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Existence and dimension of the set of mild solutions to semilinear fractional differential inclusions

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Abstract

This article studies the existence and dimension of the set for mild solutions of semilinear fractional differential inclusions. We recall and prove some new results on multivalued maps to establish our main results.

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1 Introduction

The study of fractional calculus (differentiation and integration of arbitrary order) has emerged as an important and popular field of research. It is mainly due to the extensive application of fractional differential equations in many engineering and scientific disciplines such as physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, fitting of experimental data, etc., [1-7]. Fractional derivatives are also regarded as an excellent tool for the description of memory and hereditary properties of various materials and processes [8]. Owing to these characteristics of fractional derivatives, fractional-order models are considered to be more realistic and practical than the classical integer-order models, in which such effects are not taken into account. A variety of results on initial and boundary value problems of fractional differential equations, ranging from the theoretical aspects of existence and uniqueness of solutions to the analytic and numerical methods for finding solutions, have appeared in the literature, for instance, see [9-20] and references therein.

Differential inclusions arise in the mathematical modeling of certain problems in economics, optimal control, etc., and are widely studied by many authors, see [21-23] and the references therein. For some recent development on differential inclusions of fractional order, we refer the reader to the references [24-29].

In this article, we discuss the existence and dimension of the set for the mild solutions of the following inclusion problem

$$\begin{cases} {}^cD^q x(t) \in Ax(t) + F(t, x(t)), \; t \in [0, T], \; 0 < q \le 1, T > 0, \\ \qquad\qquad x(0) + g(x) = x_0, x_0 \in \mathbb{R}^n, \end{cases} \tag{1}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, A is a sectorial operator on $\mathbb{R}^n$, g: $C([0, T], \mathbb{R}^n) \to \mathbb{R}^n$, and F: $[0, T] \times \mathbb{R}^n \to P(\mathbb{R}^n)$, where $P(\mathbb{R}^n)$ is the family of all nonempty subsets of $\mathbb{R}^n$.

2 Background material

Let us recall some basic definitions on multi-valued maps (for details, see [30,31]).

Let (X, d) be a metric space. Define $P(X) = \{Y \subseteq X: Y \neq \emptyset\}$, $P_{cl}(X) = \{Y \in P(X): Y$ is closed$\}$, $P_b(X) = \{Y \in P(X): Y$ is bounded$\}$, $P_{b,cl}(X) = \{Y \in P(X): Y$ is closed and bounded$\}$ and $P_{cp}(X) = \{Y \in P(X): Y$ is compact$\}$*:*

Consider H: $P(X) \times P(X) \to \mathbb{R} \cup \{\infty\}$ given by

$$H(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\},$$

where $d(a, B) = \inf_{b\in B} d(a, b)$. H is the (generalized) Pompeiu-Hausdorff functional. It is known that $(P_{b,cl}(X), H)$ is a metric space and $(P_{cl}(X), H)$ is a generalized metric space (see [30]).

A multivalued operator Ω: $X \to P_{cl}(X)$ is called a k-contraction if there exists $0 < k < 1$ such that

$$H(\Omega(x), \Omega(y)) \leq kd(x, y) \text{for each } x, y \in X.$$

Let C be a subset of X. A multi-valued map Ω: $C \to P(X)$ is called upper semi-continuous (u.s.c.) if $\{x \in C: \Omega(x) \subset V\}$ is open in C whenever $V \subset X$ is open. Ω is called lower semi-continuous (l.s.c.) if the set $\{y \in C: \Omega(y) \cap V \neq \emptyset\}$ is open for any open set $V \subset X$. Ω is called continuous if it is both l.s.c. and u.s.c. It is known that Ω: $X \to P_{cp}(X)$ is continuous on X if and only if Ω is continuous on X with respect to Hausdorff metric. Also, if Ω: $X \to P_{cp}(X)$ is a k-contraction, then Ω is continuous with respect to Hausdorff metric. Ω is said to be completely continuous if $\Omega(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in P_b(C)$. A mapping f: $C \to X$ is called a selection of Ω if $f(x) \in \Omega(x)$ for every $x \in C$. We say that the mapping Ω has a fixed point if there is $x \in X$ such that $x \in \Omega(x)$. The fixed points set of the multivalued operator F will be denoted by Fix (Ω). A multivalued map Ω: $[0, T] \to P_{cl}(\mathbb{R}^n)$ is said to be measurable if for every $y \in \mathbb{R}^n$, the function

$$t \mapsto d(y, \Omega(t)) = \inf\{||y - z|| : z \in \Omega(t)\}$$

is measurable.

