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Some existence results for differential inclusions of fractional order with nonlocal strip conditions

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Abstract
In this paper, the existence of solutions for differential inclusions of fractional order $q \in (1,2]$ with nonlocal strip conditions is investigated. Our study includes two cases: (i) the multivalued map involved in the problem is not necessarily convex valued, (ii) the multivalued map consists of non-convex values. We combine the nonlinear alternative of Leray-Schauder type coupled with the selection theorem of Bressan and Colombo to establish the first result, while the second result relies on Wegrzyk's fixed point theorem for generalized contractions.
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1 Introduction

Nonlocal nonlinear boundary value problems of fractional differential equations and inclusions have received considerable attention, and a great deal of work concerning a variety of boundary conditions can be found in the recent literature on the topic. It has been due to the extensive applications of fractional calculus in numerous branches of physics, economics and technical sciences [1–5]. Fractional-order differential operators are found to be effective and realistic mathematical tools for the description of memory and hereditary properties of various materials and processes. For examples and details, we refer the reader to a series of papers [6–26] and the references therein.

In this paper, we discuss the existence of solutions for a boundary value problem of differential inclusions of fractional order with nonlocal strip conditions given by

$$\begin{cases} {}^cD^q x(t) \in F(t,x(t)), \quad 0 < t < 1, 1 < q \le 2, \\ x(0) = 0, \qquad \alpha x(1) + \beta x'(1) = \eta \int_\nu^\tau x(s)\,ds, \quad 0 < \nu < \tau < 1\ (\nu \neq \tau), \end{cases} \tag{1.1}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is a family of all nonempty subsets of $\mathbb{R}$, and $\alpha, \beta, \eta \in R$ satisfy the relation $\eta \neq 2(\alpha + \beta)/(\tau^2 - \nu^2)$.

The present work is motivated by a recent paper [27] where the author studied problem (1.1) with $F(t,x(t))$ as a single-valued mapping. We establish our existence results by means of the nonlinear alternative of Leray-Schauder type, the selection theorem of Bressan and

Colombo for lower semi-continuous maps with decomposable values and Wegrzyk's fixed point theorem for generalized contraction maps.

2 Preliminaries

In this section, we present some basic concepts of multivalued maps and fixed point theorems needed in the sequel.

Let Y denote a normed space with the norm $|\cdot|$. A multivalued map $\mathcal{G}: Y \to \mathcal{P}(Y)$ is convex (closed) valued if $\mathcal{G}(u)$ is convex (closed) for all $u \in Y$. $\mathcal{G}$ is bounded on bounded sets if $\mathcal{G}(B) = \bigcup_{u\in B}\mathcal{G}(u)$ is bounded in Y for all bounded sets B in Y (*i.e.*, $\sup_{u\in B}\{|v| : v \in \mathcal{G}(u)\} < \infty$). $\mathcal{G}$ is called upper semi-continuous (u.s.c.) on Y if for each $u_0 \in Y$, the set $\mathcal{G}(u_0)$ is a nonempty closed subset of Y, and if for each open set N of Y containing $G(u_0)$, there exists an open neighborhood N_0 of u_0 such that $\mathcal{G}(N_0) \subseteq N$. $\mathcal{G}$ is said to be completely continuous if $\mathcal{G}(B)$ is relatively compact for every bounded set B in Y. If the multivalued map $\mathcal{G}$ is completely continuous with nonempty compact values, then $\mathcal{G}$ is u.s.c. if and only if $\mathcal{G}$ has a closed graph (*i.e.*, $u_n \longrightarrow u_*$, $v_n \longrightarrow v_*$, $v_n \in \mathcal{G}(u_n)$ imply $v_* \in \mathcal{G}(u_*)$). $\mathcal{G}$ has a fixed point if there is $u \in Y$ such that $u \in \mathcal{G}(u)$. The fixed point set of the multivalued operator $\mathcal{G}$ will be denoted by $\operatorname{Fix}\mathcal{G}$.

