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Extremal solutions for certain type of fractional differential equations with maxima

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Abstract

In this article, we employ the Tarski's fixed point theorem to establish the existence of extremal solutions for fractional differential equations with maxima.

1 Introduction

Fractional calculus has become an exciting new mathematical method of solution of diverse problems in mathematics, science, and engineering. Indeed, recent advances of fractional calculus are dominated by modern examples of applications in differential and integral equations and inclusions, physics, signal processing, fluid mechanics, viscoelasticity, mathematical biology, engineering, dynamical systems, control theory, electrical circuits, generalized voltage divider, computer sciences, and electrochemistry (see [1,2]).

The theory and applications of fractional differential equations received in recent years considerable interest both in pure mathematics and in applications. There exist several different definitions of fractional differentiation. Whereas in mathematical treatises on fractional differential equations the Riemann-Liouville approach to the notion of the fractional derivative is normally used [3-5], the Caputo fractional derivative often appears in applications [6], Erdèlyi-Kober fractional derivative [7] and The Weyl-Riesz fractional operators [8]. There are some advantages in studying the extremal solution for fractional differential equations, because some boundary conditions are automatically fulfilled and due to lower order differential requirements (see [9]).

Differential equations with maximum arise naturally when solving practical and phenomenon problems, in particular, in those which appear in the study of systems with automatic regulation and automatic control of various technical systems. It often occurs that the law of regulation depends on maximum values of some regulated state parameters over certain time intervals. Many studies of the existence of solutions are imposed such as periodicity, asymptotic stability and oscillatory [10-12]. In [13], the authors discusses the existence of univalent solutions for fractional integral equations with maxima in complex domain, by using technique associated with measures of non-compactness.

In this article, we establish the extreme solutions (maximal and minimal solutions) for fractional differential equation with maxima in sense of Riemann-Liouville fractional operators, by using the Tarski's fixed point theorem. Moreover, we extend the existence of extremal solutions from initial value problems to boundary value problems for infinite quasi-monotone functional systems of fractional differential equations.

2 Preliminaries

The ordered set (poset) X is called a lattice if $\sup\{x_1, x_2\}$ and $\inf\{x_1, x_2\}$ exist for all $x_1, x_2 \in X$. A lattice X is complete when each nonempty subset $Y \subset X$ has the supremum and the infimum in X. In particular, every complete lattice has the maximum and the minimum. Denoted by

$$[a, b]_X = \{x \in X : a \leq x \leq b\}.$$

The fundamental tool in our work is the following well-known Tarski's fixed point theorem which can be found in [14]:

Theorem 2.1. Every nondecreasing mapping $G : X \to X$ on a complete lattice X has a minimal, x_*, and a maximal fixed point, x^*. Moreover,

$$x_* = \min\{x \in X : Gx \leq x\}, \quad x^* = \max\{x \in X : x \leq Gx\}.$$

Let $T > 0$ and $\eta > 0$ be fixed. We denote by $AC([0, T])$ the set of all functions $x : [0, T] \to \mathbb{R}$ which are absolutely continuous and by $B([-\eta, 0])$ the set of all functions $x : [-\eta, 0] \to \mathbb{R}$ which are bounded. Let M be an arbitrary index set and for each for all $\jmath \in M$, $h_\jmath : [0, T] \to \mathbb{R}$ be a Lebesgue-integrable function and define

$$C_{h_\jmath}([0, T]) = \left\{ x : [0, T] \to \mathbb{R},\ |x(s) - x(t)| \leq \left| \int_s^t h_\jmath(\eta) d\eta \right|, \quad s, t \in J := [0, T] \right\},$$

with the property

$$x_1, x_2 \in C_{h_\jmath}([0, T]), \quad x_1 \leq x_2 \Leftrightarrow x_1(t) \leq x_2(t), \quad \forall t \in [0, T].$$

