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Existence and exponential stability of periodic solutions for a class of Hamiltonian systems on time scales

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Abstract

In this paper, by using a fixed point theorem and the theory of calculus on time scales, we obtain some sufficient conditions for the existence and exponential stability of periodic solutions for a class of Hamiltonian systems on time scales. We also present numerical examples to show the feasibility of our results. The results of this paper are completely new and complementary to the previously known results even if the time scale $\mathbb{T} = \mathbb{R}$ or \mathbb{Z} .

Keywords: periodic solution; Hamiltonian system; exponential stability; time scale

1 Introduction

Hamiltonian system, which was introduced by the Irish mathematician SWR Hamilton, is widely used in mathematical sciences, life sciences and so on. Many models in celestial mechanics, plasma physics, space science and bio-engineering are in the form of a Hamilton system. Therefore, the study of a Hamilton system is useful and meaningful in theory and practice. Recently, various Hamiltonian systems have been extensively studied (see [1–5] and references cited therein).

Most existing results on the study of Hamiltonian systems are for continuous systems. However, both discrete and continuous systems are important in applications, and it is troublesome to study the dynamical properties for continuous and discrete systems, respectively. Therefore, it is meaningful to study dynamical systems on time scales (see [6–13] and references cited therein), which helps avoid proving results twice, once for differential equations and once for difference equations. There have been some results devoted to Hamiltonian systems on time scales [9, 10, 14–17]. For example, authors in [14] introduced Hamiltonian systems on time scales and authors in [15, 16] studied the following Hamiltonian system on time scales:

$$\begin{cases} x^\Delta(t) = \alpha(t)x(\sigma(t)) + \beta(t)y(t), \\ y^\Delta(t) = -\gamma(t)x(\sigma(t)) - \alpha(t)y(t), \end{cases} \quad (1.1)$$

where $t \in \mathbb{T}$. In [15], authors obtained inequalities of Lyapunov for (1.1), and authors in [16] studied the stability of (1.1) by using Floquet theory. However, to the best of our knowledge, up to now, there have been no papers published on the existence and exponential stability of a periodic solution to (1.1).

Motivated by the above mentioned works, in this paper, we study the existence and exponential stability of periodic solutions to (1.1), in which \mathbb{T} is a periodic time scale. The main aim of this paper is to study the existence of periodic solutions to (1.1) by using a fixed point theorem. Moreover, we also study the exponential stability of the periodic solution to (1.1). Our results are new and complementary to the previously known results even if the time scale $\mathbb{T} = \mathbb{R}$ or \mathbb{Z} .

Remark 1.1 It is obvious that when $\mathbb{T} = \mathbb{R}$, (1.1) reduces to the following continuous time Hamiltonian system:

$$x'(t) = \alpha(t)x(t) + \beta(t)y(t), \quad y'(t) = -\gamma(t)x(t) - \alpha(t)y(t), \quad t \in \mathbb{R}.$$

When $\mathbb{T} = \mathbb{Z}$, (1.1) reduces to the following discrete time Hamiltonian system:

$$\Delta x(n) = \alpha(n)x(n+1) + \beta(n)y(n), \quad \Delta y(n) = -\gamma(n)x(n+1) - \alpha(n)y(n), \quad n \in \mathbb{Z}.$$

For convenience, we denote $[a, b]_{\mathbb{T}} = \{t | t \in [a, b] \cap \mathbb{T}\}$ and $\vartheta = \sup_{t \in \mathbb{T}} \mu(t)$. For an ω -periodic function $f : \mathbb{T} \rightarrow \mathbb{R}$, we denote $f^+ = \max_{t \in [0, \omega]_{\mathbb{T}}} |f(t)|$, $f^- = \min_{t \in [0, \omega]_{\mathbb{T}}} |f(t)|$ and $\bar{f} = \max_{t \in [0, \omega]_{\mathbb{T}}} f(t)$.

Throughout this paper, we assume that

(H₁) $\beta(t), \gamma(t) \in C_{rd}(\mathbb{T}, \mathbb{R})$, $\alpha(t) \in C_{rd}(\mathbb{T}, (0, +\infty))$ are all ω -periodic functions and $e_{-\alpha}(\omega, 0) \neq 1$, $-\alpha \in \mathcal{R}^+$, where $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{r : 1 + \mu(t)r(t) > 0, \forall t \in \mathbb{T}\}$.