For more details on multivalued maps, see [28–30].

Let $C([0,1],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ with the norm $\|u\| = \sup\{|u(t)| : t \in [0,1]\}$. Let $L^1([0,1],\mathbb{R})$ be the Banach space of measurable functions $u : [0,1] \longrightarrow \mathbb{R}$ which are Lebesgue integrable and normed by

$$\|u\|_{L^1} = \int_0^1 \big|u(t)\big|\,dt \quad \text{for all } u \in L^1\big([0,1],\mathbb{R}\big).$$

Let E be a Banach space, X be a nonempty closed subset of E and let $G : X \to \mathcal{P}(E)$ be a multivalued operator with nonempty closed values. G is lower semi-continuous (l.s.c.) if the set $\{x \in X : G(x) \cap B \neq \emptyset\}$ is open for any open set B in E. Let A be a subset of $[0,1] \times \mathbb{R}$. A is $\mathcal{L} \otimes B$ measurable if A belongs to the σ-algebra generated by all sets of the form $\mathcal{J} \times D$, where $\mathcal{J}$ is Lebesgue measurable in $[0,1]$ and D is Borel measurable in $\mathbb{R}$. A subset A of $L^1([0,1],\mathbb{R})$ is decomposable if for all $u, v \in A$ and $\mathcal{J} \subset [0,1]$ measurable, the function $u\chi_{\mathcal{J}} + v\chi_{[0,1]\setminus\mathcal{J}} \in A$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.

Definition 2.1 If $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map with compact values and $x(\cdot) \in C([0,1],\mathbb{R})$, then $F(\cdot,\cdot)$ is of lower semi-continuous type if

$$S_{F,x} = \big\{w \in L^1\big([0,1],\mathbb{R}\big) : w(t) \in F\big(t, x(t)\big) \text{ for a.e. } t \in [0,1]\big\}$$

is lower semi-continuous with closed and decomposable values.

Let (X,d) be a metric space associated with the metric d. The Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by

$$d_{PH}(A,B) = \max\big\{d^*(A,B), d^*(B,A)\big\}, \qquad d^*(A,B) = \sup\big\{d(a,B) : a \in A\big\},$$

where $d(x,B) = \inf_{y\in B} d(x,y)$ [31].

Definition 2.2 [26] A function $l : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a strict comparison function if it is continuous, strictly increasing and $\sum_{n=1}^{\infty} l^n(t) < \infty$ for each $t > 0$.

Definition 2.3 A multivalued operator N on X with nonempty values in X is called

(a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$d_{PH}\big(N(x), N(y)\big) \le \gamma d(x,y) \quad \text{for each } x, y \in X;$$

(b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$;
(c) a generalized contraction if and only if there is a strict comparison function $l : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$d_{PH}\big(N(x), N(y)\big) \le l\big(d(x,y)\big) \quad \text{for each } x, y \in X.$$

The following lemmas will be used in the sequel.

Lemma 2.4 [32] *Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ be a lower semi-continuous multivalued map with closed decomposable values. Then $N(\cdot)$ has a continuous selection; i.e., there exists a continuous mapping (single-valued) $g : Y \to L^1([0,1],\mathbb{R})$ such that $g(y) \in N(y)$ for every $y \in Y$.*

Lemma 2.5 (Wegrzyk's fixed point theorem [33]) *Let (X,d) be a complete metric space. If $N : X \to \mathcal{P}(X)$ is a generalized contraction with nonempty closed values, then* $\operatorname{Fix} N \neq \emptyset$.

Lemma 2.6 (Covitz and Nadler's fixed point theorem [34]) *Let (X,d) be a complete metric space. If $N : X \to \mathcal{P}(X)$ is a multivalued contraction with nonempty closed values, then N has a fixed point $z \in X$ such that $z \in N(z)$, i.e.,* $\operatorname{Fix} N \neq \emptyset$.

