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On the existence of mild solutions to the Cauchy problem for a class of fractional evolution equations

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Abstract

We are concerned with the existence of mild solutions to the Cauchy problem for fractional evolution equations of neutral type with almost sectorial operators

$$\frac{d^q}{dt^q}(x(t) - h(t, x(t))) = -A(x(t) - h(t, x(t))) + f(t, x(t)), \quad t > 0,$$

$$x(0) = x_0,$$

where $0 < q < 1$, the fractional derivative is understood in the Caputo sense, A is an almost sectorial operator on a complex Banach space, and f, h are given functions. With the help of the theory of measure of noncompactness and a fixed point theorem of Darbo type, we establish a new existence theorem of mild solutions for the Cauchy problem above. By the way, the global attractive property of the solutions is also obtained. Moreover, we give two examples to illustrate our abstract results.

Keywords: fractional evolution equations, mild solutions, almost sectorial operators, neutral type, measure of noncompactness, global attractive

1 Introduction

The fractional evolution equations have received increasing attention during recent years and have been studied extensively (see, e.g., [1-13] and references therein) since they can be used to describe many phenomena arising in engineering, physics, economy, and science.

　We mention that much of the previous research on the evolution equations was done provided that the operator in the linear part is the infinitesimal generator of a strongly continuous operator semigroup, an analytic semigroup, or a compact semigroup, or is a Hille-Yosida operator (see, e.g., [1-12,14,15] and references therein). On the other hand, when the operator in the linear part is an almost sectorial operator, for which the resolvent operators do not satisfy the required estimate to be a sectorial operator (see the example of almost sectorial operators which are not sectorial given by von Wahl in [16]), much less is known about the fractional evolution equations of neutral type with almost sectorial operators.

　In this article, we will pay our attentions to the existence of mild solutions to the following Cauchy problem for fractional evolution equations of neutral type with almost

sectorial operators

$$\frac{d^q}{dt^q}(x(t) - h(t, x(t))) = -A(x(t) - h(t, x(t))) + f(t, x(t)), \quad t > 0, \tag{1.1}$$

$$x(0) = x_0, \tag{1.2}$$

where $0 < q < 1$, the fractional derivative is understood in the Caputo sense, A is an almost sectorial operator on a complex Banach space, and f, h are given functions. We will use the theory of measure of noncompactness and a fixed point theorem of Darbo type to establish a new existence theorem for the problem (1.1)-(1.2). By the way, the global attractive property of the solutions are also obtained. Moreover, we give two examples to illustrate our abstract results.

This article is organized as follows: In Section 2, we state some basic concepts, notations and properties about fractional order operator and measure of noncompactness. A new existence result and the global attractive property of the solutions will be given and proved in Section 3. Finally, in Section 4, we present two concrete examples, whose physical background is statistical physics and fractional quantum mechanics (see, e.g., [12,13]).

2 Basic concepts, notations and lemmas

Let X be a complex Banach space with norm $||\cdot||$ and $B(x, r)$ denote the closed ball centered at x and with radius r. Suppose $\mathcal{M}_X$ denotes the family of all nonempty and bounded subsets of X and subfamily consisting of all relatively compact sets is denoted by $\mathcal{N}_X$. As usual, for a linear operator A, we denote by $D(A)$ the domain of A, by the family $R(z; A) = (zI - A)^{-1}$, $z \in \rho(A)$ of bounded linear operators the resolvent of A. Moreover, we denote by $L(X, X)$ the space of all bounded linear operators from Banach space X to X with the usual operator norm $||\cdot||_{L(X, X)}$, and we abbreviate this notation to $L(X)$.

Definition 2.1 [12] The fractional integral of order q with the lower limit zero for a function $f \in AC[0, \infty)$ is defined as

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(s) ds, \quad t > 0, \quad 0 < q < 1,$$

provided the right side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 [12] Riemann-Liouville derivative of order q with the lower limit zero for a function $f \in AC[0, \infty)$ can be written as

$$^L D^q f(t) = \frac{1}{\Gamma(1 - q)} \frac{d}{dt} \int_0^t (t - s)^{-q} f(s) ds, \quad t > 0, \quad 0 < q < 1.$$

Definition 2.3 [12] The Caputo derivative of order q for a function $f \in AC[0, \infty)$ can be written as

$$^c D^q f(t) = {}^L D^q (f(t) - f(0)), \quad t > 0, \quad 0 < q < 1.$$

where $^cD^q := \dfrac{d^q}{dt^q}$.

Next, we recall the concept of measure of noncompactness (cf. [17]).

Definition 2.4 $\mu: \mathcal{M}_X \to \mathbf{R}_+$ *is said to be a measure of noncompactness in X if it satisfies the following conditions:*

(1) *the family* $Ker\,\mu = \{\Omega \in \mathcal{M}_X; \mu\,(\Omega) = 0\}$ *is nonempty and* $Ker\,\mu \subset \mathcal{N}_X$;

(2) $\Omega \subset \Omega_0 \Rightarrow \mu(\Omega) \leq \mu(\Omega_0)$, *for* Ω *and* $\Omega_0 \in \mathcal{M}_X$;

(3) $\mu(Conv(\Omega)) = \mu(\Omega)$, *where* $Conv(\Omega)$ *denotes the convex hull of* Ω;

(4) $\mu(\bar{\Omega}) = \mu(\Omega)$, *where* $\bar{\Omega}$ *denotes the closure of* $\Omega \in \mathcal{M}_X$;

(5) $\mu(\lambda\Omega + (1 - \lambda)\Omega_0) \leq \lambda\mu(\Omega) + (1 - \lambda)\mu(\Omega_0)$, *for* $\lambda \in [0, 1]$ *and any* Ω, $\Omega_0 \in \mathcal{M}_X$;

(6) *If* $\{\Omega_n\}$ *is a sequence of sets from* $\mathcal{M}_X$ *such that* $\Omega_{n+1} \subset \Omega_n$, $\bar{\Omega}_n = \Omega_n (n = 1, 2, \ldots)$, *and if* $\lim_{n\to\infty} \mu(\Omega_n) = 0$, *then the intersection* $\Omega_\infty = \bigcap_{n=1}^\infty \Omega_n$ *is nonempty.*

The following is a fixed point theorem of Darbo type (see [17]).