2 Preliminaries

In this section, we introduce some definitions and state some preliminary results.

Definition 2.1 [6] Let \mathbb{T} be a nonempty closed subset (time scale) of \mathbb{R} . The forward and backward jump operators $\sigma, \rho : \mathbb{T} \rightarrow \mathbb{T}$ and the graininess $\mu : \mathbb{T} \rightarrow \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\} \quad \text{and} \quad \mu(t) = \sigma(t) - t.$$

Definition 2.2 [6] A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If \mathbb{T} has a left-scattered maximum m , then $\mathbb{T}^k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If \mathbb{T} has a right-scattered minimum m , then $\mathbb{T}^k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$.

Definition 2.3 [6] A function $r : \mathbb{T} \rightarrow \mathbb{R}$ is called regressive if

$$1 + \mu(t)r(t) \neq 0$$

for all $t \in \mathbb{T}^k$. If r is a regressive function, then the generalized exponential function e_r is defined by

$$e_r(t, s) = \exp \left\{ \int_s^t \xi_{\mu(\tau)}(r(\tau)) \Delta \tau \right\} \quad \text{for } s, t \in \mathbb{T},$$

with the cylinder transformation

$$\xi_h(z) = \begin{cases} \frac{\text{Log}(1+hz)}{h} & \text{if } h \neq 0, \\ z & \text{if } h = 0. \end{cases}$$

Let $p, q : \mathbb{T} \rightarrow \mathbb{R}$ be two regressive functions, we define

$$p \oplus q := p + q + \mu p q, \quad \ominus p := -\frac{p}{1 + \mu p}, \quad p \ominus q := p \oplus (\ominus q).$$

Then the generalized exponential function has the following properties.

Lemma 2.1 [6] *Assume that $p, q : \mathbb{T} \rightarrow \mathbb{R}$ are two regressive functions, then*

- (i) $e_0(t, s) \equiv 1$ and $e_p(t, t) \equiv 1$;
- (ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
- (iii) $e_p(t, s) = \frac{1}{e_p(s, t)} = e_{\ominus p}(s, t)$;
- (iv) $e_p(t, s)e_p(s, r) = e_p(t, r)$.

Lemma 2.2 [7] *Assume that $f, g : \mathbb{T} \rightarrow \mathbb{R}$ are delta differentiable at $t \in \mathbb{T}^k$, then*

- (i) $(v_1 f + v_2 g)^\Delta = v_1 f^\Delta + v_2 g^\Delta$ for any constants v_1, v_2 ;
- (ii) $(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t))$.

Lemma 2.3 [7] *Assume that $p(t) \geq 0$ for $t \geq s$, then $e_p(t, s) \geq 1$.*

Definition 2.4 [7] A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is positively regressive if $1 + \mu(t)f(t) > 0$ for all $t \in \mathbb{T}$.

Lemma 2.4 [7] *Suppose that $p \in \mathcal{R}^+$, then*

- (i) $e_p(t, s) > 0$ for all $t, s \in \mathbb{T}$;
- (ii) if $p(t) \leq q(t)$ then $e_p(t, s) \leq e_q(t, s)$ for $t, s \in \mathbb{T}$.

Lemma 2.5 [18] *If $p \in \mathcal{R}^+$ and $p(t) < 0$ for all $t \in \mathbb{T}$, then for all $s \in \mathbb{T}$ with $s \leq t$, we have*

$$0 < e_p(t, s) \leq \exp\left(\int_s^t p(u)\Delta u\right) < 1.$$

Lemma 2.6 [8] *Let $x \in \{x \in C(\mathbb{T}, \mathbb{R}) \mid x(t + \omega) = x(t)\}$. Then $\|x^\sigma\|$ exists and $\|x^\sigma\| = \|x\|$, where $\|x\| = \max_{t \in [0, \omega]_{\mathbb{T}}} |x(t)|$.*