In order to define the solution of (1.1), we consider the following lemma whose proof is given in [27].

Lemma 2.2 *For $h \in C([0,1],R)$, the unique solution of the following problem:*

$$\begin{cases} {}^cD^q x(t) = h(t), \quad 0 < t < 1, 1 < q \le 2, \\ x(0) = 0, \qquad \alpha x(1) + \beta x'(1) = \eta \int_\nu^\tau x(s)\,ds, \quad 0 < \nu < \tau < 1\ (\nu \neq \tau), \end{cases} \tag{2.1}$$

is given by

$$\begin{aligned} x(t) = {}& \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s)\,ds - \delta t \biggl[\alpha \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} h(s)\,ds \\ & + \beta \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} h(s)\,ds - \eta \int_\nu^\tau \biggl(\int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)} h(m)\,dm \biggr) ds \biggr], \end{aligned} \tag{2.2}$$

where

$$\delta = \frac{2}{2(\alpha+\beta) - \eta(\tau^2 - \nu^2)}. \tag{2.3}$$

Definition 2.7 A function $x \in AC^1([0,1])$ is a solution of problem (1.1) if there exists a function $f \in L^1([0,1],\mathbb{R})$ such that $f(t) \in F(t,x(t))$ a.e. on $[0,1]$ and

$$x(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)\,ds - \delta t\biggl[\alpha \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s)\,ds + \beta \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(s)\,ds - \eta \int_\nu^\tau \biggl(\int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)} f(m)\,dm \biggr)\,ds\biggr].$$

3 Existence of solutions

In the sequel, we set

$$\theta = \frac{1}{\Gamma(q+1)} \biggl(1 + \frac{|\delta|[(|\alpha| + q|\beta|)(q+1) + |\eta(\tau^{q+1} - \nu^{q+1})|]}{(q+1)} \biggr), \tag{3.1}$$

where δ is given by (2.3).

Our first result deals with the case when F is not necessarily convex valued. We establish this result by means of the nonlinear alternative of Leray-Schauder type together with the selection theorem of Bressan and Colombo [32] for lower semi-continuous maps with decomposable values.

Theorem 3.1 *Assume that*

(A_1) *there exists a continuous nondecreasing function* $\psi : [0,\infty) \to (0,\infty)$ *and a positive continuous function* p *such that*

$$\|F(t,x)\| := \sup\{|y| : y \in F(t,x)\} \le p(t)\psi(\|x\|) \quad \textit{for each } (t,x) \in [0,1] \times \mathbb{R};$$

(A_2) *there exists a number* $M > 0$ *such that*

$$\frac{M}{\theta \psi(M)\|p\|} > 1,$$

where θ *is given by* (3.1);

(A_3) $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is a nonempty compact-valued multivalued map such that*

(a) $(t,x) \longmapsto F(t,x)$ *is* $\mathcal{L} \otimes B$ *measurable,*

(b) $x \longmapsto F(t,x)$ *is lower semicontinuous for each* $t \in [0,1]$.

Then boundary value problem (1.1) *has at least one solution on* $[0,1]$.

Proof By the conditions (A_1) and (A_3), it follows that F is of l.s.c. type. Then from Lemma 2.4, there exists a continuous function $f : C([0,1],\mathbb{R}) \to L^1([0,1],\mathbb{R})$ such that $f(x) \in S_{F,x}$ for all $x \in C([0,1],\mathbb{R})$.

