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On the reciprocal sums of higher-order sequences

Zhengang Wu[*] and Han Zhang

[*]Correspondence:
sky.wzgfff@163.com
Department of Mathematics,
Northwest University, Xi'an, Shaanxi,
P.R. China

Abstract

Let $\{u_n\}$ be a higher-order recursive sequence. Several identities are obtained for the infinite sums and finite sums of the reciprocals of higher-order recursive sequences.
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1 Introduction

The so-called Fibonacci zeta function and Lucas zeta function defined by

$$\zeta_F(s) = \sum_{n=1}^{\infty} \frac{1}{F_n^s} \quad \text{and} \quad \zeta_L(s) = \sum_{n=1}^{\infty} \frac{1}{L_n^s},$$

where the F_n and L_n denote the Fibonacci numbers and Lucas numbers, have been considered in several different ways. Navas [1] discussed the analytic continuation of these series. Elsner *et al.* [2] obtained that for any positive distinct integer s_1, s_2, s_3, the numbers $\zeta_F(2s_1)$, $\zeta_F(2s_2)$, and $\zeta_F(2s_3)$ are algebraically independent if and only if at least one of s_1, s_2, s_3 is even.

Ohtsuka and Nakamura [3] studied the partial infinite sums of reciprocal Fibonacci numbers and proved the following conclusions:

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2} & \text{if } n \text{ is even and } n \geq 2; \\ F_{n-2} - 1 & \text{if } n \text{ is odd and } n \geq 1. \end{cases}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} F_{n-1}F_n - 1 & \text{if } n \text{ is even and } n \geq 2; \\ F_{n-1}F_n & \text{if } n \text{ is odd and } n \geq 1. \end{cases}$$

Where $\lfloor \cdot \rfloor$ denotes the floor function.

Further, Wu and Zhang [4, 5] generalized these identities to the Fibonacci polynomials and Lucas polynomials. Similar properties were also investigated in [6–8]. Related properties of the Fibonacci polynomials and Lucas polynomials can be found in [9–12].

Recently, some authors considered the nearest integer of the sums of reciprocal Fibonacci numbers and other famous sequences and obtained several new interesting identities, see [13] and [14]. Kilic and Arikan [15] defined a kth-order linear recursive sequence

$\{u_n\}$ for any positive integer $p \geq q$ and $n > k$ as follows:

$$u_n = pu_{n-1} + qu_{n-2} + u_{n-3} + \cdots + u_{n-k},$$

and they proved that there exists a positive integer n_0 such that

$$\left\| \left(\sum_{k=n}^{\infty} \frac{1}{u_k} \right)^{-1} \right\| = u_n - u_{n-1} \quad (n \geq n_0),$$

where $\| \cdot \|$ denotes the nearest integer. (Clearly, $\|x\| = \lfloor x + \frac{1}{2} \rfloor$.)

In this paper, we unify the above results by proving some theorems that include all the results, [3–8] and [13–15], as special cases. We consider the following type of higher-order recurrence sequences. For any positive integer $a_1, a_2, \ldots, a_m$, we define mth-order linear recursive sequences $\{u_n\}$ for $n > m$ as follows:

$$u_n = a_1 u_{n-1} + a_2 u_{n-2} + \cdots + a_{m-1} u_{n-m+1} + a_m u_{n-m}, \tag{1}$$

with initial values $u_i \in \mathbb{N}$ for $0 \leq i < m$ and at least one of them is not zero. If $m = 2$, $a_1 = a_2 = 1$, then $u_n = F_n$ are the Fibonacci numbers. If $m = 2$, $a_1 = 2$, $a_2 = 1$, then $u_n = P_n$ are the Pell numbers. Our main results are the following.