Lemma 2.7 [7] *If $p \in \mathcal{R}$ and $a, b, c \in \mathbb{T}$, then*

$$[e_p(c, \cdot)]^\Delta = -p[e_p(c, \cdot)]^\sigma$$

and

$$\int_a^b p(t)e_p(c, \sigma(t))\Delta t = e_p(c, a) - e_p(c, b).$$

Definition 2.5 Let $z^*(t) = (x^*(t), y^*(t))^T$ be a solution of (1.1) and $z(t) = (x(t), y(t))^T$ be an arbitrary solution of (1.1). If there exist positive constants $M > 1$ and λ with $\ominus\lambda \in \mathcal{R}^+$ such that for $t_0 \in \mathbb{T}$,

$$|z(t) - z^*(t)|_1 \leq M \|z - z^*\|_1 e_{\ominus\lambda}(t, t_0), \quad t \in \mathbb{T}, t \geq t_0,$$

where $|z(t)|_1 = \max\{|x(t) - x^*(t)|, |y(t) - y^*(t)|\}$, $\|z - z^*\|_1 = \max\{|z_1 - z_1^*|_0, |z_2 - z_2^*|_0\}$, $|z_i - z_i^*|_0 = \max_{s \in (-\infty, t_0]_{\mathbb{T}}} |z_i(s) - z_i^*(s)|$, $i = 1, 2$, then the solution $z^*(t)$ is said to be exponentially stable.

3 Existence and uniqueness

Lemma 3.1 Let (H_1) hold. Then every ω -periodic solution $(x(t), y(t))^T$ of (1.1) is of the form:

$$\begin{cases} x(t) = \frac{1}{e_{-\alpha}(\omega, 0) - 1} \int_t^{t+\omega} \beta(s)y(s)e_{-\alpha}(s, t)\Delta s, \\ y(t) = \frac{1}{1 - e_{\ominus(-\alpha)}(\omega, 0)} \int_t^{t+\omega} \frac{e_{\ominus(-\alpha)}(s, t)}{1 - \mu(s)\alpha(s)} \gamma(s)x(\sigma(s))\Delta s. \end{cases}$$

Proof Let $(x(t), y(t))^T$ be an ω -periodic solution of (1.1). We can rewrite the first equation of (1.1) as follows:

$$x^\Delta(t) - \alpha(t)x(\sigma(t)) = \beta(t)y(t).$$

Multiplying both sides of the above equation by $e_{-\alpha}(t, 0)$, we have

$$(e_{-\alpha}(t, 0)x(t))^\Delta = e_{-\alpha}(t, 0)\beta(t)y(t).$$

Integrating both sides of this equation from t to $t + \omega$ and noticing that $x(t + \omega) = x(t)$, we have

$$x(t) = \frac{1}{e_{-\alpha}(\omega, 0) - 1} \int_t^{t+\omega} \beta(s)y(s)e_{-\alpha}(s, t)\Delta s.$$

On the other hand, we rewrite the second equation of (1.1) as follows:

$$(1 - \alpha(t)\mu(t))y^\Delta(t) + \alpha(t)y(\sigma(t)) = -\gamma(t)x(\sigma(t)),$$

which is equivalent to

$$y^\Delta(t) + \ominus(-\alpha(t))y(\sigma(t)) = -\frac{\gamma(t)}{1 - \alpha(t)\mu(t)}x(\sigma(t)).$$

Multiplying both sides of the above equation by $e_{\ominus(-\alpha)}(t, 0)$, we have

$$(e_{\ominus(-\alpha)}(t, 0)y(t))^\Delta = -\gamma(t)x(\sigma(t))e_{\ominus(-\alpha)}(\sigma(t), 0).$$

