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Some identities involving Gegenbauer polynomials

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Abstract

In this paper, we derive some interesting identities involving Gegenbauer polynomials arising from the orthogonality of Gegenbauer polynomials for the inner product space $\mathbb{P}_n$ with respect to the weighted inner product $\langle p_1, p_2\rangle = \int_{-1}^{1} p_1(x)p_2(x)(1-x^2)^{\lambda-\frac{1}{2}}\,dx$.

1 Introduction

The Gegenbauer polynomials are given in terms of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ with $\alpha = \beta = \lambda - \frac{1}{2}$ ($\lambda > -\frac{1}{2}$, $\lambda \neq 0$) by

$$C_n^{(\lambda)}(x) = \frac{\Gamma(\lambda+\frac{1}{2})\Gamma(n+2\lambda)}{\Gamma(2\lambda)\Gamma(n+\lambda+\frac{1}{2})} P_n^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(x) = \binom{n+2\lambda-1}{n} \sum_{k=0}^{n} \frac{\binom{n}{k}(2\lambda+n)_k}{(\lambda+\frac{1}{2})_k}\left(\frac{x-1}{2}\right)^k, \tag{1.1}$$

where $(a)_k = a(a+1)(a+2)\cdots(a+k-1)$ (see [1, 2]).

From (1.1), we note that $C_k^{(\lambda)}(x)$ is a polynomial of degree n with real coefficients and $C_n^{(\lambda)}(1) = \binom{n+2\lambda-1}{n}$. The leading coefficient of $C_n^{(\lambda)}(x)$ is $2^n\binom{\lambda+n-1}{n}$. By the theory of Jacobi polynomials with $\alpha = \beta = \lambda - \frac{1}{2}$, $\lambda > -\frac{1}{2}$, and $\lambda \neq 0$, we get

$$C_n^{(\lambda)}(-x) = (-1)^n C_n^{(\lambda)}(x). \tag{1.2}$$

It is not difficult to show that $C_n^{(\lambda)}(x)$ is a solution of the following Gegenbauer differential equation:

$$(1-x^2)y'' - (2\lambda+1)xy' + n(n+2\lambda)y = 0.$$

The Rodrigues formula for the Gegenbauer polynomials is well known as the following:

$$(1-x^2)^{\lambda-\frac{1}{2}} C_n^{(\lambda)}(x) = \frac{(-2)^n(\lambda)_n}{n!(n+2\lambda)_n}\left(\frac{d}{dx}\right)^n (1-x^2)^{n+\lambda-\frac{1}{2}} \quad \text{(see [3, 4]).} \tag{1.3}$$

Equation (1.3) can be easily derived from the properties of Jacobi polynomials.

As is well known, the generating function of Gegenbauer polynomials is given by

$$\frac{2^{\lambda-\frac{1}{2}}}{(1-2xt+t^2)^{\frac{1}{2}}(1-xt+\sqrt{1-2xt+t^2})^{\lambda-\frac{1}{2}}}=\sum_{n=0}^{\infty}\frac{(\lambda+\frac{1}{2})_n}{(2\lambda)_n}C_n^{(\lambda)}(x)t^n. \tag{1.4}$$

Equation (1.4) can be also derived from the generating function of Jacobi polynomials.

From (1.4), we note that

$$\frac{1}{(1-2xt+t^2)^{\lambda}}=\sum_{n=0}^{\infty}C_n^{(\lambda)}(x)t^n \quad \bigl(|t|<1,|x|\leq 1\bigr). \tag{1.5}$$

The proof of (1.5) is given in the following book: Stein and Weiss, *Introduction to Fourier Analysis in Euclidean Space*, Princeton University Press, 1971.

By (1.1) and (1.2), we get

$$\int_{-1}^{1}C_n^{(\lambda)}(x)C_m^{(\lambda)}(x)\bigl(1-x^2\bigr)^{\lambda-\frac{1}{2}}\,dx=\frac{\pi 2^{1-2\lambda}\Gamma(n+2\lambda)}{n!(n+\lambda)(\Gamma(\lambda))^2}\delta_{m,n}, \tag{1.6}$$

where $\delta_{m,n}$ is the Kronecker symbol and it holds for each fixed $\lambda\in\mathbb{R}$ with $\lambda>-\frac{1}{2}$ and $\lambda\neq 0$.