Lemma 2.5 *Let* $\mathfrak{M}$ *be a nonempty, bounded, closed and convex subset of a Banach space X, and let* $H : \mathfrak{M} \to \mathfrak{M}$ *be a continuous mapping. Assume that there exists a constant* $k \in [0, 1)$, *such that*

$$\mu(H(\Omega)) \leq k\mu(\Omega)$$

for any nonempty subset Ω *of* $\mathfrak{M}$. *Then H has a fixed point in* $\mathfrak{M}$.

Let $-1 < \gamma < 0$, and S_μ^0 with $0 < \mu < \pi$ be the open sector

$$\{z \in \mathbf{C}\backslash\{0\}; |\arg z| < \mu\}$$

and $S\mu$ be its closure, that is

$$S_\mu = \{z \in \mathbf{C}\backslash\{0\}; |\arg z| \leq \mu\} \cup \{0\},$$

for more details, we refer to [18,19].

As in [18], we state the concept of almost sectorial operators as follows.

Definition 2.6 *Let* $-1 < \gamma < 0$ *and* $0 < \omega < \dfrac{\pi}{2}$. *By* $\Theta_\omega^\gamma(X)$ *we denote the family of all linear closed operators* $A: D(A) \subset X \to X$ *which satisfy*

(1) $\sigma(A) \subset S_\omega = \{z \in \mathbf{C}\backslash\{0\}; |\arg z| \leq \omega\} \cup \{0\}$ *and*

(2) *for every* $\omega < \mu < \pi$ *there exists a constant* C_μ *such that*

$$\|R(z; A)\|_{L(X)} \leq C_\mu |z|^\gamma, \quad \text{for all} \quad z \in \mathbf{C}\backslash S_\mu.$$

A linear operator A will be called an almost sectorial operator on X if $A \in \Theta_\omega^\gamma(X)$.

Remark 2.7 Let $A \in \Theta_\omega^\gamma(X)$. Then the definition implies that $0 \in \rho(A)$.

We denote the semigroup associated with A by $\{T(t)\}_{t \geq 0}$. For $t \in S^0_{\frac{\pi}{2} - \omega}$,

$$T(t) = e^{-tz}(A) = \frac{1}{2\pi i} \int_{\Gamma_\theta} e^{-tz} R(z; A) dz,$$

here $\omega < \theta < \mu < \frac{\pi}{2} - |\arg t|$, forms an analytic semigroup of growth order $1 + \gamma$.

We have the following lemma on $T(t)$ [[19], Theorem 3.9].

Lemma 2.8 *Let* $A \in \Theta^\gamma_\omega(X)$ *with* $-1 < \gamma < 0$ *and* $0 < \omega < \frac{\pi}{2}$ *Then*

(i) $T(t)$ *is analytic in* $S^0_{\frac{\pi}{2} - \omega}$ *and*

$$\frac{d^n}{dt^n} T(t) = (-A)^n T(t), \quad for\ all \quad t \in S^0_{\frac{\pi}{2} - \omega};$$

(ii) $T(s + t) = T(s)\ T(t)$ *for all* $s, t \in S^0_{\frac{\pi}{2} - \omega}$;

(ii) *There exists a constant* $C_0 = C_0(\gamma) > 0$ *such that*

$$\|T(t)\|_{L(X)} \leq C_0 t^{-\gamma - 1}, \quad for\ all \quad t > 0;$$

(iv) *The range* $R(T(t))$ *of* $T(t)$ *for each* $t \in S^0_{\frac{\pi}{2} - \omega}$ *is contained in* $D(A^\infty)$. *Particularly,*
for all $\alpha \in \mathbf{C}$ *with* $Re\beta > 0$, $R(T(t)) \subset D(A^\beta)$ *and*

$$A^\beta T(t)x = \frac{1}{2\pi i} \int_{\Gamma_\theta} z^\beta e^{-tz} R(z; A)x dz, \quad for\ all \quad x \in X,$$

and hence there exists a constant $C' = C'(\gamma, \beta) > 0$ *such that*

$$\|A^\beta T(t)\|_{L(X)} \leq C' t^{-\gamma - Re\beta - 1}, \quad for\ all \quad t > 0;$$

(v) *If* $\beta > 1 + \gamma$, *then* $D(A^\beta) \subset \Sigma_T$, *where* Σ_T *is the continuity set of the semigroup* $\{T(t)\}_{t \geq 0}$, *that is,*

$$\Sigma_T = \left\{ x \in X;\ \lim_{t \to 0;\ t > 0} T(t)x = x \right\}.$$

Clearly, we note that the condition (ii) of the Lemma 2.8 does not satisfy for $t = 0$ or $s = 0$.

The relation between the resolvent operators of A and the semigroup $T(t)$ is characterized by

Lemma 2.9 [[19], Theorem 3.13] *Let* $A \in \Theta_\omega^\gamma(X)$ *with* $-1 < \gamma < 0$ *and* $0 < \omega < \dfrac{\pi}{2}$. *Then for every* $\lambda \in \mathbf{C}$ *with* $\mathrm{Re}\lambda > 0$, *one has*

$$R(\lambda; -A) = \int_0^\infty e^{-\lambda t} T(t) dt.$$

Now, we give the definition of mild solution to (1.1)-(1.2).

