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On the stability of a functional equation deriving from additive and quadratic functions

Wang Liguang[*] and Li Jing

[*]Correspondence:
wangliguang0510@163.com
School of Mathematical Sciences,
Qufu Normal University, Qufu,
Shandong 273165, China

Abstract

In this article, we investigate the Hyers-Ulam stability of the following functional equation

$$f(x + 2y) + f(x - 2y) = f(x + y) + f(x - y) + 3f(2y) - 6f(y)$$

on quasi-β-normed spaces.

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1 Introduction

The stability problem of functional equations originated from the following question of Ulam [1] concerning the stability of group homomorphisms:

Give a group $(G_1, *)$ and a metric group $(G_2, \cdot, d)$ with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exists a $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies $d(f(x * y), f(x) \cdot f(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $g : G_1 \to G_2$ with $d(f(x), g(x)) < \varepsilon$ for all $x \in G_1$?

Hyers [2] gave the first affirmative partial answer to the question of Ulam for Banach spaces. Hyers's theorem was generalized by Aoki [3] for additive mappings and by Rassias [4] for linear mappings by considering an unbounded Cauchy difference. The study of Rassias has provided a lot of influence on the development of what we called the generalized Hyers-Ulam-Rassias stability of functional equations. In 1990, Rassias [5] asked whether such a theorem can also be proved for $p \geq 1$. In 1991, Gajda [6] gave an affirmative solution to this question when $p > 1$, but it was proved by Gajda [6] and Rassias and Semrl [7] that one cannot prove an analogous theorem when $p = 1$. In 1994, a generalization was obtained by Gavruta [8] who replaced the bound $\varepsilon(\|x\|^p + \|y\|^p)$ by a general control function $\phi(x, y)$. Beginning around 1980, the stability problems of several functional equations and approximate homomorphisms have extensively been investigated by many authors and there are many interesting results concerning this problem [9–22].

The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{1.1}$$

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is called the quadratic function equation. Function $f(x) = ax^2$ satisfies (1.1). Every solution of (1.1) is called a quadratic mapping. Skof [9] solved the Hyers-Ulam stability problem of the quadratic functional equation in Banach spaces. Kim and Rassias [10] proved the stability of the Euler-Lagrange quadratic mappings. Park [11] considered the stability of quadratic mappings on Banach modules. Moslehian *et al.* [12] considered the approximation problem of quadratic functional equation on multi-normed spaces.

Hyers [2] considered the stability of the additive functional equation

$$f(x + y) = f(x) + f(y). \tag{1.2}$$

Function $f(x) = dx$ satisfies (1.2). Every solution of (1.2) is called an additive mapping. The Cauchy type additive functional equation and its generalized Hyers-Ulam 'product-sum' stability have been studied in [21, 22].

In this article, we consider a new functional equation

$$f(x + 2y) + f(x - 2y) = f(x + y) + f(x - y) + 3f(2y) - 6f(y) \tag{1.3}$$

deriving from the quadratic functional equation (1.1) and the additive functional equation (1.2). It is not difficult to check that $f(x) = ax^2 + bx$ is a solution of (1.3).

The notion of quasi-β-normed space was introduced by Rassias and Kim [20]. This notion is similar to quasi-normed space. We fix a real number β with $0 < \beta \leq 1$ and let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let X be a linear space over $\mathbb{K}$. A quasi-β-norm $\| \cdot \|$ is a real-valued function on X satisfying the following conditions:

(1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.
(2) $\|\lambda x\| = |\lambda|^\beta \|x\|$ for all $\lambda \in \mathbb{K}$ and all $x \in X$.
(3) There is a constant $K \geq 1$ such that $\|x + y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \| \cdot \|)$ is called a quasi-β-normed space if $\| \cdot \|$ is a quasi-β-norm on X. The smallest possible K is called the modulus of concavity of $\| \cdot \|$. A quasi-β-Banach space is a complete quasi-β-normed space.

In the following, we recall some fundamental results in fixed point theory. Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We also recall the following theorem of Diaz and Margolis [23].

Theorem 1.1 [23] *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $0 < L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a nonnegative integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$;
(2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;
(3) y^* is the unique fixed point of J in the set $Y = \{y \in X : d(J^{n_0} x, y) < \infty\}$;

(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

In 2003, Cadariu and Radu [24, 25] applied the fixed-point method to the investigation of the Jensen functional equation. By using fixed point methods, the stability problems of several functional equations have extensively been investigated by a number of authors (see [26–28]).