Let $\mathfrak{C}([0, T], \mathbb{R}^n])$ denotes the Banach space of continuous functions from $[0, T]$ into $\mathbb{R}^n$ with the norm $\|x\|_\infty = \sup_{t\in[0,T]} \|x(t)\|$. Let $L^1([0, T], \mathbb{R}^n)$ be the Banach space of measurable functions x: $[0, T] \to \mathbb{R}^n$ which are Lebesgue integrable and normed by $||x||_{L^1} = \int_0^T ||x(t)||dt$. Let C be a nonempty subset of a Banach space $X: = (X, \|.\|)$. Define $P_{c,cl}(C) = \{Y \in P(C): Y$ is convex and closed$\}$, and $P_{c,cp}(C) = \{Y \in P(C): Y$ is compact and convex$\}$.

Let us recall some definitions on fractional calculus. For more details, we refer to [1,4].

Definition 2.1. For at least n-times continuously differentiable function g: $[0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$$^{c}D^{q}g(t) = \frac{1}{\Gamma(n-q)}\int_{0}^{t}(t-s)^{n-q-1}g^{(n)}(s)ds,\ n-1<q<n,\ n=[q]+1,\ q>0,$$

where $[q]$ denotes the integer part of the real number q and Γ denotes the gamma function.

Definition 2.2. The Riemann-Liouville fractional integral of order q for a continuous function g is defined as

$$I^{q}g(t) = \frac{1}{\Gamma(q)}\int_{0}^{t}\frac{g(s)}{(t-s)^{1-q}}ds, q>0,$$

provided the right-hand side is pointwise defined on $(0, \infty)$.

3 Main results

Definition 3.1. Let $A: D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a closed linear operator. A is said to be a sectorial operator of type (M, θ, μ) if there exist $0 < \theta < \pi/2$, $M > 0$, $\mu \in \mathbb{R}$ such that the resolvent of A exists outside the sector

$$\mu + S_{\theta} = \{\mu + \lambda : \lambda \in \mathbb{C},\ \mathrm{Arg}(-\lambda)| < \theta\}$$

with

$$||(\lambda I - A)^{-1}|| \le \frac{M}{|\lambda - \mu|},\ \lambda \notin \mu + S_{\theta}.$$

To define mild solutions for (1), we consider the Cauchy problem

$$\begin{cases} {}^{c}D^{q}x(t) = Ax(t) + \sigma(t),\ t \in [0, T],\ 0 < q \le 1,\ T > 0, \\ \qquad x(0) + g(x) = x_0,\ x_0 \in \mathbb{R}^n, \end{cases} \tag{2}$$

where $\sigma: [0, T] \to \mathbb{R}^n$.

The following lemma is discussed in [32]. However, for the sake of completeness, we outline its proof here.

Lemma 3.2. Let A be a sectorial operator of type (M, θ, μ). If σ satisfies a uniform Hölder condition with exponent $\beta \in (0, 1]$, then the unique solution of the Cauchy problem (2) is given by

$$x(t) = S_q(t)(x_0 - g(x)) + \int_{0}^{t} T_q(t-s)\sigma(s)ds, \tag{3}$$

where

$$S_q(t) = \frac{1}{2\pi i}\int_{\mathcal{P}} e^{\lambda t}\lambda^{q-1}R(\lambda^q, A)d\lambda,\ T_q(t) = \frac{1}{2\pi i}\int_{\mathcal{P}} e^{\lambda t}R(\lambda^q, A)d\lambda,$$

where $\mathcal{P}$ is a suitable path such that $\lambda^q \notin \mu + S_{\theta}$ for $\lambda \in \mathcal{P}$ and $R(\lambda^q, A) = (\lambda^q I - A)^{-1}$.

Proof. Taking inverse Laplace transform of (2), we get

$$\lambda^{q}(\mathcal{L}x)(\lambda) - \lambda^{q-1}(x_0 - g(x)) = A(\mathcal{L}x)(\lambda) + (\mathcal{L}\sigma)(\lambda),$$

which implies that

$$(\mathcal{L}x)(\lambda) = \lambda^{q-1}(\lambda^q I - A)^{-1}(x_0 - g(x)) + (\lambda^q I - A)^{-1}(\mathcal{L}\sigma)(\lambda). \tag{4}$$

By taking inverse Laplace transform of (4), we obtain (3). This completes the proof.