Definition 2.10 *A continuous function* $x: (0, T] \to X$ *satisfying the equation*

$$x(t) = \mathcal{S}_q(t)x_0 + h(t, x(t)) + \int_0^t (t-s)^{q-1} \mathcal{P}_q(t-s) f(s, x(s)) ds$$

for $t \in (0, T]$ *is called a mild solution of (1.1)-(1.2), where*

$$\mathcal{S}_q(t)x = \int_0^\infty \Psi_q(\sigma) T(\sigma t^q) x d\sigma, \quad t \in S_{\frac{\pi}{2}-\omega}^0, \quad x \in X,$$

$$\mathcal{P}_q(t)x = \int_0^\infty q\sigma \Psi_q(\sigma) T(\sigma t^q) x d\sigma, \quad t \in S_{\frac{\pi}{2}-\omega}^0, \quad x \in X,$$

and $\Psi_q(\sigma)$ *is the function of Wright type such that*

$$\Psi_q(z) := \sum_{n=0}^\infty \frac{(-z)^n}{n! \Gamma(-qn+1-q)}$$

$$= \frac{1}{\pi} \sum_{n=1}^\infty \frac{(-z)^n}{(n-1)!} \Gamma(nq) \sin(n\pi q), \quad z \in \mathbf{C},$$

with $0 < q < 1$.

Remark 2.11 [[13], Remark 4.1] For every $x_0 \in D(A^\beta)$ ($\beta > 1 + \gamma$), this mild solution (if any) is continuous at $t = 0$.

Remark 2.12 [13] It is not difficult to verify that for $-1 < r < \infty$, $\lambda > 0$ and $-1 < \alpha + \gamma < 0$,

(1) $\Psi_q(t) \geq 0, t > 0$;

(2) $\int_0^\infty \Psi_q(t) t^r dt = \dfrac{\Gamma(1+r)}{\Gamma(1+qr)}$.

Then we have

$$\|\mathcal{S}_q(t)x\| \leq \frac{C_0 \Gamma(-\gamma)}{\Gamma(1-q(1+\gamma))} t^{-q(1+\gamma)} \|x\|, \tag{2.1}$$

$$\|\mathcal{P}_q(t)x\| \leq \frac{q C_0 \Gamma(1-\gamma)}{\Gamma(1-q\gamma)} t^{-q(1+\gamma)} \|x\|, \tag{2.2}$$

$$
\|A^{\alpha}\mathcal{P}_q(t)x\| \le \left\| \int\limits_0^{\infty} q\sigma\Psi_q(\sigma)A^{\alpha}T(\sigma t^q)d\sigma \right\| \|x\|
$$

$$
\le qC' \int\limits_0^{\infty} \Psi_q(\sigma)t^{-q(\gamma+\alpha+1)}\sigma^{-\gamma-\alpha}d\sigma\,\|x\| \tag{2.3}
$$

$$
\le qC' \frac{\Gamma(1-\gamma-\alpha)}{\Gamma(1-q(\gamma+\alpha))}t^{-q(\gamma+\alpha+1)}\,\|x\|.
$$

Lemma 2.13 [[13], Theorem 3.2] *For $t > 0$, $\{\mathcal{S}_q(t)\}$ and $\mathcal{P}_q(t)$ are continuous in the uniform operator topology.*

Let

$$
X_{\alpha} = D(A^{\alpha}),
$$

and let $BC(\mathbf{R}_+, X_{\alpha})$ denote the Banach space consisting of all real functions defined bounded and continuous from $\mathbf{R}_+$ to X_{α} with the norm

$$
\|x\|_{\infty} = \sup_{t \in \mathbf{R}_+} \|x\|_{\alpha},
$$

for $x \in BC(\mathbf{R}_+, X_{\alpha})$.

It is clear that $D(A^{\beta}) \subset D(A^{\alpha})$.

Next, we present a measure of noncompactness introduced in [17].

For any nonempty and bounded subset Y of the space $BC(\mathbf{R}_+, X)$ and a positive number T, we denote $\omega^T(x, \varepsilon)$ as the modulus of continuity of function x on the interval $[0, T]$, where $x \in Y$ and $\varepsilon \ge 0$. Namely,

$$
\omega^T(x, \epsilon) = \sup\{\|x(t) - x(s)\| ; t, s \in [0, T], |t - s| \le \epsilon\}.
$$

We then assume additionally

$$
\omega^T(Y, \epsilon) = \sup\{\omega^T(x, \epsilon); x \in Y\},
$$

$$
\omega_0^T(Y) = \lim_{\epsilon \to 0} \omega^T(Y, \epsilon),
$$

$$
\omega_0(Y) = \lim_{T \to \infty} \omega_0^T(Y),
$$

and

$$
\operatorname{diam}(Y) = \sup\{\|x(t) - y(t)\| ; x, y \in Y\}.
$$

Finally, consider the function μ defined on the family $\mathcal{M}_{BC(\mathbf{R}_+, X)}$ by the formula:

$$
\mu(Y) = \omega_0(Y) + \limsup_{t \to \infty} \operatorname{diam}(Y). \tag{2.4}
$$

It is known that μ is a measure of noncompactness.

Definition 2.14 *The solution $x(t)$ of (1.1)-(1.2) is said to be globally attractive, if*

$$
\lim_{t \to \infty}(x(t) - y(t)) = 0,
$$

for any solution $y(t)$ of equation (1.1)-(1.2).

3 Main result

In this section, we assume $-1 < \alpha + \gamma < 0$ and $0 < \alpha < \beta < 1$.