In this article, we will consider the solution and the Hyers-Ulam stability of the functional equation (1.3) on quasi-β-normed spaces using fixed point method.

2 Solution of (1.3)

We assume X and Y are real (or complex) linear spaces in this section.

Lemma 2.1 *If an even function $f : X \to Y$ satisfies (1.3) for all $x, y \in X$, then f is a quadratic mapping.*

Proof Since f is even, $f(-x) = f(x)$. Let $x = y = 0$ in (1.3), we have $f(0) = 0$. Let $x = 0$ in (1.3), we have $f(2y) = 4f(y)$ and therefore

$$f(x + 2y) + f(x - 2y) = f(x + y) + f(x - y) + 6f(y) \tag{2.1}$$

for all $x, y \in X$. Replace x by $2x$ in (2.1), we have

$$f(2x + 2y) + f(2x - 2y) = f(2x + y) + f(2x - y) + 6f(y) \tag{2.2}$$

for all $x, y \in X$. Replace y and x by x and y in (2.1), respectively, we have

$$f(2x + y) + f(2x - y) = f(x + y) + f(x - y) + 6f(x) + 6f(y) \tag{2.3}$$

for all $x, y \in X$ since f is even and $f(2y) = 4f(y)$. It follows from (2.2) and (2.3) that

$$f(x + y) + f(x - y) = 2f(x) + 2f(y).$$

Hence f is a quadratic mapping. □

Lemma 2.2 *If an odd function $f : X \to Y$ satisfies (1.3) for all $x, y \in X$, then f is an additive mapping.*

Proof Since f is odd, we have $f(-x) = -f(x)$. Let $x = y = 0$ in (1.3) we have $f(0) = 0$. Let $x = 0$ in (1.3), we have

$$f(2y) = 2f(y) \tag{2.4}$$

for all $x, y \in X$. Thus for all $x, y \in X$, we have

$$f(x + 2y) + f(x - 2y) = f(x + y) + f(x - y). \tag{2.5}$$

Replace x by $2x$ in (2.5), we have

$$2f(x + y) + 2f(x - y) = f(2x + y) + f(2x - y). \tag{2.6}$$

Replacing y and x by x and y in (2.5), respectively, we have

$$f(2x + y) - f(2x - y) = f(x + y) - f(x - y). \tag{2.7}$$

It follows from (2.6) and (2.7) that

$$2f(2x - y) = f(x + y) + 3f(x - y). \tag{2.8}$$

Replacing y by $x - y$ in (2.8), we have

$$2f(x + y) = f(2x - y) + 3f(y) \tag{2.9}$$

for all $x, y \in X$. Hence,

$$3f(2x - y) = 3f(x + y) + 3f(x - y) - 3f(y). \tag{2.10}$$

Replacing y by $-y$ in (2.10), we have

$$3f(2x + y) = 3f(x - y) + 3f(x + y) + 3f(y). \tag{2.11}$$

By (2.7), (2.10), and (2.11), we have

$$3f(x + y) - 3f(x - y) = 6f(y). \tag{2.12}$$

Replacing y and x by x and y in (2.12), respectively, we have

$$3f(x + y) + 3f(x - y) = 6f(x). \tag{2.13}$$

It follows from (2.12) and (2.13) that

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in X$. Hence, $f : X \to Y$ is an additive mapping. $\qquad \square$

Theorem 2.1 *A function $f : X \to Y$ satisfying (1.3) for all $x, y \in X$ if and only if there exist a symmetric bi-additive mapping $B : X \times X \to X$ and an additive mapping $A : X \to Y$ such that $f(x) = B(x, x) + A(x)$ for all $x \in X$.*

Proof If there exist an symmetric bi-additive mapping $B : X \times X \to X$ and an additive mapping $A : X \to Y$ satisfying $f(x) = B(x, x) + A(x)$ for all $x \in X$, then it is not difficult to check that

$$f(x + 2y) + f(x - 2y) = f(x + y) + f(x - y) + 3f(2y) - 6f(y)$$

for all $x, y \in X$. Conversely, let

$$f_e(x) = \frac{f(x) + f(-x)}{2},$$

$$f_o(x) = \frac{f(x) + f(-x)}{2}.$$

Then

$$f(x) = f_e(x) + f_o(x)$$

for all $x \in X$. It is not difficult to check that f_e and f_0 satisfying (1.3). It follows from Lemmas 2.1 and 2.2 that f_e and f_0 are quadratic and additive mappings, respectively. Hence, there exist a symmetric bi-additive mapping $B : X \times X \to X$ such that $f_e(x) = B(x, x)$ for all $x \in X$. Let $A(x) = f_o(x)$. Then we have $f(x) = B(x, x) + A(x)$ for all $x \in X$. $\square$

3 The stability of functional equation (1.3)

Now we consider the stability of the functional equation (1.3) using fixed point method.