Also, we define the set

$$S_\jmath = \left\{ \xi : [-\eta, T] \to \mathbb{R} : \xi_{|[-\eta,0]} \in B([-\eta, 0]) \quad \text{and} \quad \xi_{|[0,T]} \in C_{h_\jmath}([0, T]) \right\}$$

satisfies

$$\xi_1, \xi_2 \in S_\jmath, \quad \xi_1 \leq \xi_2 \Leftrightarrow \xi_1(t) \leq \xi_2(t), \quad t \in [-\eta, T].$$

And set

$$S = \prod_{\jmath \in M} S_\jmath, \quad \jmath \in M$$

satisfies

$$\gamma, \lambda \in S, \quad \gamma \leq \lambda \Leftrightarrow \gamma_\jmath \leq \lambda_\jmath, \quad \jmath \in M.$$

One of the most frequently used tools in the theory of fractional calculus is furnished by the Riemann-Liouville operators (see [15]).

Definition 2.1. The fractional (arbitrary) order integral of the function f of order $\alpha > 0$ is defined by

$$I_a^\alpha f(t) = \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau) d\tau.$$

When $a = 0$, we write

$$I_a^\alpha f(t) = I^\alpha f(t) = f(t) * \phi_\alpha(t),$$

where (*) denoted the convolution product,

$$\phi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \ t > 0$$

and $\varphi_\alpha(t) = 0$, $t \le 0$ and $\varphi_\alpha \to \delta(t)$ as $\alpha \to 0$ where $\delta(t)$ is the delta function.

Definition 2.2. The fractional (arbitrary) order derivative of the function f of order $0 < \alpha < 1$ is defined by

$$D_a^\alpha f(t) = \frac{d}{dt}\int_a^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} f(\tau)d\tau = \frac{d}{dt} I_a^{1-\alpha} f(t).$$

3 Main results

We study fractional differential equations with maxima of the form

$$D^\alpha u(t) = \begin{cases} F\left(t,\ u(t),\ \max_{s\in J} u(s)\right) & \text{if } t \in J; \\ u(\theta) = \phi(\theta) & \text{if } \theta \in [-\eta,\ 0], \end{cases} \tag{1}$$

where $F : J \times \mathbb{R} \times S \to \mathbb{R}$ and $\varphi : [-\eta, 0] \to \mathbb{R}$. We denote by $||\varphi||$ the norm

$$——\phi—— = \max\left\{\phi(\theta) : \theta \in [-\eta,\ 0]\right\}.$$

Definition 3.1. We say that $u_J \in S$ is a lower solution of problem (1) if for each $J \in M$ we have

$$D^\alpha u_J(t) \le F_J\left(t,\ u(t),\ \max_{s\in J} u(s)\right), \quad t \in J; \quad u_J(\theta) \le \phi(\theta), \quad \theta \in [-\eta,\ 0]. \tag{2}$$

Analogously we say that u_j is an upper solution of (1) if the above inequalities are reversed. We say that u_J is a solution of (1) if it is both a lower and an upper solution. A solution u^* in $A \subset S$ is a maximal solution in the set A if $u^* \ge u$ for any other solution $u \in A$. The minimal solution in A is defined analogously by reversing the inequalities; when both a minimal and a maximal solution in A exist, we call them the extremal solutions in A.

Next we pose our main result

Theorem 3.1. Assume that there exist $\gamma, \lambda \in S$ with $\gamma \le \lambda$ such that the following hypotheses hold:

(i) For each $\xi \in [\gamma, \lambda]_S$ the initial value problem

$$D^\alpha z_J(t) = \begin{cases} F_J\left(t,\ z(t),\ \max_{s\in J} z(s)\right) & t \in J; \\ z_J(0) = \phi(0) \end{cases} \tag{3}$$

has a maximal solution z^* and a minimal solution z_* in $A := [\gamma_J,\ \lambda_J]_{C_{h_J}([0,T])}$

(ii) For each $\xi \in [\gamma,\ \lambda]_S$, $J \in M$ and $t \in J$ if $u(t) \le v(t)$ and $u_J = v_J$ then

$$F_J(t,\ u(t),\ \xi) \le F_J(t,\ v(t),\ \xi).$$

(iii) The function $F_J(t, u(t), .)$ is nondecreasing in $[\gamma, \lambda]_S$. Moreover, the function φ is nondecreasing in $[-\eta, 0]$.

