^{2}q_{ij}g_i(x_i(t-\sigma_{ij}(t))) + \bigvee_{i=1}^{2}G_{ij}\nu_i + J_j, \quad t \in \mathbb{T}, j = 1,2,
\end{cases}
\tag{5.1}
$$

where time delays δ_i, τ_{ji}, η_j, σ_{ij}, $i,j = 1,2$ are defined as those in system (1.1) and the coefficients are as follows:

$$
\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}, \qquad \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix},
$$

$$
(c_{ji})_{2\times2} = \begin{pmatrix} 0.01 & 0.03 \\ 0.06 & 0.04 \end{pmatrix}, \qquad (\alpha_{ji})_{2\times2} = \begin{pmatrix} 0.02 & 0 \\ -0.01 & 0.04 \end{pmatrix},
$$

$$(\beta_{ji})_{2\times 2} = \begin{pmatrix} 0.05 & -0.03 \\ 0 & 0.02 \end{pmatrix}, \qquad (d_{ij})_{2\times 2} = \begin{pmatrix} 0.03 & 0.01 \\ 0.04 & 0.02 \end{pmatrix},$$

$$(p_{ij})_{2\times 2} = \begin{pmatrix} 0.03 & -0.02 \\ -0.01 & 0.04 \end{pmatrix}, \qquad (q_{ij})_{2\times 2} = \begin{pmatrix} 0 & -0.01 \\ -0.02 & 0.03 \end{pmatrix},$$

$$\bigl(\delta_i(t)\bigr)_{2\times 2} = \begin{pmatrix} 0.001|\sin t| & 0 \\ 0 & 0.006|\cos t| \end{pmatrix}, \qquad (I_i)_{2\times 1} = \begin{pmatrix} 3 \\ 4 \end{pmatrix},$$

$$\bigl(\eta_j(t)\bigr)_{2\times 2} = \begin{pmatrix} 0.002|\sin 2t| & 0 \\ 0 & 0.004|\sin t - 1| \end{pmatrix},$$

$$(J_j)_{2\times 1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \qquad f_j(u_j) = 0.02|u_j|,$$

$$g_i(v_i) = 0.04\bigl(|v_i + 1| - |v_i - 1|\bigr), \qquad \mu_j = \nu_i = 1, \quad i,j = 1,2,$$

and $(T_{ji})_{2\times 2} = (H_{ji})_{2\times 2} = (F_{ij})_{2\times 2} = (G_{ij})_{2\times 2}$ are identity matrices. By calculating, we have $L_j^f = L_i^g = 0.02$. We can verify that for $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, all the conditions of Theorem 3.1 and Theorem 4.1 are satisfied. Hence, for $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$, (5.1) always has one unique equilibrium point, which is globally exponentially stable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors typed, read and approved the final manuscript.

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