Integrating both sides of this equation from t to $t + \omega$ and noticing that $y(t + \omega) = y(t)$, we have

$$y(t) = \frac{1}{1 - e_{\ominus(-\alpha)}(\omega, 0)} \int_t^{t+\omega} \frac{e_{\ominus(-\alpha)}(s, t)}{1 - \mu(s)\alpha(s)} \gamma(s)x(\sigma(s))\Delta s,$$

which completes the proof. □

Theorem 3.1 Assume that (H_1) and

$(H_2) \min_{s \in [0, \omega]_{\mathbb{T}}} \{1 - \mu(s)\alpha(s)\} \neq 0$ and

$$\theta = \max \left\{ \frac{\beta^+ \omega e^{\int_0^\omega |\xi_{\mu(\tau)}(-\alpha(\tau))| \Delta \tau}}{|e_{-\alpha}(\omega, 0) - 1|}, \frac{\gamma^+ \omega e^{\int_0^\omega |\xi_{\mu(\tau)}(-\alpha(\tau))| \Delta \tau}}{|1 - e_{\ominus(-\alpha)}(\omega, 0)| \min_{s \in [0, \omega]_{\mathbb{T}}} \{1 - \mu(s)\alpha(s)\}} \right\} < 1$$

hold. Then (1.1) has a unique periodic solution.

Proof Let $\mathbb{X} = \{z(t) = (x(t), y(t))^T \in C(\mathbb{T}, \mathbb{R}^2) | x(t + \omega) = x(t), y(t + \omega) = y(t)\}$ with the norm $\|z\| = \max\{|x|_0, |y|_0\}$, where $|x|_0 = \max_{t \in [0, \omega]_{\mathbb{T}}} |x(t)|$ and $|y|_0 = \max_{t \in [0, \omega]_{\mathbb{T}}} |y(t)|$. Then \mathbb{X} is a Banach space. For $z \in \mathbb{X}$, define the following operator:

$$\Phi : \mathbb{X} \rightarrow \mathbb{X}, \quad z = (x, y)^T \rightarrow \Phi z = (\Phi_1 z, \Phi_2 z)^T,$$

where

$$\Phi_1 z(t) = \frac{1}{e_{-\alpha}(\omega, 0) - 1} \int_t^{t+\omega} \beta(s) y(s) e_{-\alpha}(s, t) \Delta s$$

and

$$\Phi_2 z(t) = \frac{1}{1 - e_{\ominus(-\alpha)}(\omega, 0)} \int_t^{t+\omega} \frac{e_{\ominus(-\alpha)}(s, t)}{1 - \mu(s)\alpha(s)} \gamma(s) x(\sigma(s)) \Delta s.$$

We will show that Φ is a contraction. First we show that for any $z \in \mathbb{X}$, we have $\Phi z \in \mathbb{X}$. Note that

$$\begin{aligned} \Phi_1 z(t + \omega) &= \frac{1}{e_{-\alpha}(\omega, 0) - 1} \int_{t+\omega}^{t+2\omega} \beta(s) y(s) e_{-\alpha}(s, t + \omega) \Delta s \\ &= \frac{1}{e_{-\alpha}(\omega, 0) - 1} \int_t^{t+\omega} \beta(s + \omega) y(s + \omega) e_{-\alpha}(s + \omega, t + \omega) \Delta s \\ &= \frac{1}{e_{-\alpha}(\omega, 0) - 1} \int_t^{t+\omega} \beta(s) y(s) e_{-\alpha}(s, t) \Delta s \end{aligned}$$

and

$$\begin{aligned} \Phi_2 z(t + \omega) &= \frac{1}{1 - e_{\ominus(-\alpha)}(\omega, 0)} \int_{t+\omega}^{t+2\omega} \frac{e_{\ominus(-\alpha)}(s, t + \omega)}{1 - \mu(s)\alpha(s)} \gamma(s) x(\sigma(s)) \Delta s \\ &= \frac{1}{1 - e_{\ominus(-\alpha)}(\omega, 0)} \int_t^{t+\omega} \frac{e_{\ominus(-\alpha)}(s + \omega, t + \omega)}{1 - \mu(s + \omega)\alpha(s + \omega)} \gamma(s + \omega) x(\sigma(s + \omega)) \Delta s \\ &= \frac{1}{1 - e_{\ominus(-\alpha)}(\omega, 0)} \int_t^{t+\omega} \frac{e_{\ominus(-\alpha)}(s, t)}{1 - \mu(s)\alpha(s)} \gamma(s) x(\sigma(s)) \Delta s, \end{aligned}$$

which means that $\Phi z \in \mathbb{X}$.