Theorem 3.1 *Let* $A \in \Theta_\omega^\gamma(X)$ *and* $0 < \omega < \dfrac{\pi}{2}$. *Assume that*

(H1) *f*: $\mathbf{R}_+ \times X_\alpha \to X$ *is continuous, and there exists a positive function* $v(\cdot)$: $\mathbf{R}_+ \to \mathbf{R}_+$ *such that*

$$
\begin{cases}
\|f(t, x)\| \le v(t), \\
\text{the function } s \mapsto \dfrac{v(s)}{(t - s)^{1 + q(\gamma + \alpha)}} \text{ belongs to } L^1([0, t], \mathbf{R}_+),
\end{cases}
\tag{3.1}
$$

$$
\lim_{t \to \infty} \eta(t) := \lim_{t \to \infty} \int_0^t \frac{v(s)}{(t - s)^{1 + q(\gamma + \alpha)}} = 0.
\tag{3.2}
$$

(H2) *The function* $h \in BC(\mathbf{R}_+, X_\alpha)$ *and there exists a constant* $L \in (0, 1)$ *such that*

$$
\|h(t_1, x(t_1)) - h(t_2, x(t_2))\|_\alpha \le L(|t_1 - t_2| + \|x(t_1) - x(t_2)\|_\alpha).
$$

(H3) *For each nonempty, bounded set* $D \subset BC(\mathbf{R}_+, X_\alpha)$, *the family of functions*

$$
\{t \to h(t, \varphi); \varphi \in D\}
$$

is equicontinuous.
Then

(1) *for every* $x_0 \in D(A^\beta)$ *with* $\beta > 1 + \gamma$, *the problem* (1.1)-(1.2) *has at least a mild solution on* $BC(\mathbf{R}_+, X_\alpha)$;
(2) *all solutions are globally attractive.*

Proof. Consider the operator as follows:

$$
(Hx)(t) = \mathcal{S}_q(t)x_0 + h(t, x(t)) + \int_0^t (t - s)^{q-1} \mathcal{P}_q(t - s) f(s, x(s)) ds, \quad t \ge 0.
$$

Step 1: We prove that there exists a ball

$$
B_r = \{x \in BC(\mathbf{R}_+, X_\alpha); \ \|x\|_\infty \le r\}
$$

with radius r and centered at 0, such that $H(B_r) \subseteq B_r$.
For any $r > 0$ and $x \in B_r$, in view of (H2),

$$
\begin{aligned}
\|h(t, x(t))\|_\alpha &\le \|h(t, x(t)) - h(t, 0)\|_\alpha + \|h(t, 0)\|_\alpha \\
&\le Lr + M_1,
\end{aligned}
\tag{3.3}
$$

where

$$
M_1 = \sup_{t \in \mathbf{R}_+} \|h(t, 0)\|_\alpha.
$$

By (3.2), we get

$$\sup\{\eta(t)\} \leq K$$

for a positive constant K.

Moreover, for arbitrary $x \in B_r$, by (2.3) and (3.1) we have

$$\|(Hx)(t)\|_\alpha \leq \|\mathcal{S}_q(t)x_0\|_\alpha + \|h(t, x(t))\|_\alpha + \int_0^t (t-s)^{q-1} \|\mathcal{P}_q(t-s)f(s, x(s))\|_\alpha ds$$

$$\leq \|\mathcal{S}_q(t)x_0\|_\alpha + Lr + M_1 + qC' \frac{\Gamma(1-\gamma-\alpha)}{\Gamma(1-q(\gamma+\alpha))} \int_0^t (t-s)^{-1-q(\gamma+\alpha)} v(s) ds$$

$$\leq \sup_{t \in \mathbf{R}_+} \|\mathcal{S}_q(t)A^\alpha x_0\| + Lr + M_1 + qC' \frac{\Gamma(1-\gamma-\alpha)}{\Gamma(1-q(\gamma+\alpha))} K$$

Choose r such that

$$r \geq \frac{\displaystyle\sup_{t \in \mathbf{R}_+} \|\mathcal{S}_q(t)A^\alpha x_0\| + M_1 + \frac{qC'K\Gamma(1-\gamma-\alpha)}{\Gamma(1-q(\gamma+\alpha))}}{1-L}.$$

Then

$$\|(Hx)(t)\|_\alpha \leq r,$$

that is $H(B_r) \subset B_r$.

Step 2: We prove that the operator H is continuous on B_r.

Let $\{x_n\}$ be a sequence of B_r such that $x_n \to x$ in B_r as $n \to \infty$. Then

$$f(s, x_n(s)) \to f(s, x(s)), \quad \text{as } n \to \infty \tag{3.4}$$

since the function f is continuous on $\mathbf{R}_+ \times X_\alpha$.

For every $t \in [0, T]$, using (H2) and (2.3), we obtain

$$\|(Hx_n)(t) - (Hx)(t)\|_\alpha$$

$$\leq \|h(t, x_n(t)) - h(t, x(t))\|_\alpha + \left\| \int_0^t (t-s)^{q-1} \mathcal{P}_q(t-s)[f(s, x_n(s)) - f(s, x(s))]ds \right\|_\alpha$$

$$\leq L\|x_n - x\|_\infty + \int_0^t (t-s)^{q-1} \|\mathcal{P}_q(t-s)[f(s, x_n(s)) - f(s, x(s))]\|_\alpha ds \tag{3.5}$$

$$\leq L\|x_n - x\|_\infty + M_2 \int_0^t (t-s)^{-1-q(\gamma+\alpha)} \|f(s, x_n(s)) - f(s, x(s))\| ds,$$

where

$$M_2 = qC' \frac{\Gamma(1-\gamma-\alpha)}{\Gamma(1-q(\gamma+\alpha))}.$$

Clearly, the first term of (3.5) tends to zero as $n \to \infty$. From the fact that

$$\|f(s, x_n(s)) - f(s, x(s))\| \leq 2v(s), \quad s \in \mathbf{R}_+,$$

(3.4), and the Lebesgue Dominated Convergence Theorem, it follows that the second term of (3.5) tends to zero too as $n \to \infty$.