It has been shown in [32] that

$$\sup_{t\in[0,T]} ||S_q(t)|| = \widetilde{M}_S, \ ||T_q|| \le t^{q-1}\widetilde{M}_T, \tag{5}$$

with $\widetilde{M}_S = \sup_{t\in[0,T]}||S_q(t)||_{L(\mathbb{R}^n)}$, and $\widetilde{M}_T = \sup_{t\in[0,T]}Ce^{\mu t}(1+t^{1-q})$, where $L(\mathbb{R}^n)$ is the Banach space of bounded linear operators from $\mathbb{R}^n$ into $\mathbb{R}^n$ equipped with natural topology and C, μ are appropriate constants (for more details see Equation (3.1) in [32]).

Remark 3.3. The definition of the mild solution used in [33] is not appropriate as it does not correspond to the classical case due to the failure of the Leibniz product rule for the Caputo fractional derivative. For more details, see [32].

Definition 3.4. A function $x \in \mathfrak{C}([0, T], \mathbb{R}^n])$ is a mild solution of the problem (1) if there exists a function $f \in L,^1([0, T], \mathbb{R}^n)$ such that $f(t) \in F(t, x(t))$ a.e. on $[0, T]$ and

$$x(t) = S_q(t)(x_0 - g(x)) + \int_0^t T_q(t-s)f(s)ds.$$

Let $\mathcal{S}_{x_0}([0, \alpha])$ denotes the set of all solutions of (1) on the interval $[0, \alpha]$, where $0 < \alpha \le T$.

To prove the existence of solutions for (1), we need the following lemma due to Nadler and Covitz [34].

Lemma 3.5. Let (X, d) be a complete metric space. If $\Omega: X \to P_{cl}(X)$ is a k-contraction, then $\mathrm{Fix}(\Omega) \neq \varnothing$.

Theorem 3.6. Assume that

$(\mathbf{A_1})$ F: $[0, T] \times \mathbb{R}^n \to P_{cp}(\mathbb{R}^n)$ is such that $F(., x)$: $[0, T] \to P_{c,\ cp}(\mathbb{R}^n)$ is measurable for each $x \in \mathbb{R}^n$;

$(\mathbf{A_2})$ $H(F(t, x), F(t,\bar{x})) \le \kappa_1(t)||x - \bar{x}||$ for almost all $t \in [0, T]$ and $x, \bar{x} \in \mathbb{R}^n$ with $\kappa_1 \in \mathfrak{C}([0,T],\mathbb{R}_+)$ and $\|F(t, x)\| = \sup\{\|v\|: v \in F(t, x)\} \le \kappa_1(t)$ for almost all $t \in [0, T]$ and $x \in \mathbb{R}^n$;

$(\mathbf{A_3})$ g: $\mathfrak{C}([0, T], \mathbb{R}^n) \to \mathbb{R}^n$ is continuous and $\|g(x) - g(y)\| \le \kappa_2\|x - y\|_\infty$ for all x, $y \in \mathfrak{C}([0, T], \mathbb{R}^n)$ with some $\kappa_2 > 0$.

Then the Cauchy problem (1) has at least one solution on $[0, T]$ if

$$(\widetilde{M}_S\kappa_2 + \widetilde{M}_T(T^q/q)||\kappa_1||_\infty) < 1$$

($\widetilde{M}_S$ and $\widetilde{M}_T$ are given by (5)).

Proof. For each $y \in \mathfrak{C}([0, T], \mathbb{R}^n)$, define the set of selections of F by

$$S_{F,y} := \{v \in L^1([0,T],\mathbb{R}^n) : v(t) \in F(t, y(t)) \text{ for a.e. } t \in [0,T]\}.$$