Next, we prove that Φ is a contraction mapping. Together with

$$e_{-\alpha}(s, t) = e^{\int_t^s \xi_{\mu(\tau)}(-\alpha(\tau)) \Delta \tau} \leq e^{\int_t^{t+\omega} |\xi_{\mu(\tau)}(-\alpha(\tau))| \Delta \tau} = e^{\int_0^\omega |\xi_{\mu(\tau)}(-\alpha(\tau))| \Delta \tau}, \quad t \leq s \leq t + \omega, s \in \mathbb{T},$$

for any $z_1(t) = (x_1(t), y_1(t))^T, z_2(t) = (x_2(t), y_2(t))^T \in \mathbb{X}$, we have

$$\begin{aligned} |\Phi_1 z_1 - \Phi_1 z_2|_0 &= \max_{t \in [0, \omega]_{\mathbb{T}}} \left| \frac{1}{e_{-\alpha}(\omega, 0) - 1} \int_t^{t+\omega} (\beta(s)y_1(s)e_{-\alpha}(s, t) - \beta(s)y_2(s)e_{-\alpha}(s, t)) \Delta s \right| \\ &\leq \max_{t \in [0, \omega]_{\mathbb{T}}} \frac{1}{|e_{-\alpha}(\omega, 0) - 1|} \int_t^{t+\omega} e_{-\alpha}(s, t) |\beta(s)| |y_1(s) - y_2(s)| \Delta s \\ &\leq \frac{\omega \beta^+ e^{\int_0^\omega |\xi_{\mu(\tau)}(-\alpha(\tau))| \Delta \tau}}{|e_{-\alpha}(\omega, 0) - 1|} |y_1 - y_2|_0 \end{aligned}$$

and

$$\begin{aligned} |\Phi_2 z_1 - \Phi_2 z_2|_0 &= \max_{t \in [0, \omega]_{\mathbb{T}}} \left| \frac{1}{1 - e_{\ominus(-\alpha)}(\omega, 0)} \int_t^{t+\omega} \frac{e_{\ominus(-\alpha)}(s, t)}{1 - \mu(s)\alpha(s)} \gamma(s) (x_1(\sigma(s)) - x_2(\sigma(s))) \Delta s \right| \\ &\leq \max_{t \in [0, \omega]_{\mathbb{T}}} \frac{1}{|1 - e_{\ominus(-\alpha)}(\omega, 0)|} \int_t^{t+\omega} \frac{e_{\ominus(-\alpha)}(s, t)}{|1 - \mu(s)\alpha(s)|} |\gamma(s)| |x_1(\sigma(s)) - x_2(\sigma(s))| \Delta s \\ &\leq \frac{\omega \gamma^+ e^{\int_0^\omega |\xi_{\mu(\tau)}(-\alpha(\tau))| \Delta \tau}}{|1 - e_{\ominus(-\alpha)}(\omega, 0)| \min_{s \in [0, \omega]_{\mathbb{T}}} \{ |1 - \mu(s)\alpha(s)| \}} |x_1 - x_2|_0. \end{aligned}$$

Hence, we have

$$\|\Phi z_1 - \Phi z_2\| \leq \theta \|z_1 - z_2\| < \|z_1 - z_2\|.$$

It follows that Φ is a contraction. Therefore Φ has a fixed point in \mathbb{X} , that is, (1.1) has a unique periodic solution in \mathbb{X} . This completes the proof. \square

4 Exponential stability of periodic solution

In this section, we study the exponential stability of the periodic solution to (1.1).

Theorem 4.1 *Assume that (H₁) and (H₂) hold. Suppose further that $\alpha^- > \gamma^+$. Then the periodic solution of (1.1) is exponentially stable.*

Proof By Theorem 3.1, one can see that (1.1) has a unique ω -periodic solution $z^*(t) = (x^*(t), y^*(t))^T$. Suppose that $z(t) = (x(t), y(t))^T$ is an arbitrary solution of (1.1). Denote $w(t) = (u(t), v(t))^T$, where $u(t) = x(t) - x^*(t)$, $v(t) = y(t) - y^*(t)$. Then it follows from (1.1) that

$$\begin{cases} u^\Delta(t) = \alpha(t)u(\sigma(t)) + \beta(t)v(t), \\ v^\Delta(t) = -\gamma(t)u(\sigma(t)) - \alpha(t)v(t). \end{cases} \quad (4.1)$$