Therefore, H is continuous on B_r.

Step 3: Let Ω be arbitrary nonempty subset of B_r, we prove that

$$\mu(H(\Omega)) \leq \mu(\Omega).$$

Let us choose $x \in \Omega$ and t_1, t_2 with $|t_2 - t_1| < \varepsilon$. Without loss of generality we may assume that $t_1 < t_2$.

For any $T > 0$, when $0 = t_1 < t_2 \leq T$, we have

$$\int_0^{t_2} \left\| (t_2 - s)^{q-1} \mathcal{P}_q(t_2 - s) f(s, x(s)) \right\|_\alpha ds \leq M_2 \int_0^{t_2} (t_2 - s)^{-1-q(\gamma+\alpha)} v(s) ds.$$

Hence $||(Hx)(t_2)||$ is small as t_2 is small independently of $x \in \Omega$.

For $0 < t_1 < t_2 \leq T$, taking into account our assumptions, we get

$$
\begin{aligned}
\left\| (Hx)(t_2) - (Hx)(t_1) \right\|_\alpha \leq\ & \left\| (\mathcal{S}_q(t_2) - \mathcal{S}_q(t_1)) x_0 \right\|_\alpha \\
& + \left\| h(t_2, x(t_2)) - h(t_1, x(t_1)) \right\|_\alpha \\
& + \left\| \int_0^{t_1} [(t_2 - s)^{q-1} - (t_1 - s)^{q-1}] \mathcal{P}_q(t_2 - s) f(s, x(s)) ds \right\|_\alpha \\
& + \left\| \int_{t_1}^{t_2} (t_2 - s)^{q-1} \mathcal{P}_q(t_2 - s) f(s, x(s)) ds \right\|_\alpha \\
& + \left\| \int_0^{t_1} (t_1 - s)^{q-1} [\mathcal{P}_q(t_2 - s) - \mathcal{P}_q(t_1 - s)] f(s, x(s)) ds \right\|_\alpha \\
=\ & I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}
\tag{3.6}
$$

As a consequence of the continuity of $\{\mathcal{S}_q(t)\}$ in the uniform operator topology for $t > 0$, we know that

$$I_1 \to 0, \quad \text{as} \quad t_2 \to t_1.$$

By (H3), we see that

$$I_2 \to 0, \quad \text{as} \quad t_2 \to t_1.$$

Using (2.3) and (H1), we have

$$
\begin{aligned}
I_3 &= \left\| \int_0^{t_1} [(t_2 - s)^{q-1} - (t_1 - s)^{q-1}] \mathcal{P}_q(t_2 - s) f(s, x(s)) ds \right\|_\alpha \\
&\leq qC' \frac{\Gamma(1-\gamma-\alpha)}{\Gamma(1-q(\gamma+\alpha))} \int_0^{t_1} \frac{\left| (t_2 - s)^{q-1} - (t_1 - s)^{q-1} \right|}{(t_2 - s)^{q-1}} \frac{v(s)}{(t_2 - s)^{1+q(\gamma+\alpha)}} ds.
\end{aligned}
$$

Therefore, by (3.2), we get

$$I_3 \to 0, \quad \text{as} \quad t_2 \to t_1.$$

Moreover, we have

$$
I_4 = \left\| \int_{t_1}^{t_2} (t_2 - s)^{q-1} \mathcal{P}_q(t_2 - s) f(s, x(s)) ds \right\|_\alpha
$$

$$
\leq q C' \frac{\Gamma(1 - \gamma - \alpha)}{\Gamma(1 - q(\gamma + \alpha))} \int_{t_1}^{t_2} (t_2 - s)^{-1 - q(\gamma + \alpha)} v(s) ds
$$

$$
\to 0, \quad \text{as} \quad t_2 \to t_1.
$$

Finally, for $\varepsilon > 0$ small enough, we obtain

$$
I_5 = \left\| \int_0^{t_1} (t_1 - s)^{q-1} [\mathcal{P}_q(t_2 - s) - \mathcal{P}_q(t_1 - s)] f(s, x(s)) ds \right\|_\alpha
$$

$$
\leq q \int_0^{t_1} \int_0^\infty \sigma \Psi_q(\sigma) (t_1 - s)^{q-1} \left\| T((t_2 - s)^q \sigma) - T((t_1 - s)^q \sigma) \right\|_\alpha \cdot v(s) d\sigma ds
$$

$$
\leq q \int_0^{t_1 - 2\varepsilon} \int_0^\infty \sigma \Psi_q(\sigma) (t_1 - s)^{q-1} \left\| T((t_2 - s)^q \sigma - \varepsilon^q \sigma) - T((t_1 - s)^q \sigma - \varepsilon^q \sigma) \right\|
$$

$$
\cdot \left\| A^\alpha T(\varepsilon^q \sigma) \right\| v(s) d\sigma ds
$$

$$
+ M_2 \int_{t_1 - 2\varepsilon}^{t_1} \left(\frac{(t_1 - s)^{q-1}}{(t_1 - s)^{q(\alpha + \gamma + 1)}} + \frac{(t_1 - s)^{q-1}}{(t_2 - s)^{q(\alpha + \gamma + 1)}} \right) v(s) ds
$$