Theorem 1 *Let $\{u_n\}$ be an mth-order sequence defined by* (1) *with the restriction $a_1 \geq a_2 \geq \cdots \geq a_m \geq 1$. For any positive real number $\beta > 2$, there exists a positive integer n_1 such that*

$$\left\| \left(\sum_{k=n}^{\lfloor \beta n \rfloor} \frac{1}{u_k} \right)^{-1} \right\| = u_n - u_{n-1} \quad (n \geq n_1).$$

Taking $\beta \to +\infty$, from Theorem 1 we may immediately deduce the following.

Corollary 1 *Let $\{u_n\}$ be an mth-order sequence defined by* (1) *with the restriction $a_1 \geq a_2 \geq \cdots \geq a_m \geq 1$. Then there exists a positive integer n_2 such that*

$$\left\| \left(\sum_{k=n}^{\infty} \frac{1}{u_k} \right)^{-1} \right\| = u_n - u_{n-1} \quad (n \geq n_2).$$

For a positive real number $1 < \beta \leq 2$, whether there exits an identity for

$$\left\| \left(\sum_{k=n}^{\lfloor \beta n \rfloor} \frac{1}{u_k} \right)^{-1} \right\|$$

is an interesting open problem.

2 Several lemmas
To complete the proof of our theorem, we need the following.

Lemma 1 *Let $a_1, a_2, \ldots, a_m$ be positive integers with $a_1 \geq a_2 \geq \cdots \geq a_m \geq 1$ and $m \in \mathbb{N}$ with $m \geq 2$. Then, for the polynomial*

$$f(x) = x^m - a_1 x^{m-1} - a_2 x^{m-2} - \cdots - a_{m-1} x - a_m,$$

we have

  (I) *Polynomial $f(x)$ has exactly one positive real zero α with $a_1 < \alpha < a_1 + 1$.*

 (II) *Other $m-1$ zeros of $f(x)$ lie within the unit circle in the complex plane.*

Proof For any positive integer $a_1 \geq a_2 \geq \cdots \geq a_m \geq 1$ and $m \geq 2$, we have

$$f(a_1) = a_1^m - a_1^m - a_2 a_1^{m-2} - \cdots - a_{m-1} a_1 - a_m$$

$$= -a_2 a_1^{m-2} - \cdots - a_{m-1} a_1 - a_m < 0,$$

and

$$f(a_1 + 1) = (a_1 + 1)^m - a_1 (a_1 + 1)^{m-1} - a_2 (a_1 + 1)^{m-2} - \cdots - a_m$$

$$> (a_1 + 1)^m - a_1 \big((a_1 + 1)^{m-1} + (a_1 + 1)^{m-2} + \cdots + 1 \big)$$

$$= (a_1 + 1)^m - a_1 \cdot \frac{(a_1 + 1)^m - 1}{a_1} = 1 > 0.$$

Thus there exits a positive real zero α of $f(x)$ with $a_1 < \alpha < a_1 + 1$. According to Descarte's rule of signs, $f(x) = 0$ has at most one positive real root. So, $f(x)$ has exactly one positive real zero α with $a_1 < \alpha < a_1 + 1$. This completes the proof of (I) in Lemma 1.

Observe that from (I) in Lemma 1 we have

$$\text{if } x \in \mathbb{R} \text{ such that } x > \alpha, \text{ then } f(x) > 0, \tag{2}$$

$$\text{if } x \in \mathbb{R} \text{ such that } 0 < x < \alpha, \text{ then } f(x) < 0. \tag{3}$$

Let

$$g(x) = (x - 1) f(x)$$

$$= x^{m+1} - (a_1 + 1) x^m + (a_1 - a_2) x^{m-1} + (a_2 - a_3) x^{m-2} + \cdots + (a_{m-1} - a_m) x + a_m.$$

Since $f(x)$ has exactly one positive real zero α, $g(x)$ has two positive real zeros α and 1. Observe that

$$\text{if } x \in \mathbb{R} \text{ such that } x > \alpha, \text{ then } g(x) > 0, \tag{4}$$

$$\text{if } x \in \mathbb{R} \text{ such that } 1 < x < \alpha, \text{ then } g(x) < 0. \tag{5}$$