Observe that by the assumptions $(\mathbf{A_1})$ and $(\mathbf{A_2})$, $F(t, x(t))$ is measurable and has a measurable selection $v(t)$ (see [[35], Theorem III.6]). Also $\kappa_1 \in \mathfrak{C}([0, T], \mathbb{R}_+)$ and

$$||v(t)|| \le ||F(t, x(t))|| \le \kappa_1(t).$$

Thus the set $S_{F,x}$ is nonempty for each $x \in \mathfrak{C}([0, T], \mathbb{R}^n)$. Let us define an operator Ω by

$$\Omega(x) = \left\{ h \in \mathfrak{C}([0,T],\mathbb{R}^n) : h(t) = S_q(t)(x_0 - g(x)) + \int_0^t T_q(t-s)f(s)ds, f \in S_{F,x} \right\},$$

and show that it satisfies the conditions of Lemma 3.5. As a first step, we show that $\Omega(x) \in P_{cl}(\mathfrak{C}([0, T], \mathbb{R}^n))$ for each $x \in \mathfrak{C}([0, T], \mathbb{R}^n)$. Let $\{u_n\}_{n\geq 0} \in \Omega(x)$ be such that $u_n \to u$ $(n \to \infty)$ in $\mathfrak{C}([0, T], \mathbb{R}^n)$. Then $u \in \mathfrak{C}([0, T], \mathbb{R}^n)$ and there exists $v_n \in S_{F,x}$ such that, for each $t \in [0, T]$,

$$u_n(t) = S_q(t)(x_0 - g(x)) + \int_0^t T_q(t-s)v_n(s)ds$$

As F has compact values, we pass to a subsequence to obtain that v_n converges to v in $\mathfrak{C}([0, T], \mathbb{R}^n)$. Thus, $v \in S_{F,x}$ and for each $t \in [0, T]$,

$$u_n(t) \to u(t) = v(t).$$

Hence $u \in \Omega(x)$.

Next we show that there exists a $\gamma \in (0, 1)$ such that

$$H(\Omega(x), \Omega(\bar{x})) \leq \gamma \|x - \bar{x}\|_\infty \text{ for each } x, \bar{x} \in \mathfrak{C}([0,T],\mathbb{R}^n).$$

Let $x, \bar{x} \in \mathfrak{C}([0,T],\mathbb{R}^n)$ and $h_1 \in \Omega(x)$. Then there exists $v_1(t) \in S_{F,x}$ such that, for each $t \in [0, T]$,

$$h_1(t) = S_q(t)\,(x_0 - g(x)) + \int_0^t T_q(t-s)v_1(s)ds.$$

By $(\mathbf{A_2})$, we have

$$H(F(t,\ x),\ F(t,\bar{x})) \leq \kappa_1(t)||x(t) - \bar{x}(t)||.$$

So, there exists $w \in F(t, \bar{x}(t))$ such that

$$||v_1(t) - w|| \leq \kappa_1(t)||x(t) - \bar{x}(t)||, \quad t \in [0,\ T].$$

Define V: $[0, T] \to P(\mathbb{R}^n)$ by

$$V(t) = \{w \in \mathbb{R}^n : ||v_1(t) - w|| \leq \kappa_1(t)||x(t) - \bar{x}(t)||\}$$

Since the nonempty closed valued operator $V(t) \cap F(t, \bar{x}(t))$ is measurable [[35], Proposition III.4], there exists a function $v_2(t)$ which is a measurable selection for $V(t) \cap F(t, \bar{x}(t))$. So $v_2(t) \in F(t, \bar{x}(t))$ and for each $t \in [0, T]$, we have $\|v_1(t) - v_2(t)\| \leq \kappa_1(t)\,\|x(t) - \bar{x}(t)\|$. For each $t \in [0, T]$, let us define

$$h_2(t) = S_q(t)\,(x_0 - g(x)) + \int_0^t T_q(t-s)v_2(s)ds.$$

Thus

$$||h_1(t) - h_2(t)|| \leq ||S_q(t)||||g(x) - g(\bar{x}))|| + \int_0^t \|T_q(s-t)\|\,\|v_1(s) - v_2(s)\|\,ds.$$

In view of (5), it follows that

$$\begin{aligned}||h_1 - h_2||_{\infty} &\le \widetilde{M}_S \kappa_2 ||x - \bar{x}||_{\infty} + \widetilde{M}_T (T^q/q) ||\kappa_1||_{\infty} ||x - \bar{x}||_{\infty} \\ &= (\widetilde{M}_S \kappa_2 + \widetilde{M}_T (T^q/q) ||\kappa_1||_{\infty}) ||x - \bar{x}||_{\infty}.\end{aligned}$$

Analogously, interchanging the roles of x and $\bar{x}$, we obtain

$$H(\Omega(x), \ \Omega(\bar{x})) \le \gamma ||x - \bar{x}||_{\infty} \text{for each } x, \ \bar{x} \in \mathfrak{C}([0, \ T], \ \mathbb{R}^n),$$

where $\gamma = (\widetilde{M}_S \kappa_2 + \widetilde{M}_T (T^q/q) ||\kappa_1||_{\infty}) < 1$. Since Ω is a contraction, it follows by Lemma 3.5 that Ω has a fixed point x which is a solution of (1). This completes the proof.