Then problem (1) has a maximal solution, u^*, and a minimal one, u_*, in $[\gamma, \lambda]_S$.

Proof. We shall prove the existence of the maximal solution since the existence of the minimal solution follows from the dual arguments.

Firstly we consider the mapping

$$\Phi_J : [\gamma, \lambda]_S \to [\gamma_J, \lambda_J]_{S_J}$$

then in virtue of condition (i) we can define

$$(\Phi_J \xi) = \begin{cases} \phi_\xi(\theta) \text{ if } \theta \in [-\eta, 0]; \\ \xi^*(t) \text{ if } t \in J, \end{cases} \tag{4}$$

where ξ^* is the the maximal solution in $[\gamma_J, \lambda_J]_{C_{h_J}([0,T])}$ of the problem (3). Therefore $(\Phi_J \xi) \in [\gamma_J, \lambda_J]_{S_J}$. Secondly, we impose the mapping

$$\Phi : [\gamma, \lambda]_S \to [\gamma, \lambda]_S.$$

Next we proceed to prove that Φ satisfies the conditions of Theorem 2.1.

Step 1. Φ: $[\gamma, \lambda]_S \to [\gamma, \lambda]_S$ is nondecreasing.

Let $\xi_1, \xi_2 \in [\gamma, \lambda]_S$ and fix $J \in M$. By (iii) we have

$$(\Phi_J \xi_1)(\theta) = \phi_{\xi_1}(\theta) \le \phi_{\xi_2}(\theta) = (\Phi_J \xi_2)(\theta), \quad \theta \in [-\eta, 0].$$

On the other hand, $\Phi_J \xi \in A$ and in view of conditions (ii) and (iii) we obtain that

$$(\Phi_J \xi_1) \le (\Phi_J \xi_2), \quad \text{on } J.$$

Since $J \in M$ is arbitrary we conclude that $(\Phi \xi_1) \le (\Phi \xi_2)$.

Step 2. $[\gamma, \lambda]_S$ is a complete lattice.

It suffices to prove that for each $J \in M$ the set $[\gamma_J, \lambda_J]_{S_J}$ is a complete lattice. Let $B \subset [\gamma_J, \lambda_J]_{S_J}$ this implies that $B \neq \emptyset$ and B has the supremum and the infimum. Define

$$\xi^*(t) = \sup \left\{ \xi(t) : \xi \in B, \quad t \in [-\eta, T] \right\}.$$

It is clear that $\xi^*(t)$ is well defined for all $t \in [-\eta, T]$ and satisfies $\gamma_J \le \xi^* \le \lambda_J$ i.e, ξ^* is bounded on $[-\eta, 0]$. Finally we shall prove that $\xi^* \in A$. For fix $t, s \in J$ and $\xi \in B$ we observe that

$$\xi(s) \le |\xi(s) - \xi(t)| + \xi(t) \le \left| \int_s^t h_J(r) dr \right| + \xi^*(t)$$

$$\Rightarrow \sup \xi(s) \le \left| \int_s^t h_J(r) dr \right| + \xi^*(t)$$

$$\Rightarrow \xi^*(s) \le \left| \int_s^t h_J(r) dr \right| + \xi^*(t)$$

$$\Rightarrow \xi^*(t) \le \left| \int_t^s h_J(r) dr \right| + \xi^*(s)$$

$$\Rightarrow |\xi^*(s) - \xi^*(t)| \le \left| \int_s^t h_J(r) dr \right|.$$

Therefor $\xi^* \in [\gamma_J, \lambda_J]_{S_J}$ and $\zeta^* = \sup B$. The existence of inf B is proved by similar manner. Hence $[\gamma_J, \lambda_J]_{S_J}$ is a complete lattice and consequently $[\gamma, \lambda]_S = \prod_{J\in M} [\gamma_J, \lambda_J]_{S_J}$.