To complete the proof of (II) in Lemma 1, it is sufficient to show that there is no zero on and outside of the unit circle.   □

Claim 1 *$f(x)$ has no complex zero z_1 with $|z_1| > \alpha$.*

Proof Assume that there exits such a zero. So, we have

$$f(z_1) = z_1^m - a_1 z_1^{m-1} - a_2 z_1^{m-2} - \cdots - a_{m-1} z_1 - a_m = 0,$$

then we obtain

$$\left| z_1^m \right| \le a_1 \left| z_1^{m-1} \right| + a_2 \left| z_1^{m-2} \right| + \cdots + a_{m-1} |z_1| + a_m,$$

$$f\left(|z_1| \right) = \left| z_1^m \right| - a_1 \left| z_1^{m-1} \right| - a_2 \left| z_1^{m-2} \right| - \cdots - a_{m-1} |z_1| - a_m \le 0.$$

This contradicts with (2). □

Claim 2 $f(x)$ *has no complex zero* z_2 *with* $1 < |z_2| < \alpha$.

Proof Assume that there exits such a zero. Since $f(z_2) = 0$,

$$g(z_2) = z_2^{m+1} - (a_1 + 1)z_2^m + (a_1 - a_2)z_2^{m-1} + \cdots + (a_{m-1} - a_m)z_2 + a_m = 0,$$

then we obtain

$$(a_1 + 1)|z_2|^m \le |z_2|^{m+1} + (a_1 - a_2)|z_2|^{m-1} + \cdots + (a_{m-1} - a_m)|z_2| + a_m.$$

So, we have $g(|z_2|) \ge 0$, which contradicts with (5). □

Claim 3 *On the circle* $|z_3| = \alpha$ *and* $|z_3| = 1$, $f(x)$ *has the unique zero* α.

Proof If $f(z_3) = 0$, then

$$g(z_3) = z_3^{m+1} - (a_1 + 1)z_3^m + (a_1 - a_2)z_3^{m-1} + \cdots + (a_{m-1} - a_m)z_3 + a_m = 0,$$

then we obtain

$$(a_1 + 1)|z_3|^m \le |z_3|^{m+1} + (a_1 - a_2)|z_3|^{m-1} + \cdots + (a_{m-1} - a_m)|z_3| + a_m. \tag{6}$$

If $z_3 = \alpha$ or $z_3 = 1$, then $g(z_3) = 0$, so (6) must be an equality. Therefore, $z_3^{m+1}, (a_1 - a_2)z_3^{m-1}, (a_2 - a_3)z_3^{m-2}, \ldots, (a_{m-1} - a_m)z_3$ and a_m all lie on the same ray issuing from the origin. Since $(a_1 - a_2), (a_2 - a_3), \ldots, a_m$, are all the elements of $\mathbb{R}^+$, $z_3^{m+1}, z_3^{m-1}, z_3^{m-2}, \ldots, z_3$ must be the elements of $\mathbb{R}^+$. Therefore we obtain $f(z_3) \in \mathbb{R}^+$. On the circle $|z_3| = \alpha$ and $|z_3| = 1$, there are two conditions $z_3 = 1$ or $z_3 = \alpha$. Since $f(1) \ne 0$, α is the unique zero of $f(x)$, Claim 3 holds.

From the three claims, (II) in Lemma 1 is proven. □

Lemma 2 *Let* $m \ge 2$ *and let* $\{u_n\}_{n \ge 0}$ *be an integer sequence satisfying the recurrence formula* (1). *Then the closed formula of* u_n *is given by*

$$u_n = c\alpha^n + \mathcal{O}\left(d^{-n} \right) \quad (n \to \infty),$$

where $c > 0$, $d > 1$, *and* $a_1 < \alpha < a_1 + 1$ *is the positive real zero of* $f(x)$.