For $t_0 \in \mathbb{T}$, by Theorem 2.74 and Theorem 2.77 in [8], we have

$$u(t) = e_{\ominus(-\alpha)}(t, t_0)u(t_0) + \int_{t_0}^t e_{\ominus(-\alpha)}(t, s)\beta(s)v(s)\Delta s \quad (4.2)$$

and

$$v(t) = e_{-\alpha}(t, t_0)v(t_0) - \int_{t_0}^t e_{-\alpha}(t, \sigma(s))\gamma(s)u(\sigma(s))\Delta s. \quad (4.3)$$

Take a constant $\lambda > 0$ with $-\lambda \in \mathcal{R}^+$ such that $\lambda > \beta^+(1 + \vartheta\lambda)$ and $\alpha^-\epsilon_1 > \gamma^+$, where $\epsilon_1 = \min_{t \in [0, \omega]_{\mathbb{T}}} e_{\ominus\lambda}(t, t_0)$. Let

$$M > \max \left\{ \frac{\lambda\epsilon_2}{\lambda - \beta^+(1 + \vartheta\lambda)}, \frac{\alpha^-}{\alpha^-\epsilon_1 - \gamma^+} \right\},$$

where $\epsilon_2 = \max_{t \in [0, \omega]_{\mathbb{T}}} e_{\lambda}(t, t_0)$. It is easy to verify that $M > 1$ and hence, we have

$$|w(t)| \leq \|w\|_1 < Me_{\ominus\lambda}(t, t_0)\|w\|_1, \quad \forall t \in (-\infty, t_0]_{\mathbb{T}}.$$

We claim that

$$|w(t)| < Me_{\ominus\lambda}(t, t_0)\|w\|_1, \quad \forall t \in (t_0, +\infty)_{\mathbb{T}}, \tag{4.4}$$

which means that

$$|u(t)| < Me_{\ominus\lambda}(t, t_0)\|w\|_1, \quad \forall t \in (t_0, +\infty)_{\mathbb{T}} \tag{4.5}$$

and

$$|v(t)| < Me_{\ominus\lambda}(t, t_0)\|w\|_1, \quad \forall t \in (t_0, +\infty)_{\mathbb{T}}. \tag{4.6}$$

By way of contradiction, assume that (4.4) does not hold, then we have the following three cases.

Case one: (4.6) is true and (4.5) is not true. Then there exists $t_1 \in (t_0, +\infty)_{\mathbb{T}}$ such that

$$|u(t_1)| \geq Me_{\ominus\lambda}(t_1, t_0)\|w\|_1, \quad |u(t)| < Me_{\ominus\lambda}(t, t_0)\|w\|_1, \quad t \in (t_0, t_1)_{\mathbb{T}}.$$

Hence, there must be a constant $p \geq 1$ such that

$$|u(t_1)| = pMe_{\ominus\lambda}(t_1, t_0)\|w\|_1, \quad |u(t)| < Me_{\ominus\lambda}(t, t_0)\|w\|_1, \quad t \in (t_0, t_1)_{\mathbb{T}}.$$

In view of (4.2), we have

$$\begin{aligned} |u(t_1)| &= \left| e_{\ominus(-\alpha)}(t_1, t_0)u(t_0) + \int_{t_0}^{t_1} e_{\ominus(-\alpha)}(t_1, s)\beta(s)v(s)\Delta s \right| \\ &\leq e_{\ominus(-\alpha)}(t_1, t_0)|u(t_0)| + \int_{t_0}^{t_1} e_{\ominus(-\alpha)}(t_1, s)|\beta(s)||v(s)|\Delta s \\ &\leq e_{\ominus(-\alpha)}(t_1, t_0)\|w\|_1 + \beta^+pM\|w\|_1 \int_{t_0}^{t_1} e_{\ominus(-\alpha)}(t_1, s)e_{\ominus\lambda}(s, t_0)\Delta s \\ &= e_{\ominus(-\alpha)}(t_1, t_0)\|w\|_1 + \beta^+pM\|w\|_1 e_{\ominus\lambda}(t_1, t_0) \int_{t_0}^{t_1} e_{-\alpha\ominus\lambda}(s, t_1)\Delta s \\ &= pM\|w\|_1 e_{\ominus\lambda}(t_1, t_0) \left(\frac{1}{pM} e_{-\alpha\ominus\lambda}(t_0, t_1) + \beta^+ \int_{t_0}^{t_1} e_{-\alpha\ominus\lambda}(s, t_1)\Delta s \right) \\ &\leq pM\|w\|_1 e_{\ominus\lambda}(t_1, t_0) \left(\frac{1}{M} e_{\ominus\lambda}(t_0, t_1) + \beta^+ \int_{t_0}^{t_1} e_{\ominus\lambda}(s, t_1)\Delta s \right) \end{aligned}$$