In this section, we always assume that X is a complex (or real) linear space and Y is a quasi-β-Banach space with norm $\| \cdot \|$. Suppose K is the modulus of concavity of $\| \cdot \|$. For a mapping $f : X \to Y$, we define

$$Df(x, y) = f(x + 2y) + f(x - 2y) - f(x + y) - f(x - y) - 3f(2y) + 6f(y)$$

for all $x, y \in X$.

Theorem 3.1 *Suppose $f : X \to Y$ is an odd mapping and $\varphi : X^2 \to [0, \infty)$ is a mapping. If there exist a constant L ($0 < L < 1$) satisfying*

$$\left\| Df(x, y) \right\| \leq \varphi(x, y), \tag{3.1}$$

$$\varphi(2x, 2y) \leq 2^{\beta} L \varphi(x, y) \tag{3.2}$$

for all $x, y \in X$, then there is a unique additive mapping $A : X \to Y$ such that

$$\left\| f(x) - A(x) \right\| \leq \frac{1}{6^{\beta}(1 - L)} \varphi(0, x) \tag{3.3}$$

for all $x \in X$. The mapping $A : X \to Y$ is defined by

$$A(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n}. \tag{3.4}$$

Proof Consider the set $\Omega = \{g : X \to Y\}$ and define a generalized metric on Ω by

$$d(g, h) = \inf \left\{ C : C \in \mathbb{R}, C \geq 0, \left\| g(x) - h(x) \right\| \leq C\varphi(0, x), x \in X \right\}.$$

Then it is not difficult to check that (Ω, d) is complete. Consider the mapping $\Lambda : \Omega \to \Omega$ defined by

$$\Lambda g(x) = \frac{1}{2} g(2x). \tag{3.5}$$

For any C such that $\|g(x) - h(x)\| \le C\varphi(0, x)$ for all $x \in X$, we have

$$\left\|\Lambda g(x) - \Lambda h(x)\right\| = \frac{1}{2^\beta}\left\|g(2x) - h(2x)\right\| \le \frac{1}{2^\beta}C\varphi(0, 2x) \le LC\varphi(0, x).$$

Hence $d(\Lambda g, \Lambda h) \le LC$ and therefore $d(\Lambda g, \Lambda h) \le Ld(g, h)$. Let $x = 0$ in (3.1), we have

$$\left\|3f(2y) - 6f(y)\right\| \le \varphi(0, y) \tag{3.6}$$

for all $y \in X$. Let $y = x$ in (3.6), we have

$$\left\|\frac{1}{2}f(2x) - f(x)\right\| \le \frac{1}{6^\beta}\varphi(0, x) \tag{3.7}$$

for $x \in X$. Thus $d(f, \Lambda f) \le \frac{1}{6^\beta}$. According to Theorem 1.1, the sequence $\Lambda^n f$ converges to a unique fixed point A of Λ on the set $Y = \{g \in X : d(f, g) < \infty\}$, *i.e.*,

$$A(x) = \lim_{n \to \infty} \Lambda^n f(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n}.$$

Also we have

$$A(2x) = 2A(x)$$

for all $x \in X$. Since $d(f, A) \le \frac{1}{6^\beta(1-L)}$, we have

$$\left\|f(x) - A(x)\right\| \le \frac{1}{6^\beta(1 - L)}\varphi(0, x)$$

for all $x \in X$ and (3.3) holds true. For all $x, y \in X$, we have

$$\begin{aligned}
\left\|DA(x, y)\right\| &= \lim_{n \to \infty}\left\|\frac{1}{2^n}Df\left(2^n x, 2^n y\right)\right\| = \lim_{n \to \infty}\frac{1}{2^{n\beta}}\left\|Df\left(2^n x, 2^n y\right)\right\| \\
&\le \lim_{n \to \infty}\frac{1}{2^{n\beta}}\varphi\left(2^n x, 2^n y\right) \le \lim_{n \to \infty}\frac{(2^\beta L)^n}{2^{n\beta}}\varphi(x, y) \\
&= \lim_{n \to \infty} L^n \varphi(x, y) = 0.
\end{aligned}$$