Proof Let $\alpha, \alpha_1, \ldots, \alpha_t$ be the distinct roots of $f(x) = 0$, where $f(x) = 0$ is the characteristic equation of the recurrence formula (1). From Lemma 1 we know that α is the simple root of $f(x) = 0$, then let r_j, for $j = 1, 2, \ldots, t$, denote the multiplicity of the root α_j. From the properties of mth-order linear recursive sequences, u_n can be expressed as follows:

$$u_n = c\alpha^n + \sum_{i=1}^{t} P_i(n)\alpha_i^n, \tag{7}$$

where

$$P_i(n) \in \mathbb{R}[n], \quad \deg P_i(n) = r_i - 1, r_1 + r_2 + \cdots + r_t = m - 1, \quad \text{and} \quad c \in \mathbb{R}.$$

For example, for positive integers $1 \le u, v, w \le t$, if α_u is the simple root of $f(x)$, then $P_u(n) = g_1$, where $g_1 \in \mathbb{R}$, and $\deg P_u(n) = 0$; if α_v is the double root of $f(x)$, then $P_v(n) = g_2 n + g_3$, where $g_2, g_3 \in \mathbb{R}$, and $\deg P_v(n) = 1$; if α_w is the multiple root of $f(x)$ with the multiplicity r_w, then $P_w(n) = b_1 n^{r_w-1} + b_2 n^{r_w-2} + \cdots + b_{r_w-1} n + b_{r_w}$, where $b_1, b_2, \ldots, b_{r_w} \in \mathbb{R}$, and $\deg P_w(n) = r_w - 1$.

From Lemma 1 we have $|\alpha_i| < 1$ for $1 \le i \le t$. Since each term of tail in (7) goes to 0 as $n \to \infty$, we can find the constant $M \in \mathbb{R}$ and $d \in \mathbb{R}$ with $d > 1$ for $n > n_0$ such that

$$\left| \sum_{i=1}^{t} P_i(n)\alpha_i^n \right| \le \sum_{i=1}^{t} \left| P_i(n)\alpha_i^n \right| \le M d^{-n},$$

which completes the proof (note that if all the roots of $f(x)$ are distinct, we can choose $d^{-1} = \max\{|\alpha_1|, |\alpha_2|, \ldots, |\alpha_{m-1}|\}$ and $M = m - 1$). $\qquad\square$

3 Proof of Theorem 1

In this section, we shall complete the proof of Theorem 1. From the geometric series as $\epsilon \to 0$, we have

$$\frac{1}{1 \pm \epsilon} = 1 \mp \epsilon + \mathcal{O}(\epsilon^2) = 1 + \mathcal{O}(\epsilon).$$

Using Lemma 2, we have

$$\begin{aligned}
\frac{1}{u_k} &= \frac{1}{c\alpha^k + \mathcal{O}(d^{-k})} = \frac{1}{c\alpha^k(1 + \mathcal{O}((\alpha d)^{-k}))} \\
&= \frac{1}{c\alpha^k}\left(1 + \mathcal{O}((\alpha d)^{-k})\right) \\
&= \frac{1}{c\alpha^k} + \mathcal{O}((\alpha^2 d)^{-k}).
\end{aligned} \tag{8}$$

Thus

$$\begin{aligned}
\sum_{k=n}^{\lfloor \beta n \rfloor} \frac{1}{u_k} &= \frac{1}{c} \sum_{k=n}^{\lfloor \beta n \rfloor} \frac{1}{\alpha^k} + \mathcal{O}\left(\sum_{k=n}^{\lfloor \beta n \rfloor} (\alpha^2 d)^{-k} \right) \\
&= \frac{\alpha}{c(\alpha - 1)} \cdot \alpha^{-n} - \frac{1}{c(\alpha - 1)} \cdot \alpha^{-\lfloor \beta n \rfloor} + \mathcal{O}(\alpha^{-2n} d^{-n})
\end{aligned}$$

$$= \frac{\alpha}{c(\alpha - 1)} \cdot \alpha^{-n} + \mathcal{O}\big(\alpha^{-2n}\alpha^{-\lfloor \beta n \rfloor + 2n}\big) + \mathcal{O}\big(\alpha^{-2n}d^{-n}\big)$$

$$= \frac{\alpha}{c(\alpha - 1)}\alpha^{-n} + \mathcal{O}\big(\alpha^{-2n}h\big),$$

where $h = \max\{\alpha^{-\lfloor \beta n \rfloor + 2n}, d^{-n}\}$.