Equation (1.6) implies the orthogonality of $C_n^{(\lambda)}(x)$ and equation (1.6) is important in deriving our results in this paper. From (1.5), we can derive the following derivative of Gehenbauer polynomials $C_n^{(\lambda)}(x)$:

$$\frac{d}{dx}C_n^{(\lambda)}(x)=2\lambda C_{n-1}^{(\lambda+1)}(x), \quad \text{for } n\geq 1. \tag{1.7}$$

By (1.7), we get

$$\frac{d^k}{dx^k}C_n^{(\lambda)}(x)=2^k\lambda^k C_{n-k}^{(\lambda+k)}(x). \tag{1.8}$$

As is well known, the Bernoulli polynomials $B_n(x)$ are defined by the generating function to be

$$\frac{t}{e^t-1}e^{xt}=e^{B(x)t}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!} \quad \text{(see [5–13])}, \tag{1.9}$$

with the usual convention about replacing $B^n(x)$ by $B_n(x)$. In the special case, $x=0$, $B_n(0)=B_n$ are called the nth Bernoulli numbers.

From (1.9), we note that

$$B_n(x)=(B+x)^n=\sum_{l=0}^{n}\binom{n}{l}B_{n-l}x^l \quad \text{(see [9–13])}, \tag{1.10}$$

and

$$B_n'(x)=\frac{d}{dx}B_n(x)=nB_{n-1}(x). \tag{1.11}$$

The Euler polynomials $E_n(x)$ are also defined by the generating function to be

$$\frac{2}{e^t+1}e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \quad \text{(see [4, 14–18])}, \tag{1.12}$$

with the usual convention about replacing $E^n(x)$ by $E_n(x)$. In the special case, $x = 0$, $E_n(0) = E_n$ are called the nth Euler numbers. By (1.12), we see that the recurrence formula for E_n is given by

$$E_0 = 1, \qquad (E+1)^n + E_n = 2\delta_{0,n} \quad \text{(see [19–21])}. \tag{1.13}$$

For each fixed $\lambda \in \mathbb{R}$ with $\lambda > -\frac{1}{2}$ and $\lambda \neq 0$, let $\mathbb{P}_n = \{p(x) \in \mathbb{R}[x] \mid \deg p(x) \leq n\}$ be an inner product space with respect to the inner product

$$\langle p_1(x), p_2(x)\rangle = \int_{-1}^{1} \left(1-x^2\right)^{\lambda-\frac{1}{2}} p_1(x)p_2(x)\,dx, \tag{1.14}$$

where $p_1(x), p_2(x) \in \mathbb{P}_n$. The information entropy of Gegenbauer polynomials is relevant since it is related to the angular part of the information entropies of certain quantum mechanical systems such as the harmonic oscillator and the hydrogen atom in D dimensions. In [7], Buyarov, Lopez-Artes, Martinez-Finkelshtein, and Van Assche gave an effective method to compute the entropy for Gegenbauer polynomials with an integer parameter and obtain the first few terms in the asymptotic expansion as the degree of the polynomial tends to infinity. That is, an efficient method was provided for evaluating, in a closed form, the information entropy of the Gegenbauer polynomials $C_n^{(\lambda)}(x)$ in the case when $\lambda = l \in \mathbb{N}$. For given values of n and l, this method requires the computation by means of recurrence relations of two auxiliary polynomials, $P(x)$ and $H(x)$, of degrees $2l-2$ and $2l-4$, respectively (see [18]). In [18], Sanchez-Ruiz showed that $P(x)$ is related to the coefficients of the Gaussian quadrature formula for the Gegenbauer weights $w_l(x) = (1-x^2)^{l-\frac{1}{2}}$, and this fact is used to obtain the explicit expression of $P(x)$. The position and momentum information entropies of D-dimensional quantum systems with central potentials, such as the isotropic harmonic oscillator and the hydrogen atom, depend on the entropies of the (hyper)spherical harmonics (see [2]). In turn, these entropies are expressed in terms of the entropies of the Gegenbauer (ultraspherical) polynomials $C_n^{(\lambda)}(x)$, the parameter λ being either an integer or a half-integer number. Up to now, however, the exact analytical expression of the entropy of Gegenbauer polynomials of arbitrary degree n has only been obtained for the particular values of the parameter $\lambda = 0, 1, 2$ (see [2]). In [2], de Vicente, Gandy, Sanchez-Ruiz presented a novel approach to the evaluation of the information entropy of Gegenbauer polynomials, which makes use of trigonometric representations for these polynomials and complex integration techniques. Using this method, we are able to find the analytical expression of the entropy for arbitrary values of both n and $\lambda \in \mathbb{N}$ (see [2]). The Gegenbauer polynomial seems to be interesting and important in the area of mathematical physics. Recently, many authors have studied Gegenbauer polynomials related to mathematical physics (see [1–5, 7, 10, 11, 14, 18, 21, 22]). In this paper, we derive some interesting identities involving Gegenbauer polynomials arising from the orthogonality of those for the inner product space $\mathbb{P}_n$ with respect to the weighted inner product $\langle p_1, p_2\rangle = \int_{-1}^{1} p_1(x)p_2(x)(1-x^2)^{\lambda-\frac{1}{2}}\,dx$.