Let us consider the problem

$$\begin{cases} {}^cD^q x(t) = f(x)(t), & t \in [0,1], 1 < q \le 2, \\ x(0) = 0, \quad \alpha x(1) + \beta x'(1) = \eta \int_\nu^\tau x(s)\,ds, & 0 < \nu < \tau < 1\ (\nu \ne \tau). \end{cases} \tag{3.2}$$

Note that if $x \in C^2([0,1],\mathbb{R})$ is a solution of (3.2), then x is a solution to problem (1.1). In order to transform problem (3.2) into a fixed point problem, we define the operator

$\mathcal{H}: C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$

$$\begin{aligned}\mathcal{H}(x)(t) = {}& \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(x)(s)\,ds - \delta t\biggl[\alpha \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(x)(s)\,ds \\ &+ \beta \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(x)(s)\,ds - \eta \int_\nu^\tau \biggl(\int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)} f(x)(m)\,dm\biggr)\,ds\biggr].\end{aligned}$$

The proof consists of several steps.

(i) $\mathcal{H}$ *is continuous*. Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C([0,1],\mathbb{R})$. Then

$$\begin{aligned}\bigl|\mathcal{H}(y_n)(t) - \mathcal{H}(y)(t)\bigr| = {}& \biggl|\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\bigl[f(y_n)(s) - f(y)(s)\bigr]\,ds \\ &- \delta t\biggl[\alpha \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\bigl[f(y_n)(s) - f(y)(s)\bigr]\,ds \\ &+ \beta \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}\bigl[f(y_n)(s) - f(y)(s)\bigr]\,ds \\ &- \eta \int_\nu^\tau \biggl(\int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)}\bigl[f(y_n)(m) - f(y)(m)\bigr]\,dm\biggr)\,ds\biggr]\biggr| \\ \le {}& \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\bigl|f(y_n)(s) - f(y)(s)\bigr|\,ds \\ &+ |\delta| t\biggl[|\alpha| \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\bigl|f(y_n)(s) - f(y)(s)\bigr|\,ds \\ &+ |\beta| \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}\bigl|f(y_n)(s) - f(y)(s)\bigr|\,ds \\ &+ |\eta| \int_\nu^\tau \biggl(\int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)}\bigl|f(y_n)(m) - f(y)(m)\bigr|\,dm\biggr)\,ds\biggr].\end{aligned}$$

Hence

$$\bigl\|\mathcal{H}(y_n) - \mathcal{H}(y)\bigr\| = \sup_{t\in[0,1]}\bigl|\mathcal{H}(y_n)(t) - \mathcal{H}(y)(t)\bigr| \to 0 \quad \text{as } n \to \infty.$$

Thus $\mathcal{H}$ is continuous.

(ii) $\mathcal{H}$ *maps bounded sets into bounded sets in* $C([0,1],\mathbb{R})$. Indeed, it is enough to show that there exists a positive constant ν_1 such that, for each $x \in B_r = \{x \in C([0,1],\mathbb{R}) : \|x\| \le r\}$, we have $\|\mathcal{H}(x)\| \le \nu_1$. From (A_1) we have

$$\begin{aligned}&\bigl|\mathcal{H}(x)(t)\bigr| \\ &\quad\le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} p(s)\psi\bigl(\|x\|\bigr)\,ds + |\delta| t\biggl[|\alpha| \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} p(s)\psi\bigl(\|x\|\bigr)\,ds \\ &\qquad+ |\beta| \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} p(s)\psi\bigl(\|x\|\bigr)\,ds + |\eta| \int_\nu^\tau \biggl(\int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)} p(m)\psi\bigl(\|x\|\bigr)\,dm\biggr)\,ds\biggr] \\ &\quad\le \frac{1}{\Gamma(q+1)}\biggl(1 + \frac{|\delta|[(|\alpha| + q|\beta|)(q+1) + |\eta(\tau^{q+1} - \nu^{q+1})|]}{(q+1)}\biggr)\|p\|\psi\bigl(\|x\|\bigr).\end{aligned}$$