Taking reciprocal, we get

$$\left(\sum_{k=n}^{\lfloor \beta n \rfloor} \frac{1}{u_k}\right)^{-1} = \frac{1}{\frac{\alpha}{c(\alpha-1)}\alpha^{-n}(1 + \mathcal{O}(\alpha^{-n}h))}$$

$$= \frac{\alpha - 1}{\alpha}c\alpha^n\big(1 + \mathcal{O}\big(\alpha^{-n}h\big)\big)$$

$$= \frac{\alpha - 1}{\alpha}c\alpha^n + \mathcal{O}(h)$$

$$= u_n - u_{n-1} + \mathcal{O}(h).$$

Since $h = \max\{\alpha^{-\lfloor \beta n \rfloor + 2n}, d^{-n}\} < 1$, there exists $n \geq n_1$ sufficient large so that the modulus of the last error term becomes less than $1/2$, which completes the proof.

Proof of Corollary 1 From identity (8), we have

$$\frac{1}{u_k} = \frac{1}{c\alpha^k} + \mathcal{O}\big((\alpha^2 d)^{-k}\big).$$

Thus

$$\sum_{k=n}^{\infty} \frac{1}{u_k} = \frac{1}{c}\sum_{k=n}^{\infty} \frac{1}{\alpha^k} + \mathcal{O}\left(\sum_{k=n}^{\infty}(\alpha^2 d)^{-k}\right) = \frac{\alpha}{c(\alpha - 1)}\alpha^{-n} + \mathcal{O}\big((\alpha^2 d)^{-n}\big).$$

Taking reciprocal, we get

$$\left(\sum_{k=n}^{\infty} \frac{1}{u_k}\right)^{-1} = \frac{1}{\frac{\alpha}{c(\alpha-1)}\alpha^{-n}(1 + \mathcal{O}((\alpha d)^{-n}))}$$

$$= \frac{\alpha - 1}{\alpha}c\alpha^n\big(1 + \mathcal{O}\big((\alpha d)^{-n}\big)\big)$$

$$= \frac{\alpha - 1}{\alpha}c\alpha^n + \mathcal{O}\big(d^{-n}\big)$$

$$= u_n - u_{n-1} + \mathcal{O}\big(d^{-n}\big).$$

So, there exists $n \geq n_2$ sufficiently large so that the modulus of the last error term becomes less than $1/2$, which completes the proof. □

4 Related results

The following results are obtained similarly.

Theorem 2 *Let $\{u_n\}$ be an mth-order sequence defined by* (1) *with the restriction $a_1 \geq a_2 \geq \cdots \geq a_m \geq 1$. Let p and q be positive integers with $0 \leq q < p$. For any real number $\beta > 2$,*

there exist positive integers n_3, n_4 and n_5 depending on $a_1, a_2, \ldots,$ and a_m such that

(a) $\left\| \left(\sum_{k=n}^{\lfloor \beta n \rfloor} \frac{(-1)^k}{u_k} \right)^{-1} \right\| = (-1)^n (u_n + u_{n-1}) \quad (n \geq n_3),$

(b) $\left\| \left(\sum_{k=n}^{\lfloor \beta n \rfloor} \frac{1}{u_{pk+q}} \right)^{-1} \right\| = u_{pn+q} - u_{pn-p+q} \quad (n \geq n_4),$

(c) $\left\| \left(\sum_{k=n}^{\lfloor \beta n \rfloor} \frac{(-1)^k}{u_{pk+q}} \right)^{-1} \right\| = (-1)^n (u_{pn+q} + u_{pn-p+q}) \quad (n \geq n_5).$

For $\beta \to +\infty$, we deduce the following identity of infinite sum as a special case of Theorem 2.