Lemma 3.7. Let F: $[0, T] \times \mathbb{R}^n \to P_{c, \ cp}(\mathbb{R}^n)$ satisfy ($\mathbf{A_1}$), ($\mathbf{A_2}$), and ($\mathbf{A_3}$) and suppose that Ω: $\mathfrak{C}([0, T], \mathbb{R}^n) \to P(\mathfrak{C}([0, T], \mathbb{R}^n))$ is defined by

$$\Omega(x) = \left\{ h \in \mathfrak{C}([0, \ T], \ \mathbb{R}^n) : h(t) = x_0 - g(x) + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s) ds, \ f \in S_{F,x} \right\}.$$

Then $\Omega(x) \in P_{c, \ cp}(\mathfrak{C}([0, T], \mathbb{R}^n))$ for each $x \in \mathfrak{C}([0, T], \mathbb{R}^n)$.

Proof. First we show that $\Omega(x)$ is convex for each $x \in ([0, T], \mathbb{R}^n)$. For that, let h_1, $h_2 \in \Omega(x)$. Then there exist $f_1, f_2 \in S_{F, \ x}$ such that for each $t \in [0, T]$, we have

$$h_i(t) = S_q(t)(x_0 - g(x)) + \int_0^t T_q(t-s) f_i(s) ds, \quad i = 1, 2.$$

Let $0 \le \lambda \le 1$. Then, for each $t \in [0, T]$, we have

$$\begin{aligned}&[\lambda h_1 + (1 - \lambda) h_2](t) \\ &\quad = S_q(t)(x_0 - g(x)) + \int_0^t T_q(t-s)[\lambda f_1(s) + (1 - \lambda) f_2(s)] ds.\end{aligned}$$

Since $S_{F, \ x}$ is convex (F has convex values), therefore it follows that $\lambda h_1 + (1-\lambda) h_2 \in \Omega(x)$. Next, we show that Ω maps bounded sets into bounded sets in $\mathfrak{C}([0, T], \mathbb{R}^n)$. For a positive number r, let $B_r = \{x \in \mathfrak{C}([0, T], \mathbb{R}^n): \|x\|_{\infty} \le r\}$ be a bounded set in $\mathfrak{C}([0, T], \mathbb{R}^n)$. Then, for each $h \in \Omega(x)$, $x \in B_r$, there exists $f \in S_{F, \ x}$ such that

$$h(t) = S_q(t)(x_0 - g(x)) + \int_0^t T_q(t-s) f(s) ds$$

and in view of (H_1), we have

$$\begin{aligned}||h(t)|| &\le |S_q(t)|(||x_0|| + \sup_{x \in B_r} ||g(x)||) + \int_0^t |T_q(t-s)| ||f(s)|| ds \\ &\le |S_q(t)|(||x_0|| + \sup_{x \in B_r} ||g(x)||) + \int_0^T |T_q(t-s)| \kappa_1(s) ds.\end{aligned}$$

Thus,

$$||h||_\infty \leq |S_q(t)|(||x_0|| + \sup_{x\in B_r} ||g(x)||) + \int_0^T |T_q(t-s)|\kappa_1(s)ds.$$

Now we show that Ω maps bounded sets into equicontinuous sets in $\mathfrak{C}([0, T], \mathbb{R}^n)$. Let t', $t'' \in [0, T]$ with $t' < t''$ and $x \in B_r$, where B_r is a bounded set in $\mathfrak{C}([0, T], \mathbb{R}^n)$. For each $h \in \Omega(x)$, we obtain

$$\begin{aligned} &||h(t'') - h(t')|| \\ &\leq ||(S_q(t'') - S_q(t'))(x_0 - g(x))|| + ||\int_0^{t''} T_q(t''-s)f(s)ds - \int_0^{t'} T_q(t'-s)f(s)ds|| \\ &\leq ||(S_q(t'') - S_q(t'))(x_0 - g(x))|| + ||\int_0^{t'} [T_q(t''-s) - T_q(t'-s)]\kappa_1(s)ds|| \\ &+ ||\int_{t'}^{t''} T_q(t''-s)\kappa_1(s)ds||. \end{aligned}$$