Hence, $DA(x, y) = 0$ for all $x, y \in X$. It follows from Lemma 2.2 that $A : X \to Y$ is an additive mapping. This completes the proof. □

Corollary 3.1 *Suppose X is a normed linear space, $\beta = 1$, θ and r are nonnegative numbers with $r < 1$, $f : X \to Y$ is an odd mapping and*

$$\left\|Df(x, y)\right\| \le \theta\left(\|x\|^r + \|y\|^r + \|x\|^{\frac{r}{2}}\|y\|^{\frac{r}{2}}\right)$$

for all $x, y \in X$. Then there is a unique additive mapping $A : X \to Y$ such that

$$\left\|f(x) - A(x)\right\| \le \frac{\theta\|x\|^r}{6(1 - 2^{r-1})}$$

for all $x \in X$.

Proof For all $x, y \in X$, let $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r + \|x\|^{\frac{r}{2}}\|y\|^{\frac{r}{2}})$. Then the results follows from Theorem 3.1. $\qquad\square$

Similar to the proof of Theorem 3.1 and Corollary 3.2, we have Theorem 3.2 and Corollary 3.2 whose proofs are omitted.

Theorem 3.2 *Suppose* $f : X \to Y$ *is an odd mapping,* $\varphi : X^2 \to [0, \infty)$ *is a mapping and there exist a constant L ($0 < L < 1$) such that*

$$\|Df(x, y)\| \leq \varphi(x, y),$$
$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \leq \frac{L}{2^\beta}\varphi(x, y)$$

for all $x, y \in X$. *Then there exists a unique additive mapping* $A : X \to Y$ *such that*

$$\|f(x) - A(x)\| \leq \frac{1}{6^\beta(1 - L)}\varphi(0, x)$$

for all $x \in X$. *The mapping* $A : X \to Y$ *is defined by*

$$A(x) = \lim_{n \to \infty} 2^n f\left(\frac{x}{2^n}\right).$$

Corollary 3.2 *Suppose* X *is a normed linear space,* $\beta = 1$, θ *and* r *are nonnegative numbers with* $r > 1$, $f : X \to Y$ *is an odd mapping and*

$$\|Df(x, y)\| \leq \theta\left(\|x\|^r + \|y\|^r + \|x\|^{\frac{r}{2}}\|y\|^{\frac{r}{2}}\right)$$

for all $x, y \in X$. *Then there is a unique additive mapping* $A : X \to Y$ *such that*

$$\|f(x) - A(x)\| \leq \frac{\theta \|x\|^r}{6(1 - 2^{1-r})} \tag{3.8}$$

for all $x \in X$.

Theorem 3.3 *Suppose* $f : X \to Y$ *is an even mapping with* $f(0) = 0$ *and* $\varphi : X^2 \to [0, \infty)$ *is a mapping. If there exists a constant L ($0 < L < 1$) such that*

$$\|Df(x, y)\| \leq \varphi(x, y),$$
$$\varphi(2x, 2y) \leq 2^{2\beta}L\varphi(x, y) \tag{3.9}$$

for all $x, y \in X$, *then there is a unique quadratic mapping* $Q : X \to Y$ *such that*

$$\|f(x) - Q(x)\| \leq \frac{1}{2^{2\beta}(1 - L)}\varphi(0, x) \tag{3.10}$$

for all $x \in X$. *The mapping* $Q : X \to Y$ *is defined by*

$$Q(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^{2n}}$$

for all $x \in X$.