Taking norm and using (3.1), we get

$$\|\mathcal{H}(x)\| \leq \theta\|p\|\psi(r) := \nu_1.$$

(iii) *$\mathcal{H}$ maps bounded sets into equicontinuous sets in* $C([0,1],\mathbb{R})$. Let $t_1, t_2 \in [0,1]$, $t_1 < t_2$ and B_r be a bounded set in $C([0,1],\mathbb{R})$. Then

$$\begin{aligned}
&\left|\mathcal{H}(x)(t_2) - \mathcal{H}(x)(t_1)\right| \\
&\quad \leq \left|\int_0^{t_1} \frac{(t_2-s)^{q-1} - (t_1-s)^{q-1}}{\Gamma(q)} p(s)\psi\left(\|x\|\right) ds\right| + \left|\int_{t_1}^{t_2} \frac{(t_2-s)^{q-1}}{\Gamma(q)} p(s)\psi\left(\|x\|\right) ds\right| \\
&\qquad + |\delta||t_2 - t_1|\left[|\alpha|\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} p(s)\psi\left(\|x\|\right) ds\right. \\
&\qquad \left. + |\beta|\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} p(s)\psi\left(\|x\|\right) ds + |\eta|\int_\nu^\tau \left(\int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)} p(m)\psi\left(\|x\|\right) dm\right) ds\right].
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero independently of $x \in B_r$. Therefore it follows by the Arzelá-Ascoli theorem that $\mathcal{H} : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is completely continuous.

(iv) Finally, we discuss *a priori bounds on solutions*. Let x be a solution of (3.2). In view of (A_1), for each $t \in [0,1]$, we obtain

$$\begin{aligned}
\left|x(t)\right| &\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} p(s)\psi\left(\|x\|\right) ds + |\delta|t\left[|\alpha|\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} p(s)\psi\left(\|x\|\right) ds\right. \\
&\quad + |\beta|\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} p(s)\psi\left(\|x\|\right) ds \\
&\quad \left. + |\eta|\int_\nu^\tau \left(\int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)} p(m)\psi\left(\|x\|\right) dm\right) ds\right] \\
&\leq \frac{1}{\Gamma(q+1)}\left(1 + \frac{|\delta|[(|\alpha| + q|\beta|)(q+1) + |\eta(\tau^{q+1} - \nu^{q+1})|]}{(q+1)}\right)\|p\|\psi\left(\|x\|\right),
\end{aligned}$$

which, on taking norm and using (3.1), yields

$$\frac{\|x\|}{\theta\|p\|\psi(\|x\|)} \leq 1.$$

In view of (A_2), there exists M such that $\|x\| \neq M$. Let us set

$$U = \left\{x \in C\left([0,1],\mathbb{R}\right) : \|x\| < M\right\}.$$

Note that the operator $\mathcal{H} : \overline{U} \to C([0,1],\mathbb{R})$ is upper semicontinuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \mu\mathcal{H}(x)$ for some $\mu \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type [35], we deduce that $\mathcal{H}$ has a fixed point $x \in \overline{U}$, which is a solution of problem (3.2). Consequently, it is a solution to problem (1.1). This completes the proof. □

Next, we show the existence of solutions for problem (1.1) with a non-convex valued right-hand side by applying Lemma 2.5 due to Wegrzyk.

Theorem 3.2 *Suppose that*

(A_4) $F : [0,1] \times \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$ *has nonempty compact values and* $F(\cdot,u)$ *is measurable for each* $u \in \mathbb{R}$;

(A_5) $d_{PH}(F(t,x),F(t,\overline{x})) \leq k(t)\ell(|x-\overline{x}|)$ *for almost all* $t \in [0,1]$ *and* $x,\overline{x} \in \mathbb{R}$ *with* $k \in C([0,1],\mathbb{R}_+)$ *and* $d_{PH}(0,F(t,0)) \leq k(t)$ *for almost all* $t \in [0,1]$, *where* $\ell : \mathbb{R}_+ \to \mathbb{R}_+$ *is strictly increasing.*

Then BVP (1.1) *has at least one solution on* $[0,1]$ *if* $\gamma\ell : \mathbb{R}_+ \to \mathbb{R}_+$ *is a strict comparison function, where* $\gamma = \theta\|k\|$ (θ *is given by* (3.1)).