Steps 1 and 2 imply that Φ satisfies the conditions of Tarski's fixed point theorem and then Φ has the maximal fixed point x^* which satisfies

$$x^* = \max\left\{x \in [\gamma, \lambda]_S : x \leq \Phi x\right\}. \tag{5}$$

Step 3. X^* is the maximal solution of problem (1) in $[\gamma, \lambda]_S$.

By the definition of Φ we have u^* is a solution for the problem (1). Suppose now that $u := u_J)_{J\in M} \in [\gamma, \lambda]_S$ is a lower solution for (1) i.e.

$$D^\alpha u_J(t) \leq \begin{cases} F_J\ \left(t,\ u(t),\ \max_{s\in J} u(s)\right) & \text{if } t\ \in J; \\ u_J(\theta) \leq \phi(\theta) & \text{if } \theta \in [-\eta,\ 0]. \end{cases} \tag{6}$$

Then by (5) it follows that for every solution x of the problem (1) satisfies $x \leq x^*$. This completes the proof of Theorem 3.1.

Remark 3.1. Note that Condition (i) in Theorem 3.1 looks difficult to verify but it is useful for applying the Theorem 2.1. however, there are in the literature a lot of sufficient conditions which imply the existence of extremal solutions. Condition (ii) is called quasimonotonicity. This property is important for extremal fixed points of discontinuous maps. Moreover, the functional boundary condition $u(\theta) = \varphi(\theta)$, $\theta \in [-\eta, 0]$ includes the initial condition $u(0) = \varphi(0) := u_0$, where $\theta = 0$. As well as several types of periodic conditions, which have more interest, such as the ordinary periodic condition $u(\theta) = \varphi(\theta) := u(T)$ for fixed θ which probably takes the value $\theta = 0$. Moreover, the functional periodic condition $x(\theta) = \varphi(\theta) := x(\theta + T)$, $\theta \in [-\eta, 0]$. Finally, $\varphi(t)$ can represented as integral initial condition such as

$$u(0) = \int_0^T u_J(s)ds.$$

Additional condition on $\xi \in S$, for all $J \in M$ if ξ_J is Lebesgue-measurable on $[-\eta, 0]$ leads to suggest the initial condition

$$u(0) = \int_{-\eta}^{T/2} u_J(s)ds.$$

Next we replace the condition (i) by assuming F in the set of $L^1_X(J,\ \mathbb{R}\times\mathbb{R})$-*Carathéodory.*

Definition 3.2. A mapping $p : J \times \mathbb{R} \to \mathbb{R}$ is said to be *Carathéodory* if

(C1) $t \to p\ (t, u)$ is measurable for each $u \in \mathbb{R}$,

(C2) $u \to p\ (t, u)$ is continuous a.e. for $t \in J$.

A *Carathéodory* function $p\ (t, u)$ is called $L^1\ (J, \mathbb{R})$-*Carathéodory* if (C3) for each number $r > 0$ there exists a function $h_r \in L^1(J, \mathbb{R})$ such that $|p(t, u)| \leq h_r\ (t)$ a.e $t \in J$ for all $u \in \mathbb{R}$ with $|u| = r$.

A *Carathéodory* function $p\ (t,\ u)$ is called $L^1_X(J,\mathbb{R})$- *Carthéodory* if (C4) there exists a function $h \in L^1(J,\ \mathbb{R})$ such that $|p\ (t,\ u)| \leq h\ (t)$ a.e $t \in J$ for all $u \in \mathbb{R}$ where h is called the bounded function of p.

Theorem 3.2. Let F be $L^1_X(J,\mathbb{R})$- *Carathéodory*. If the assumptions (ii) and (iii) hold then the problem (1) has at least one solution $u\ (t)$ on J.