$$
\leq \frac{q C'}{\varepsilon^{q(\gamma + \alpha + 1)}} \int_0^{t_1 - 2\varepsilon} \int_0^\infty \sigma^{-\gamma - \alpha} \Psi_q(\sigma) \left\| T((t_2 - s)^q \sigma - \varepsilon^q \sigma) - T((t_1 - s)^q \sigma - \varepsilon^q \sigma) \right\|
$$

$$
\cdot \frac{v(s)}{(t_1 - s)^{1 - q}} d\sigma ds
$$

$$
+ M_2 \int_{t_1 - 2\varepsilon}^{t_1} \left(\frac{(t_1 - s)^{q-1}}{(t_1 - s)^{q(\alpha + \gamma + 1)}} + \frac{(t_1 - s)^{q-1}}{(t_2 - s)^{q(\alpha + \gamma + 1)}} \right) v(s) ds
$$

$$
= I_5' + I_5''.
$$

The continuity of the function $t \to ||T(t)||k$ for $t \in (0, T)$ implies that

$$
I_5' \to 0, \quad \text{as} \quad t_2 \to t_1.
$$

Furthermore, it is easy to see that

$$
I_5'' \to 0, \quad \text{as} \quad \varepsilon \to 0.
$$

Thus, we obtain

$$
\omega_0^T(H\Omega) = 0.
$$

Consequently, we have

$$
\omega_0(H\Omega) = 0. \tag{3.7}
$$

Now, by our assumptions, for arbitrarily fixed $t \in \mathbf{R}_+$ and $x, y \in \Omega$ we deduce that

$$\left\| (Hx)(t) - (Hy)(t) \right\|_\alpha$$

$$\leq \left\| h(t, x(t)) - h(t, y(t)) \right\|_\alpha + \int_0^t (t-s)^{q-1} \left\| \mathcal{P}_q(t-s)[f(s, x(s)) - f(s, y(s))] \right\|_\alpha ds$$

$$\leq L \left\| x(t) - y(t) \right\|_\alpha + 2M_2 \eta(t).$$

By (3.2), we have

$$\limsup_{t \to \infty} \operatorname{diam}(H\Omega)(t) \leq L \limsup_{t \to \infty} \operatorname{diam}\Omega(t). \tag{3.8}$$

Therefore, using the measure of noncompactness μ defined by the formula (2.4) and keeping in mind (3.7) and (3.8), we obtain

$$\mu(H\Omega) \leq L\mu(\Omega). \tag{3.9}$$

Step 4: We prove that the conclusion (1) is true.

Since $0 < L < 1$, in view of (3.9) and Lemma 2.5, we deduce that the operator H has a fixed point x in the ball B_r. Hence equation (1.1)-(1.2) has at least one mild solution $x(t)$.

Step 5: We prove that the conclusion (2) is true.

Clearly, for any other mild solution $y(t)$ of Equation (1.1)-(1.2), we have

$$\begin{aligned} \left\| x(t) - y(t) \right\|_\alpha &= \left\| (Hx)(t) - (Hy)(t) \right\|_\alpha \\ &\leq L \left\| x(t) - y(t) \right\|_\alpha + 2M_2 \eta(t). \end{aligned}$$

Then by (3.2) we have

$$\lim_{t \to \infty} \left\| x(t) - y(t) \right\|_\alpha \leq \frac{2M_2}{1-L} \lim_{t \to \infty} \eta(t) = 0.$$

That is, all mild solutions of (1.1)-(1.2) are globally attractive. □

From the proof of Theorem 3.1, we can also see that the following theorem holds.

Theorem 3.2 *Let* $A \in \Theta_\omega^\gamma(X)$ *and* $0 < \omega < \dfrac{\pi}{2}$. *If the maps f and h satisfy*

(H1) *The function f:* $\mathbf{R}_+ \times X \to X$ *is continuous, and there exists a positive function* v (·): $\mathbf{R}_+ \to \mathbf{R}_+$ *such that*

$$\left\| f(t, x) \right\| \leq v(t), \text{ the function } s \mapsto \frac{v(s)}{(t-s)^{1+q\gamma}} \text{ belongs to } L^1([0, t], \mathbf{R}_+),$$

$$\lim_{t \to \infty} \eta(t) := \lim_{t \to \infty} \int_0^t \frac{v(s)}{(t-s)^{1+q\gamma}} ds = 0.$$

(H2) *The function* $h \in BC(\mathbf{R}_+, X)$ *and there exists a constant* $L \in (0, 1)$ *such that*

$$\left\| h(t_1, x(t_1)) - h(t_2, x(t_2)) \right\| \leq L(|t_1 - t_2| + \left\| x(t_1) - x(t_2) \right\|), \quad t_1, t_2 \geq 0.$$

(H3) *For each nonempty, bounded set* $D \subset BC(\mathbf{R}_+, X)$, *the family of functions*

$$\{t \to h(t, \varphi); \varphi \in D\}$$

is equicontinuous.

Then for every $x_0 \in D(A^\beta)$ with $\beta > 1 + \gamma$, the problem (1.1)-(1.2) has at least a mild solution on BC $(\mathbf{R}_+, X)$ and all solutions are globally attractive.