Proof Suppose that $\gamma\ell : \mathbb{R}_+ \to \mathbb{R}_+$ is a strict comparison function. Observe that by the assumptions (A_4) and (A_5), $F(\cdot,x(\cdot))$ is measurable and has a measurable selection $v(\cdot)$ (see Theorem III.6 [36]). Also, $k \in C([0,1],\mathbb{R}_+)$ and

$$\begin{aligned}\left|v(t)\right| &\leq d_{PH}\bigl(0,F(t,0)\bigr) + d_{PH}\bigl(F(t,0),F\bigl(t,x(t)\bigr)\bigr)\\ &\leq k(t) + k(t)\ell\bigl(\left|x(t)\right|\bigr)\\ &\leq \bigl(1+\ell\bigl(\|x\|\bigr)\bigr)k(t).\end{aligned}$$

Thus the set $S_{F,x}$ is nonempty for each $x \in C([0,1],\mathbb{R})$.

Transform problem (1.1) into a fixed point problem. Consider the operator $\mathcal{H} : C([0,1],\mathbb{R}) \to \mathcal{P}(C([0,1],\mathbb{R}))$ defined by

$$\mathcal{H}(x) = \left\{\begin{array}{l} y \in C([0,1],\mathbb{R}) : \\ y(t) = \left\{\begin{array}{l} \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s)\,ds \\ -\,\delta t[\alpha\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s)\,ds \\ +\,\beta\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(s)\,ds \\ -\,\eta\int_\nu^\tau (\int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)} f(m)\,dm)\,ds], \end{array}\right. \end{array}\right\}$$

for $f \in S_{F,x}$. We shall show that the map $\mathcal{H}$ satisfies the assumptions of Lemma 2.5. To show that the map $\mathcal{H}(x)$ is closed for each $x \in C([0,1],\mathbb{R})$, let $(x_n)_{n\geq 0} \in \mathcal{H}(x)$ such that $x_n \longrightarrow \tilde{x}$ in $C([0,1],\mathbb{R})$. Then $\tilde{x} \in C([0,1],\mathbb{R})$ and there exists $y_n \in S_{F,x_n}$ such that, for each $t \in [0,1]$,

$$\begin{aligned}x_n(t) = {}& \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} y_n(s)\,ds - \delta t\biggl[\alpha\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} y_n(s)\,ds\\ &+ \beta\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} y_n(s)\,ds - \eta\int_\nu^\tau\biggl(\int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)} y_n(m)\,dm\biggr)\,ds\biggr].\end{aligned}$$

As F has compact values, we pass onto a subsequence to obtain that y_n converges to y in $L^1([0,1],\mathbb{R})$. Thus, $y \in S_{F,x}$ and for each $t \in [0,1]$,

$$\begin{aligned}x_n(t) &\longrightarrow \tilde{x}(t)\\ &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} y(s)\,ds - \delta t\biggl[\alpha\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} y(s)\,ds\\ &\quad+ \beta\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} y(s)\,ds - \eta\int_\nu^\tau\biggl(\int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)} y(m)\,dm\biggr)\,ds\biggr].\end{aligned}$$

So, $\tilde{x} \in \mathcal{H}(x)$ and hence $\mathcal{H}(x)$ is closed.

Next, we show that

$$d_{PH}\big(\mathcal{H}(x), \mathcal{H}(\overline{x})\big) \leq \gamma \ell\big(\|x - \overline{x}\|\big) \quad \text{for each } x, \overline{x} \in C\big([0,1], \mathbb{R}\big).$$

Let $x, \overline{x} \in C([0,1], \mathbb{R})$ and $h_1 \in \mathcal{H}(x)$. Then there exists $y_1(t) \in F(t, x(t))$ such that for each $t \in [0,1]$,

$$\begin{aligned} h_1(t) = {} & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y_1(s)\, ds - \delta t \bigg[\alpha \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} y_1(s)\, ds \\ & + \beta \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} y_1(s)\, ds - \eta \int_\nu^\tau \bigg(\int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)} y_1(m)\, dm \bigg)\, ds \bigg]. \end{aligned}$$