Proof. Operating equation (1) by I^α and using the properties of the fractional operators (see [9,15]), we have

$$u(t)\ = \phi(\theta) + \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} F(\tau,\ u,\ v)d\tau.$$

Define an operator P as follows:

$$(Pu)\ (t) := \phi\ (\theta) + \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} F(\tau,\ u,\ v)d\tau. \tag{7}$$

Then by the assumption of the theorem and the properties of the fractional calculus we obtain that

$$\begin{aligned}
|(Pu)(t)| &\leq ||\phi|| + \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} |F(\tau,\ u,\ v)|d\tau \\
&\leq |\phi(\theta)| + \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\ h(\tau)\, d\tau \\
&\leq ||\phi|| + ||h||_{L^1} \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\ d\tau \\
&\leq ||\phi|| + \frac{||h||_{L^1} T^\alpha}{\Gamma(\alpha+1)} \\
&:= \rho.
\end{aligned}$$

This further implies that

$$||Pu||_{\mathcal{C}}\ \leq \rho,$$

where $\mathcal{C}\,[(J,\ \mathbb{R}\times\mathbb{R})]$ is the space of all continuous real valued functions on J with a supremum norm $||.||_{\mathcal{C}}$ that is $P: B_\rho \to B_\rho$. Therefore, P maps B_ρ into itself. In fact, P maps the convex closure of $P\ [B_\rho]$ into itself. Since f is bounded on B_ρ, thus $P\ [B_\rho]$ is equicontinuous and the Schauder fixed point theorem shows that P has at least one fixed point $u \in A$ such that $Pu = u$, which is corresponding to solution of the problem (1). To obtain the maximal and minimal solutions, we use the same arguments in Theorem 3.1.

Moreover condition (i) can replaced by letting F in the set of all functions which are μ *- Lipschitz*. We have the following definition:

Definition 3.3. A function $F\ (t,\ u,\ v)$: $J \times \mathbb{R} \times S \to \mathbb{R}$ is called

(i) a μ *- Lipschitz* if and only if there exists a positive constant μ such that

$$\left|F(t,\ u_1,\ v_1) - F(t,\ u_2,\ v_2)\right| \leq \mu\,[||u_1-u_2|| + ||v_1-v_2||],$$

where

$$||.|| = \sup_{t,s\in J}\{|\,.\,|\},$$

and the constant μ is called a Lipschitz constant.

(ii) A *contraction* if and only if it is μ - *Lipschitz* with $\mu < 1$.

Theorem 3.3. Let F be μ - *Lipschitz*. If $\frac{\mu T^\alpha}{\Gamma(\alpha+1)} < 1$, then (1) has a unique solution $u(t)$ on J.

Proof. Assume the operator P defined in Equation (6) then we have

$$\begin{aligned}
|(Pu_1)(t) - (Pu_2)(t)| &\leq \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}|F(\tau,\ u_1,\ v_1) - F(\tau,\ u_2,\ v_2)|d\tau \\
&\leq \mu(||u_1 - u_2|| + ||v_1 - v_2||)\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\,d\tau \\
&\leq \frac{\mu T^\alpha}{\Gamma(\alpha+1)}(||u_1 - u_2|| + ||v_1 - v_2||).
\end{aligned}$$

Hence by the assumption of the theorem we have that P is a contraction mapping then in view of the Banach fixed point theorem, P has a unique fixed point which is corresponding to the solution of Equation (1). In this case $u(t) = u^*(t) = u_*(t)$.