4 Applications

Example 4.1: Let Ω be a bounded domain in $\mathbf{R}^N$ $(N \geq 1)$ with boundary $\partial\Omega$ of class C^4. Let $X = C^l(\bar{\Omega})(0 < l < 1)$. Set

$$\widetilde{A} = -\Delta, \qquad D(\widetilde{A}) = \{v \in C^{2+l}(\bar{\Omega}); v = 0 \text{ on } \partial\Omega\}.$$

It follows from [[13], Example 1.2] that there exist $v, \varepsilon > 0$ such that

$$\tilde{A} + v \in \Theta^\gamma_{\frac{\pi}{2} - \varepsilon} (C^l(\bar{\Omega})), \quad \gamma = \frac{l}{2} - 1.$$

We consider the fractional initial boundary value problem

$$\begin{cases} \dfrac{\partial^q}{\partial t^q} \left[u(t, x) - h(t, u(t, x)) \right] = \Delta \left[u(t, x) - h(t, u(t, x)) \right] + f(t, u(t, x)), & x \in \Omega, \\ (u - h)|_{\partial\Omega} = 0, \\ u(0, x) = u_0(x), & x \in \Omega, \end{cases} \tag{4.1}$$

where

$$h(t, u(t, x)) = \arctan t \cdot \tilde{A}^{-\alpha} \sin(1 + |\tilde{A}^\alpha u(t, x)|) \int_0^\infty \frac{\zeta(t)}{1 + |u(t, x)|} dt,$$

$$f(t, u(t, x)) = (t + r_0)^a \cdot \frac{|u(t, x)|}{1 + |u(t, x)|},$$

here $t > 0$, r_0 is a positive constant,

$$\frac{l}{2} < \alpha < 1, \quad 0 < \alpha + \frac{l}{2} < 1, \quad -1 < a < q\left(\alpha + \frac{l}{2} - 1\right),$$

$\zeta(\cdot) \in L^1(\mathbf{R}_+, \mathbf{R})$ and $\dfrac{\pi}{2} \int_0^\infty |\zeta(t)| dt \leq L < 1$.

The problem (4.1) can be written abstractly as (1.1)-(1.2).

Moreover, for $t \geq 0$, we can see

$$\|f(t, u(t))\| \leq v(t),$$

where $v(t) := (t + r_0)^a$.

It is clear that the function $s \to \dfrac{v(s)}{(t-s)^{1+q(\gamma+\alpha)}}$ belongs to $L^1([0, t], \mathbf{R}_+)$ and

$$\int_0^t \frac{v(s)}{(t - s)^{1+q(\gamma+\alpha)}} ds \leq \int_0^t \frac{s^a}{(t - s)^{1+q(\gamma+\alpha)}} ds$$

$$= t^{a-q(\gamma+\alpha)} \int_0^1 s^a (1 - s)^{-1-q(\gamma+\alpha)} ds$$

$$= t^{a-q(\gamma+\alpha)} B(a + 1, -q(\gamma + \alpha))$$

$$\to 0, \quad t \to \infty.$$

where $B(\cdot, \cdot)$ is the Beta function.

Moreover, for $t_1, t_2 \geq 0$ we have

$$
\left\| h(t_1, u(t_1)) - h(t_2, u(t_2)) \right\|_\alpha
$$

$$
= \left\| \arctan t_1 \cdot \sin\left(1 + \left|\tilde{A}^\alpha u(t_1, x)\right|\right) \int_0^\infty \frac{\zeta(t)}{1 + |u(t, x)|} dt \right.
$$

$$
\left. - \arctan t_2 \cdot \sin\left(1 + \left|\tilde{A}^\alpha u(t_2, x)\right|\right) \int_0^\infty \frac{\zeta(t)}{1 + |u(t, x)|} dt \right\|
$$

$$
\leq \left[|\arctan t_1 - \arctan t_2| \cdot \left\| \sin\left(1 + \left|\tilde{A}^\alpha u(t_1, x)\right|\right) \right\| \right.
$$

$$
\left. + |\arctan t_2| \cdot \left\| \sin\left(1 + \left|\tilde{A}^\alpha u(t_1, x)\right|\right) - \sin\left(1 + \left|\tilde{A}^\alpha u(t_2, x)\right|\right) \right\| \right] \cdot \int_0^\infty |\zeta(t)| dt
$$

$$
\leq \left[|t_1 - t_2| + \frac{\pi}{2} \|u(t_1) - u(t_2)\|_\alpha \right] \cdot \int_0^\infty |\zeta(t)| \, dt
$$

$$
\leq L\left(|t_1 - t_2| + \|u(t_1) - u(t_2)\|_\alpha\right).
$$

Consequently, it follows from Theorem 3.1 that, for every $u_0 \in D(\tilde{A}^{\alpha+\beta})$ with $1 > \beta > \alpha > \frac{1}{2}$, the Equation (4.1) has at least a mild solution on $BC(\mathbf{R}_+, X_\alpha)$ and all solutions are globally attractive.

For example, if we put

$$
l = \frac{1}{12}, \quad \alpha = \frac{1}{8}, \quad a = -\frac{8}{9}, \quad q = \frac{1}{2}, \quad \zeta(t) = e^{-\pi t},
$$

then the assumptions can be satisfied.

Example 4.2: Let

$$
\widehat{A} = (-i\Delta + \sigma)^{\frac{1}{2}}, \quad D(\widehat{A}) = W^{1,3}(\mathbf{R}^2) \quad \text{(a Sobolev space)},
$$

where $i\Delta$ is the Schro" dinger operator, $\sigma > 0$ is a suitable constant.