Proof Consider the $\Omega = \{g : X \to Y : g(0) = 0\}$ and define a generalized metric on Ω by

$$d(g, h) = \inf\{C : C \in \mathbb{R}, C \geq 0, \|g(x) - h(x)\| \leq C\varphi(0, x), \forall x \in X\}.$$

Then it is easy to check that (Ω, d) is complete. Define $\Lambda : \Omega \to \Omega$ by $\Lambda g(x) = \frac{g(2x)}{2^2}$. If C is a constant such that $\|g(x) - h(x)\| \leq C\varphi(0, x)$ for all $x \in X$, then we have

$$\|\Lambda g(x) - \Lambda h(x)\| = \frac{1}{2^{2\beta}}\|g(2x) - h(2x)\| \leq \frac{1}{2^{2\beta}}C\varphi(0, 2x) \leq LC\varphi(0, x),$$

i.e., $d(\Lambda g, \Lambda h) \leq LC$, hence we have $d(\Lambda g, \Lambda h) \leq Ld(g, h)$. Let $x = 0$ in (3.9), we have

$$\|f(2y) - 4f(y)\| \leq \varphi(0, y)$$

for all $y \in X$. Thus

$$\left\|\frac{1}{4}f(2x) - f(x)\right\| \leq \frac{1}{4^{\beta}}\varphi(0, x)$$

for all $x \in X$. Hence $d(f, \Lambda f) \leq \frac{1}{4^{\beta}}$. By Theorem 1.1 there is a mapping $Q : X \to Y$ which is the fixed point of Λ and satisfies

$$d(f, Q) \leq \frac{1}{1 - L}d(\Lambda f, f) \leq \frac{1}{4^{\beta}(1 - L)}.$$

Note Q is defined by

$$Q(x) = \lim_{n \to \infty}\frac{f(2^n x)}{2^{2n}}$$

for all $x \in X$. Similar to the proof of Theorem 3.1, we have $DQ(x, y) = 0$ for all $x, y \in X$. Since Q is also even, Q is a quadratic mapping. This completes the proof. $\square$

Corollary 3.3 *Suppose X is a normed linear space, $\beta = 1$, θ and r are nonnegative numbers with $r < 2$, $f : X \to Y$ is an even mapping, $f(0) = 0$ and*

$$\|Df(x, y)\| \leq \theta\left(\|x\|^r + \|y\|^r + \|x\|^{\frac{r}{2}}\|y\|^{\frac{r}{2}}\right)$$

for all $x, y \in X$. Then there is a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{\theta\|x\|^r}{4(1 - 2^{r-2})}$$

for all $x \in X$.

Proof The proof is similar to that of Corollary 3.1 and we omit it. $\square$

Similar to the proof of Theorem 3.3 and Corollary 3.3, we have the following Theorem 3.4 and Corollary 3.4.

Theorem 3.4 *Suppose $f : X \to Y$ is an even mapping with $f(0) = 0$ and $\varphi : X^2 \to [0, \infty)$ is a mapping. If there exists a constant L $(0 < L < 1)$ such that*

$$\left\| Df(x, y) \right\| \leq \varphi(x, y),$$

$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \leq \frac{L}{2^{2\beta}} \varphi(x, y)$$

for all $x, y \in X$, then there is a unique quadratic mapping $Q : X \to Y$ such that

$$\left\| f(x) - Q(x) \right\| \leq \frac{1}{2^{2\beta}(1 - L)} \varphi(0, x)$$

for all $x \in X$. The mapping $Q : X \to Y$ is defined by

$$Q(x) = \lim_{n \to \infty} 2^{2n} f\left(\frac{x}{2^n}\right)$$

for all $x \in X$.

Corollary 3.4 *Suppose X is a normed linear space, $\beta = 1$, θ and r are nonnegative numbers with $r > 2$, $f : X \to Y$ is an even mapping, $f(0) = 0$ and*

$$\left\| Df(x, y) \right\| \leq \theta \left(\|x\|^r + \|y\|^r + \|x\|^{\frac{r}{2}} \|y\|^{\frac{r}{2}} \right)$$

for all $x, y \in X$. Then there is a unique quadratic mapping $Q : X \to Y$ such that

$$\left\| f(x) - Q(x) \right\| \leq \frac{\theta \|x\|^r}{4(1 - 2^{2-r})}$$

for all $x \in X$.

Theorem 3.5 *Suppose $f : X \to Y$ is a mapping satisfying $f(0) = 0$ and $\varphi : X^2 \to [0, \infty)$ is a mapping. If there exists a constant L $(0 < L < 1)$ such that*

$$\left\| Df(x, y) \right\| \leq \varphi(x, y),$$

$$\varphi(2x, 2y) \leq 2^{\beta} L \varphi(x, y)$$

for all $x, y \in X$, then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that

$$\left\| f(x) - A(x) - Q(x) \right\| \leq \frac{K^2}{2^{\beta}} \left(\frac{1}{2^{2\beta}(1 - L)} + \frac{1}{6^{\beta}(1 - L)} \right) \left[\varphi(0, x) + \varphi(0, -x) \right] \tag{3.11}$$

for all $x \in X$.