Obviously the right-hand side of the above inequality tends to zero independently of $x \in B_{r'}$ as $t'' - t' \to 0$. By the Arzela-Ascoli Theorem, $\Omega:\mathfrak{C}([0, T], \mathbb{R}^n)\to P(\mathfrak{C}([0, T], \mathbb{R}^n))$ is completely continuous. As in Theorem 3.6, Ω is closed-valued. Consequently, $\Omega(x) \in P_{c,cp}(\mathfrak{C}([0, T], \mathbb{R}^n))$ for each $x \in \mathfrak{C}([0, T], \mathbb{R}^n)$.

For $0 < \alpha \leq T$, let us consider the operator

$$\Omega(x) = \left\{ h \in \mathfrak{C}([0, \alpha], \mathbb{R}^n) : h(t) = S_q(t)(x_0 - g(x)) + \int_0^t T_q(t-s)f(s)ds, f \in S_{F,x} \right\}.$$

It is well-known that $\mathrm{Fix}(\Omega) = \mathcal{S}_{x_0}([0,\alpha])$ and, in view of Theorem 3.6, it is nonempty for each $0 < \alpha \leq T$.

The following results are useful in the sequel.

Lemma 3.8 (Dzedzej and Gelman [36]) Let F: $[0, \alpha] \to P_{c,cp}(\mathbb{R}^n)$ be a measurable map such that the Lebesgue measure μ of the set $\{t: \dim F(t) < 1\}$ is zero. Then there are arbitrarily many linearly independent measurable selections $x_1(\cdot), x_2(\cdot), \ldots, x_m(\cdot)$ of F.

Lemma 3.9. (Dzedzej and Gelman [36]) (see also, [29,37] for general versions) Let C be a nonempty closed convex subset of a Banach space X. Suppose that Ω: $C \to P_{c,cp}(C)$ is a k-contraction. If f: $C \to C$ is a continuous selection of Ω, then Fix(f) is nonempty.

Lemma 3.10. (Michael's selection theorem) [38] Let C be a metric space, X be a Banach space and Ω: $C \to P_{c,cl}(C)$ a lower semicontinuous map. Then there exists a continuous selection f: $C \to X$ of Ω.

Lemma 3.11. (Saint Raymond [39]) Let K be a compact metric space with $\dim K < n$, X a Banach space and Ω: $K \to P_{c,cp}(X)$ a lower semicontinuous map such that $0 \in \Omega(x)$ and $\dim \Omega(x) \geq n$ for every $x \in K$. Then there exists a continuous selection f of Ω such that $f(x) \neq 0$ for each $x \in K$.

Theorem 3.12. Let F: $[0, \alpha] \times \mathbb{R}^n \to P_{c,cp}(\mathbb{R}^n)$ satisfy ($\mathbf{A_1}$), ($\mathbf{A_2}$), and ($\mathbf{A_3}$) and suppose that the Lebesgue measure μ of the set $\{t: \dim F(t, x) < 1$ for some $x \in \mathbb{R}^n\}$ is zero. Then for each α, $0 < \alpha < \min\left\{\left(\frac{(1-\widetilde{M}_S\kappa_2)q}{\widetilde{M}_{T\|\kappa_1\|_\infty}}\right)^{\frac{1}{q}}, T\right\}$, the set $\mathcal{S}_{x_0}([0,\alpha])$ of solutions of (1) has an infinite dimension for any x_0.

Proof. Define the operator Ω by

$$\Omega(x) = \left\{h \in \mathfrak{C}([0,\ \alpha],\ \mathbb{R}^n) : h(t) = S_q(t)(x_0 - g(x)) + \int_0^t T_q(t-s)f(s)ds, f \in S_{F,x}\right\}.$$