Corollary 2 *Let $\{u_n\}$ be an mth-order sequence defined by (1) with the restriction $a_1 \geq a_2 \geq \cdots \geq a_m \geq 1$. Let p and q be positive integers with $0 \leq q < p$. Then there exist positive integers n_6, n_7 and n_8 depending on $a_1, a_2, \ldots,$ and a_m such that*

(e) $\left\| \left(\sum_{k=n}^{\infty} \frac{(-1)^k}{u_k} \right)^{-1} \right\| = (-1)^n (u_n + u_{n-1}) \quad (n \geq n_6),$

(f) $\left\| \left(\sum_{k=n}^{\infty} \frac{1}{u_{pk+q}} \right)^{-1} \right\| = u_{pn+q} - u_{pn-p+q} \quad (n \geq n_7),$

(g) $\left\| \left(\sum_{k=n}^{\infty} \frac{(-1)^k}{u_{pk+q}} \right)^{-1} \right\| = (-1)^n (u_{pn+q} + u_{pn-p+q}) \quad (n \geq n_8).$

Proof We shall prove only (c) in Theorem 2 and other identities are proved similarly. From Lemma 2 we have

$$\frac{(-1)^k}{u_{pk+q}} = \frac{(-1)^k}{c\alpha^{pk+q} + \mathcal{O}(d^{-pk-q})} = \frac{(-1)^k}{c\alpha^{pk+q}} \left(1 + \mathcal{O}\big((\alpha d)^{-pk-q}\big) \right).$$

Thus

$$\sum_{k=n}^{\lfloor \beta n \rfloor} \frac{(-1)^k}{u_{pk+q}} = \frac{(-1)^n \alpha^p}{c\alpha^{pn+q}(\alpha^p + 1)} + \frac{(-1)^n \alpha^p}{c\alpha^{p\lfloor \beta n \rfloor + q}(\alpha^p + 1)} + \mathcal{O}\big((\alpha^2 d)^{-pn-q}\big)$$

$$= \frac{(-1)^n \alpha^p}{c\alpha^{pn+q}(\alpha^p + 1)} + \mathcal{O}\big(\alpha^{-p\lfloor \beta n \rfloor - q}\big) + \mathcal{O}\big(\alpha^{-2pn-2q} d^{-pn-q}\big)$$

$$= \frac{(-1)^n \alpha^p}{c\alpha^{pn+q}(\alpha^p + 1)} + \mathcal{O}\big(\alpha^{-2pn} \alpha^{-p\lfloor \beta n \rfloor + 2pn}\big) + \mathcal{O}\big(\alpha^{-2pn} d^{-pn}\big)$$

$$= \frac{(-1)^n \alpha^p}{c\alpha^{pn+q}(\alpha^p + 1)} + \mathcal{O}\big(\alpha^{-2pn} h^p\big),$$

where $h = \max\{\alpha^{-\lfloor \beta n \rfloor + 2n}, d^{-n}\}$.

Taking reciprocal, we get

$$\left(\sum_{k=n}^{\lfloor \beta n \rfloor} \frac{(-1)^k}{u_{pk+q}}\right)^{-1} = (-1)^n \left(c\alpha^{pn+q} + c\alpha^{pn-p+q}\right)\left(1 + \mathcal{O}\left(\alpha^{-pn}h^p\right)\right)$$

$$= (-1)^n (u_{pn+q} + u_{pn-p+q}) + \mathcal{O}(h^p).$$

Since $h = \max\{\alpha^{-\lfloor \beta n \rfloor + 2n}, d^{-n}\} < 1$, there exists $n \geq n_5$ sufficiently large so that the modulus of the last error term becomes less than $1/2$, which completes the proof. $\qquad\square$

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