$$\begin{aligned}
 &= pM\|w\|_1 e_{\ominus\lambda}(t_1, t_0) \left(\frac{1}{M} e_{\ominus\lambda}(t_0, t_1) + \frac{\beta^+}{\ominus\lambda} \int_{t_0}^{t_1} \ominus\lambda e_{\ominus\lambda}(s, t_1) \Delta s \right) \\
 &\leq pM\|w\|_1 e_{\ominus\lambda}(t_1, t_0) \left(\frac{1}{M} e_{\ominus\lambda}(t_0, t_1) + \frac{\beta^+}{\lambda} (1 + \vartheta\lambda) e_{\ominus\lambda}(t_0, t_1) \right) \\
 &< pM\|w\|_1 e_{\ominus\lambda}(t_1, t_0),
 \end{aligned} \tag{4.7}$$

which is a contradiction.

Case two: (4.5) is true and (4.6) is not true. Then there exists $t_2 \in (t_0, +\infty)_{\mathbb{T}}$ such that

$$|v(t_2)| \geq M e_{\ominus\lambda}(t_2, t_0) \|w\|_1, \quad |v(t)| < M e_{\ominus\lambda}(t, t_0) \|w\|_1, \quad t \in (t_0, t_2)_{\mathbb{T}}.$$

In view of (4.3), we have

$$\begin{aligned}
 |v(t_2)| &= \left| e_{-\alpha}(t_2, t_0) v(t_0) - \int_{t_0}^{t_2} e_{-\alpha}(t_2, \sigma(s)) \gamma(s) u(\sigma(s)) \Delta s \right| \\
 &\leq e_{-\alpha}(t_2, t_0) |v(t_0)| + \int_{t_0}^{t_2} e_{-\alpha}(t_2, \sigma(s)) |\gamma(s)| |u(\sigma(s))| \Delta s \\
 &\leq \|w\|_1 + \gamma^+ M \|w\|_1 \int_{t_0}^{t_2} e_{-\alpha^-}(t_2, \sigma(s)) \Delta s \\
 &= \|w\|_1 + \frac{\gamma^+ M \|w\|_1}{-\alpha^-} \int_{t_0}^{t_2} (-\alpha^-) e_{-\alpha}(t_2, \sigma(s)) \Delta s \\
 &= \|w\|_1 + \frac{\gamma^+ M \|w\|_1}{-\alpha^-} (e_{-\alpha}(t_2, t_0) - 1) \\
 &\leq \|w\|_1 + \frac{\gamma^+ M \|w\|_1}{\alpha} \\
 &= M \|w\|_1 e_{\ominus\lambda}(t_2, t_0) \left(\frac{1}{M e_{\ominus\lambda}(t_2, t_0)} + \frac{\gamma^+}{\alpha e_{\ominus\lambda}(t_2, t_0)} \right) \\
 &< M \|w\|_1 e_{\ominus\lambda}(t_2, t_0),
 \end{aligned}$$

which is also a contradiction.

Case three: Both (4.5) and (4.6) are untrue. By case one and case two, we can obtain a contradiction. Therefore, (4.4) holds. Hence, we have that

$$|z(t) - z^*(t)|_1 \leq M \|z - z^*\|_1 e_{\ominus\lambda}(t, t_0), \quad t \in \mathbb{T}, t \geq t_0,$$

which means that the periodic solution $z^*(t)$ of (1.1) is exponentially stable. This completes the proof. \square

By Theorem 2.1 in [16], we have the following corollary.