Proof If we decompose f into the even and the odd parts by putting $f_e(x) = \frac{f(x) + f(-x)}{2}$ and $f_o(x) = \frac{f(x) - f(-x)}{2}$ for all $x \in X$, then

$$f(x) = f_e(x) + f_o(x).$$

Then if it is not difficult to check that

$$\|Df_e(x, y)\| \leq \frac{K}{2^\beta}[\varphi(x, y) + \varphi(-x, -y)]$$

for all $x, y \in X$. Let

$$\psi(x, y) = \frac{K}{2^\beta}[\varphi(x, y) + \varphi(-x, -y)]$$

for all $x, y \in X$. Then $\psi(2x, 2y) \leq 2^{2\beta} L\psi(x, y)$ for all $x, y \in X$. It follows from Theorem 3.3 that there is a unique quadratic mapping $Q : X \to Y$ such that

$$\|f_e(x) - Q(x)\| \leq \frac{1}{2^{2\beta}(1 - L)}\psi(0, x) \tag{3.12}$$

for all $x \in X$. Similarly it follows from Theorem 3.1 that there is a unique additive mapping $A : X \to Y$ such that

$$\|f_0(x) - A(x)\| \leq \frac{1}{6^\beta(1 - L)}\psi(0, x). \tag{3.13}$$

Hence

$$\|f(x) - Q(x) - A(x)\| \leq K\left(\frac{1}{2^{2\beta}(1 - L)} + \frac{1}{6^\beta(1 - L)}\right)\psi(0, x)$$

$$= \frac{K^2}{2^\beta}\left(\frac{1}{2^{2\beta}(1 - L)} + \frac{1}{6^\beta(1 - L)}\right)[\varphi(0, x) + \varphi(0, -x)]$$

for all $x \in X$. This completes the proof. □

Corollary 3.5 *Suppose X is a normed linear space, $\beta = 1$, θ, and r are nonnegative numbers with $r < 1$, $f : X \to Y$ is a mapping and*

$$\|Df(x, y)\| \leq \theta\left(\|x\|^r + \|y\|^r + \|x\|^{\frac{r}{2}}\|y\|^{\frac{r}{2}}\right)$$

for all $x, y \in X$. Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - A(x) - Q(x)\| \leq \frac{5K^2}{12}\frac{\theta\|x\|^r}{(1 - 2^{r-1})}$$

for all $x \in X$.

The proofs of Theorem 3.6 and Corollary 3.6 are similar to that of Theorem 3.5 and Corollary 3.5 and we omit them.

Theorem 3.6 *Suppose $f : X \to Y$ is a mapping satisfying $f(0) = 0$ and $\varphi : X^2 \to [0, \infty)$ is a mapping. If there exists a constant L ($0 < L < 1$) such that*

$$\|Df(x, y)\| \leq \varphi(x, y),$$

$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \leq \frac{L}{2^{2\beta}}\varphi(x, y)$$

for all $x, y \in X$, *then there exist a unique additive mapping* $A : X \to Y$ *and a unique quadratic mapping* $Q : X \to Y$ *such that*

$$\left\| f(x) - A(x) - Q(x) \right\| < \frac{K^2}{2^\beta}\left(\frac{1}{6^\beta(1-L)} + \frac{1}{2^{2\beta}(1-L)}\right)\left[\varphi(0, x) + \varphi(0, -x)\right]$$

for all $x \in X$.

Corollary 3.6 *Suppose* X *is a normed linear space,* $\beta = 1$, θ *and* r *are nonnegative numbers with* $r > 2$, $f : X \to Y$ *is a mapping and*

$$\left\| Df(x, y) \right\| \leq \theta\left(\|x\|^r + \|y\|^r + \|x\|^{\frac{r}{2}}\|y\|^{\frac{r}{2}}\right)$$

for all $x, y \in X$. *Then there exist a unique additive mapping* $A : X \to Y$ *and a unique quadratic mapping* $Q : X \to Y$ *such that*

$$\left\| f(x) - A(x) - Q(x) \right\| \leq \frac{5K^2}{12}\frac{\theta\|x\|^r}{(1 - 2^{2-r})}$$

for all $x \in X$.

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