Example 3.1. Let $J = [0, 1]$ denote a closed and bounded interval in $\mathbb{R}$. Consider the problem

$$D^\alpha u(t) = \begin{cases} 0, & \text{if } u < 0; \\ \left[h(t),\ h(t)\exp^{\frac{u(t)}{2}}\right], & \text{if } u \geq 0 \\ u(0) = h(0) = 0. \end{cases} \tag{8}$$

It is clear that F is $L^1_X(J,\mathbb{R})$- *Carathéodory* with any decreasing growth function $h \in L^1(J, \mathbb{R}^+)$ such that $||F(t, u)|| \leq h(t)$ a.e $t \in J$ for all $u \in \mathbb{R}$. Therefore in view of Theorem 3.2, the problem (8) has maximal and minimal solutions.

Example 3.2. Let S be any nonmeasurable set such that $S \subset [0, 1]$. Consider the problem

$$D^\alpha u(t) = \begin{cases} 1, & \text{if } u > t,\ t \in J; \\ 1, & \text{if } u = t,\ t \in S \\ 0, & \text{otherwise.} \\ u(0) = 0. \end{cases} \tag{9}$$

Obviously F does not satisfy the condition (i) of Theorem 3.1, and hence the problem (9) hasn't extremal solutions.

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Competing interests
The authors declare that they have no competing interests.

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References

1. Lewandowski, R, Chorazyczewski, B: Identification of the parameters of the KelvinVoigt and the Maxwell fractional models, used to modeling of viscoelastic dampers. Comput Struct. **88**, 1–17 (2010). doi:10.1016/j.compstruc.2009.09.001
2. Yu, F: Integrable coupling system of fractional soliton equation hierarchy. Phys Lett A. **373**, 3730–3733 (2009). doi:10.1016/j.physleta.2009.08.017
3. Ibrahim, RW: Existence of convex and non convex local solutions for class of fractional differential inclusion, Electron. J Diff Equ. **2009**(15), 1–15 (2009)
4. Ibrahim, RW: Continuous solutions for fractional integral inclusion in locally convex topological space. Appl Math J Chin Univ. **24**(2), 175–183 (2009). doi:10.1007/s11766-009-1904-y
5. Ibrahim, RW: On the existence for diffeo-integral inclusion of Sobolev-type of fractional order with applications. ANZIAM J. **52**(E), 1–21 (2010)
6. Ibrahim, RW: Initial and boundary value problems for inclusions involving Caputo's fractional derivatives. Pure Math Appl. **18**(1), 1–14 (2007)
7. Ibrahim, RW, Momani, S: On the existence and uniqueness of solutions of a class of fractional differential equations. J Math Anal Appl. **334**, 1–10 (2007). doi:10.1016/j.jmaa.2006.12.036
8. Momani, SM, Ibrahim, RW: On a fractional integral equation of periodic functions involving Weyl-Riesz operator in Banach algebras. J Math Anal Appl. **339**, 1210–1219 (2008). doi:10.1016/j.jmaa.2007.08.001
9. Lakshmikantham, V, Leela, S, Vasundhara, J: Theory of Fractional Dynamic Systems. Cambridge Academic Publishers, Cambridge (2009)
10. Golev, A, Hristova, S, Rahnev, A: An algorithm for approximate solving of differential equations with maxima. Comput Math Appl. **60**(10), 2771–2778 (2010). doi:10.1016/j.camwa.2010.09.030
11. Mishev, DP, Musa, SM: Distribution of the zeros of the solutions of hyperbolic differential equations with maxima. Rocky Mt J Math. **37**(4), 1271–1281 (2007). doi:10.1216/rmjm/1187453110
12. Otrocol, D, Rus, IA: Functional differential equations with maxima of mixed type. Fixed Point Theory. **9**(1), 207–220 (2008)
13. Ibrahim, RW, Darus, M: A Note on the existence of fractional integral equations. Adv Stud Theory Phys. **4**(14), 653–663 (2010)
14. Tarski, A: A lattice-theoretical fixed point theorem and its applications. Pac J Math. **5**, 285–309 (1955)
15. Kilbas, AA, Srivastava, HM, Trujillo, JJ: Theory and Applications of Fractional Differential Equations. North-Holland, Mathematics Studies. Elsevier, Amsterdam (2006)

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