Corollary 4.1 *Assume that (H₁) and (H₂) hold. Suppose further that $\beta(t) \geq 0$, $\beta(t) \neq 0$, $t \in [0, \omega]_{\mathbb{T}}$ and $\int_0^\omega (\gamma(t) - \frac{\alpha^2(t)}{\beta(t)}) \Delta t > 0$. If*

$$\int_0^\omega \beta(t) \Delta t \int_0^\omega \gamma_+(t) \Delta t < 4 \exp\left(-\int_0^\omega |\xi_{\mu(t)}(-\alpha(t))| \Delta t\right)$$

or

$$\int_0^\omega |\alpha(t)| \Delta t + \left(\int_0^\omega \beta(t) \Delta t \right)^{1/2} \left(\int_0^\omega \gamma_+(t) \Delta t \right)^{1/2} < 2, \quad \gamma_+(t) = \max\{0, \gamma(t)\}.$$

Then (1.1) has a stable ω -periodic solution.

5 Examples

In this section, we present two examples to illustrate the feasibility of our results obtained in previous sections.

Example 5.1 Consider the following Hamiltonian system on $\mathbb{T} = \mathbb{R}$:

$$\begin{cases} x'(t) = \alpha(t)x(t) + \beta(t)y(t), \\ y'(t) = -\gamma(t)x(t) - \alpha(t)y(t), \end{cases} \quad t \in \mathbb{R}, \tag{5.1}$$

in which we take the coefficients as follows:

$$\alpha(t) = 2 + \sin t, \quad \beta(t) = 0.03 \sin t, \quad \gamma(t) = 0.05 \cos t.$$

Since $\mathbb{T} = \mathbb{R}$, then $\mu(t) = 0$. By calculating, we have

$$\alpha^+ = \bar{\alpha} = 3, \quad \alpha^- = 1, \quad \beta^+ = 0.03, \quad \gamma^+ = 0.05,$$

$$\exp\left\{ \int_0^\omega (-\alpha(s)) \, ds \right\} = e^{-4\pi}$$

and $\theta \approx 0.134 < 1$. All the conditions in Theorem 3.1 and Theorem 4.1 are satisfied. Hence, (5.1) has an exponentially stable 2π -periodic solution.

Example 5.2 Consider the following Hamiltonian system on $\mathbb{T} = \mathbb{Z}$:

$$\begin{cases} \Delta x(n) = \alpha(n)x(n+1) + \beta(n)y(n), \\ \Delta y(n) = -\gamma(n)x(n+1) - \alpha(n)y(n), \end{cases} \quad n \in \mathbb{Z}, \tag{5.2}$$

in which we take the coefficients as follows:

$$\alpha(n) = 0.7 + 0.2 \sin \frac{n\pi}{3}, \quad \beta(n) = 0.02 \cos \frac{n\pi}{3}, \quad \gamma(n) = 0.04 \sin \frac{n\pi}{3}.$$

Since $\mathbb{T} = \mathbb{Z}$, then $\mu(t) = 1$. By calculating, we have $\alpha^+ = \bar{\alpha} = 0.9$, $\alpha^- = 0.5$, $\beta^+ = 0.02$, $\gamma^+ = 0.04$ and

$$\frac{|1 - \prod_{k=1}^{\omega-1} (1 - \alpha(k))| \exp\{\sum_0^{\omega-1} \log |1 - \alpha(k)|\}}{|\prod_{k=1}^{\omega-1} (1 - \alpha(k))|} \approx 1.147, \quad \theta \approx 0.1634 < 1.$$

All the conditions in Theorem 3.1 and Theorem 4.1 are satisfied. Hence, (5.2) has an exponentially stable 6-periodic solution.

Remark 5.1 Since in (5.1) and (5.2), $\beta(t)$ or $\beta(n)$ may be negative, Theorem 2.1 in [16] is not suitable for our examples. But from our results, we can obtain that both (5.1) and (5.2) have exponentially stable ω -periodic solutions.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the manuscript and typed, read and approved the final manuscript.

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