From (A_5) it follows that

$$d_{PH}\big(F\big(t, x(t)\big), F\big(t, \overline{x}(t)\big)\big) \leq k(t) \ell\big(\big|x(t) - \overline{x}(t)\big|\big).$$

So, there exists $w \in F(t, \overline{x}(t))$ such that

$$\big|y_1(t) - w\big| \leq k(t)\ell\big(\big|x(t) - \overline{x}(t)\big|\big), \quad t \in [0,1].$$

Define $U : [0,1] \to \mathcal{P}(\mathbb{R})$ as

$$U(t) = \big\{ w \in \mathbb{R} : \big|y_1(t) - w\big| \leq k(t)\ell\big(\big|x(t) - \overline{x}(t)\big|\big)\big\}.$$

Since the multivalued operator $U(t) \cap F(t, \overline{x}(t))$ is measurable (see Proposition III.4 in [36]), there exists a function $y_2(t)$ which is a measurable selection for $U(t) \cap F(t, \overline{x}(t))$. So, $y_2(t) \in F(t, \overline{x}(t))$, and for each $t \in [0,1]$,

$$\big|y_1(t) - y_2(t)\big| \leq k(t)\ell\big(\big|x(t) - \overline{x}(t)\big|\big).$$

For each $t \in [0,1]$, let us define

$$\begin{aligned} h_2(t) = {} & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y_2(s)\, ds - \delta t \bigg[\alpha \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} y_2(s)\, ds \\ & + \beta \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} y_2(s)\, ds - \eta \int_\nu^\tau \bigg(\int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)} y_2(m)\, dm \bigg)\, ds \bigg]. \end{aligned}$$

Then

$$\begin{aligned} \big|h_1(t) - h_2(t)\big| \leq {} & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \big|y_1(s) - y_2(s)\big|\, ds \\ & + |\delta| t \bigg[|\alpha| \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \big|y_1(s) - y_2(s)\big|\, ds \\ & + |\beta| \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} \big|y_1(s) - y_2(s)\big|\, ds \end{aligned}$$

$$+ |\eta| \int_{\nu}^{\tau} \left(\int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)} \big| y_1(m) - y_2(m) \big| \, dm \right) ds \Bigg]$$
$$\leq \theta \|k\| \ell \big(\|x - \overline{x}\| \big).$$

Thus

$$\|h_1 - h_2\| \leq \theta \|k\| \ell \big(\|x - \overline{x}\| \big) = \gamma \ell \big(\|x - \overline{x}\| \big).$$

By an analogous argument, interchanging the roles of x and $\overline{x}$, we obtain

$$d_{PH} \big(\mathcal{H}(x), \mathcal{H}(\overline{x}) \big) \leq \theta \|k\| \ell \big(\|x - \overline{x}\| \big) = \gamma \ell \big(\|x - \overline{x}\| \big)$$

for each $x, \overline{x} \in C([0,1], \mathbb{R})$. So, $\mathcal{H}$ is a generalized contraction and thus, by Lemma 2.5, $\mathcal{H}$ has a fixed point x which is a solution to (1.1). This completes the proof. □

Remark 3.3 It is important to note that the condition (A_5) reduces to

$$d_{PH} \big(F(t,x), F(t,\overline{x}) \big) \leq k(t) |x - \overline{x}|$$

for $\ell(t) = t$, where a contraction principle for a multivalued map due to Covitz and Nadler [34] (Lemma 2.6) is applicable under the condition $\theta \|k\| < 1$. Thus, our result dealing with a non-convex valued right-hand side of (1.1) is more general. Furthermore, Theorem 3.2 holds for several values of the function ℓ.

Competing interests
The author did not provide this information.

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