Then by Lemma 3.7, $\Omega(x) \in P_{c,\ cp}(\mathfrak{C}([0,\alpha]), \mathbb{R}^n))$ for each $x \in \mathfrak{C}([0,\ \alpha], \mathbb{R}^n)$ and as in the proof of Theorem 3.6, it is a contraction if $(\widetilde{M_S}\kappa_2 + \widetilde{M_T}(\alpha^q/q)||\kappa_1||_\infty) < 1$ or $\alpha < \left(\frac{(1-\widetilde{M}_S\kappa_2)q}{\widetilde{M}_{T\|\kappa_1\|_\infty}}\right)^{\frac{1}{q}}$. We shall show that dim $\Omega(x) \geq m$ for any $x \in \mathfrak{C}([0,\alpha], \mathbb{R}^n)$ and arbitrary $m \in \mathbb{N}$. Consider $G(t) = F(t, x(t))$. By Lemma 3.8, there exist linearly independent measurable selections $x_1(.)$, $x_2(.)$, . . . , $x_m(.)$ of G. Set $y_i(t) = S_q(t)(x_0 - g(x)) + \int_0^t T_q(t-s)x_i(s)ds \in \Omega(x)$. Assume that $\sum_{i=1}^m a_i y_i(t) = 0$ a.e. in [0, α]. Taking Caputo derivatives a.e. in [0, α], we have $\sum_{i=1}^m a_i x_i(t) = 0$ a.e. in [0, α] and hence $a_i = 0$ for all i. As a result, $y_i(.)$ are linearly independent. Thus Ω(x) contains an m-dimensional simplex. So dim $\Omega(x) \geq m$. As in Theorem 3.6, Fix(Ω) is nonempty. Since Ω is condensing with respect to the Hausdorff measure of noncompactness χ [40] and Fix(Ω) ⊂ Ω(Fix(Ω)), we have

$$\mathcal{X}(\text{Fix}(\Omega)) \leq \mathcal{X}(\Omega(\text{Fix}(\Omega))).$$

This implies that Fix(Ω) is compact. Consider the map I - Ω: Fix(Ω) → $P_{c,\ cp}(\mathbb{R}^n)$, where I is the identity operator. Assume that dim Fix(Ω) < n. Then, by Lemma 3.11, there is a continuous selection g of I - Ω such that $g(x) \neq 0$ for each $x \in$ Fix(Ω). This implies that there exists a continuous selection h of F: Fix(F) → $P_{c,\ cp}(\mathbb{R}^n)$ without fixed points. Define T: $\mathbb{R}^n \to P_{c,\ cp}(\mathbb{R}^n)$ by

$$T(x) = \begin{cases} \Omega(x), & x \in \mathbb{R}^n \setminus \text{Fix}(\Omega) \\ h(x), & x \in \text{Fix}(\Omega). \end{cases}$$

Since T is lower semicontinuous, Michael's selection result (Lemma 3.10) guarantees that T admits a continuous selection f: $\mathbb{R}^n \to \mathbb{R}^n$. Thus f: $\mathbb{R}^n \to \mathbb{R}^n$ is a continuous selection of Ω with no fixed points and $f = h$ on Fix(Ω), which contradicts Lemma 3.9. Consequently, $Fix(\Omega) = \mathcal{S}_{x_0}([0,\ \alpha])$ is infinite dimensional.

Recall that a metric space X is an AR-space if, whenever it is nonempty closed subset of another metric space Y, then there exists a continuous retraction r: $Y \to X$, $r(x) = x$ for $x \in X$. In particular, it is contractible (and hence connected).

Lemma 3.13. [41] Let C be a nonempty closed convex subset of a Banach space X and Ω: $C \to P_{c,\ cp}(C)$ a contraction. Then Fix(Ω) is a nonempty AR-space.

Theorem 3.12 together with Lemma 4.13 yields the following result.

Corollary 3.14. Let F: $[0, \alpha] \times \mathbb{R}^n \to P_{c,\ cp}(\mathbb{R}^n)$ satisfy $(\mathbf{A_1})$, $(\mathbf{A_2})$, and $(\mathbf{A_3})$ and suppose that the Lebesgue measure μ of the set $\{t$: dim $F\ (t, x) < 1$ for some $x \in \mathbb{R}^n\}$ is zero. Then for each $\alpha, 0 < \alpha < \min\left\{\left(\frac{(1-\widetilde{M}_S\kappa_2)q}{\widetilde{M}_T\|\kappa_1\|_\infty}\right)^{\frac{1}{q}}, T\right\}$, the set $\mathcal{S}_{x_0}([0,\ \alpha])$ of solutions of (1) is an infinite dimensional AR-space.

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Authors' contributions
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