Then $i\Delta$ generates a $\tilde{\beta}$-times integrated semigroup $S^{\tilde{\beta}}(t)$ with $\tilde{\beta} = \frac{5}{12}$ on $L^3(\mathbf{R}^2)$ such that

$$
\left\| S^{\tilde{\beta}}(t) \right\|_{L(L^3(\mathbf{R}^2))} \leq \widehat{M} t^{\tilde{\beta}}
$$

for all $t \geq 0$ and some constant $\widehat{M} > 0$ (see [20]). Therefore, by virtue of [[21], Theorem 1.3.5 (P. 15)], [[21], Definition 1.3.1 (P. 12)] for $C = I$, we deduce that the operator $i\Delta + \sigma$ belongs to $\Theta_{\frac{\pi}{2}}^{\tilde{\beta}-1}(L^3(\mathbf{R}^2))$, which denotes the family of all linear closed operators $A: D(A) \subset L^3(\mathbf{R}^2) \to L^3(\mathbf{R}^2)$ satisfying

$$
\sigma(A) \subset S_{\frac{\pi}{2}} = \left\{ z \in \mathbf{C}\backslash\{0\}; |\arg z| \leq \frac{\pi}{2} \right\} \cup \{0\},
$$

and for every $\frac{\pi}{2} < \mu < \pi$ there exists a constant $C\mu$ such that

$$\left\| R(z; A) \right\| \leq C_\mu |z|^{\tilde{\beta}-1}$$

for all $z \in \mathbf{C} \backslash S_\mu$. Thus, it follows from [[19], Proposition 3.6] that $\widehat{A} \in \Theta_\omega^\gamma(L^3(\mathbf{R}^2))$ for some $0 < \omega < \dfrac{\pi}{2}$, where

$$\gamma = -1 + 2\tilde{\beta} = -\frac{1}{6}.$$

Let $X = L^3(\mathbf{R}^2)$, we consider the following equation:

$$\begin{cases} \dfrac{\partial^q}{\partial t^q}\left[u(t, x) - \sin t \cdot e^{-(1+|u(t,x)|)} \displaystyle\int_0^\infty k(t)\dfrac{|u(t, x)|}{1 + |u(t, x)|}dt \right] = \\ \qquad\qquad -\widehat{A}\left[u(t, x) - \sin t \cdot e^{-(1+|u(t,x)|)} \displaystyle\int_0^\infty k(t)\dfrac{|u(t, x)|}{1 + |u(t, x)|}dt \right] \\ \qquad\qquad + (t + 1)^b \cdot \cos(1 + |u(t, x)|), \quad x \in \mathbf{R}^2, \\ u(0, x) = u_0(x), \quad x \in \mathbf{R}^2, \end{cases} \tag{4.2}$$

where $t > 0$, $-1 < b < q\gamma$ and $k(\cdot) \in L^1(\mathbf{R}_+, \mathbf{R})$ and $\displaystyle\int_0^\infty |k(t)dt| \leq L < 1$.

Set

$$u(t)(x) = u(t, x),$$

$$h(t, u(t))(x) = \sin t \cdot e^{-(1+|u(t,x)|)} \int_0^\infty k(t)\frac{|u(t, x)|}{1 + |u(t, x)|}dt,$$

$$f(t, u(t))(x) = (t + 1)^b \cdot \cos(1 + |u(t, x)|).$$

Then the above Equation (4.2) can be reformulated as the abstract (1.1)-(1.2). Moreover, for $t \geq 0$, we can see

$$\left\| f(t, u(t)) \right\| \leq v(t),$$

where $v(t) := (t + 1)^b$.

It is clear that the function $s \rightarrow \dfrac{v(s)}{(t-s)^{1+q\gamma}}$ belongs to $L^1([0, t], \mathbf{R}_+)$ and

$$\int_0^t \frac{v(s)}{(t - s)^{1+q\gamma}}ds \leq \int_0^t \frac{s^b}{(t - s)^{1+q\gamma}}ds$$

$$= t^{b-q\gamma}\int_0^1 s^b(1 - s)^{-1-q\gamma}ds$$

$$= t^{b-q\gamma}B(b + 1, -q\gamma)$$

$$\rightarrow 0, \quad t \rightarrow \infty.$$

Moreover, for t_1, $t_2 \geq 0$ we have

$$\left\| h(t_1, u(t_1)) - h(t_2, u(t_2)) \right\|$$

$$= \left\| \sin t_1 \cdot e^{-(1+|u(t_1,x)|)} \int_0^\infty k(t) \frac{|u(t,x)|}{1 + |u(t,x)|} dt \right.$$

$$\left. - \sin t_2 \cdot e^{-(1+|u(t_2,x)|)} \int_0^\infty k(t) \frac{|u(t,x)|}{1 + |u(t,x)|} dt \right\|$$

$$\leq \left[|\sin t_1 - \sin t_2| \cdot \left\| e^{-(1+|u(t_1,x)|)} \right\| \right.$$

$$\left. + |\sin t_2| \cdot \left\| e^{-(1+|u(t_1,x)|)} - e^{-(1+|u(t_2,x)|)} \right\| \right] \cdot \int_0^\infty |k(t)| dt$$

$$\leq \left[|t_1 - t_2| + \|u(t_1) - u(t_2)\| \right] \cdot \int_0^\infty |k(t)| dt$$

$$\leq L \big(|t_1 - t_2| + \|u(t_1) - u(t_2)\| \big).$$

Consequently, it follows from Theorem 3.2 that, for every $u_0 \in D(\widehat{A}^\beta)$ with $1 > \beta > \dfrac{5}{6}$, the Equation (4.2) has at least a mild solution on $BC(\mathbf{R}_+, X)$ and all solutions are globally attractive.

For example, if we put

$$q = \frac{1}{2}, \quad b = -\frac{1}{2}, \quad k(t) = e^{-2t},$$

then the assumptions can be satisfied.

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Authors' contributions
JL made the main contribution to Theorem 3.1 and Example 4.2, and drafted the manuscript. S-HY participated in the writing of Section 1 and made contributions to give some estimates in the proof of Theorem 3.1. FL made the main contribution to Example 4.1 and participated in giving the list of references. T-WH participated in the study of Example 4.2 and Definition 2.10. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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