Our methods used in this paper are useful in finding some new identities and relations on the Bernoulli and Euler polynomials involving Gegenbauer polynomials.

2 Some identities involving Gegenbauer polynomials

Let us take $p(x) = \sum_{k=0}^{n} d_k C_k^{(\lambda)}(x) \in \mathbb{P}_n$, $d_k \in \mathbb{R}$. Then, by (1.6) and (1.14), we get

$$\begin{aligned}\langle p(x), C_k^{(\lambda)}(x)\rangle &= d_k \langle C_k^{(\lambda)}(x), C_k^{(\lambda)}(x)\rangle \\ &= d_k \int_{-1}^{1} (1-x^2)^{\lambda-\frac{1}{2}} C_k^{(\lambda)}(x) C_k^{(\lambda)}(x)\,dx = d_k \frac{\pi 2^{1-2\lambda}\Gamma(k+2\lambda)}{k!(k+\lambda)(\Gamma(\lambda))^2}.\end{aligned} \tag{2.1}$$

Thus, from (2.1), we have

$$d_k = \frac{(\Gamma(\lambda))^2 k!(k+\lambda)}{\pi 2^{1-2\lambda}\Gamma(k+2\lambda)} \int_{-1}^{1} (1-x^2)^{\lambda-\frac{1}{2}} p(x) C_k^{(\lambda)}(x)\,dx. \tag{2.2}$$

By (1.3) and (2.2), we get

$$\begin{aligned} d_k &= \frac{(\Gamma(\lambda))^2 k!(k+\lambda)}{\pi 2^{1-2\lambda}\Gamma(k+2\lambda)} \times \frac{(-2)^k(\lambda)_k}{k!(k+2\lambda)_k} \int_{-1}^{1} \left(\frac{d^k}{dx^k}(1-x^2)^{k+\lambda-\frac{1}{2}}\right) p(x)\,dx \\ &= \frac{(k+\lambda)\Gamma(\lambda)}{(-2)^k\sqrt{\pi}\,\Gamma(k+\lambda+\frac{1}{2})} \int_{-1}^{1} \left(\frac{d^k}{dx^k}(1-x^2)^{k+\lambda-\frac{1}{2}}\right) p(x)\,dx. \end{aligned} \tag{2.3}$$

Therefore, by (2.3), we obtain the following proposition.

Proposition 2.1 *For $p(x) \in \mathbb{P}_n$, let*

$$p(x) = \sum_{n=0}^{n} d_k C_k^{(\lambda)}(x) \quad (d_k \in \mathbb{R}).$$

Then

$$d_k = \frac{(k+\lambda)\Gamma(\lambda)}{(-2)^k\sqrt{\pi}\,\Gamma(k+\lambda+\frac{1}{2})} \int_{-1}^{1} \left(\frac{d^k}{dx^k}(1-x^2)^{k+\lambda-\frac{1}{2}}\right) p(x)\,dx.$$

For example, let $p(x) = x^n \in \mathbb{P}_n$. From Proposition 2.1, we note that

$$\begin{aligned} d_k &= \frac{(k+\lambda)\Gamma(\lambda)}{(-2)^k\sqrt{\pi}\,\Gamma(k+\lambda+\frac{1}{2})} \int_{-1}^{1} \left(\frac{d^k}{dx^k}(1-x^2)^{k+\lambda-\frac{1}{2}}\right) x^n\,dx \\ &= (-n) \int_{-1}^{1} \left(\frac{d^{k-1}}{dx^{k-1}}(1-x^2)^{k+\lambda-\frac{1}{2}}\right) x^{n-1}\,dx \times \left(\frac{(k+\lambda)\Gamma(\lambda)}{\sqrt{\pi}(-2)^k\Gamma(k+\lambda+\frac{1}{2})}\right) \\ &= \cdots \\ &= \frac{(k+\lambda)n!\Gamma(\lambda)}{(n-k)!2^k\sqrt{\pi}(k+\frac{1}{2}+\lambda)} \int_{-1}^{1} (1-x^2)^{k+\lambda-\frac{1}{2}} x^{n-k}\,dx \\ &= \left(1+(-1)^{n-k}\right) \frac{(k+\lambda)n!\Gamma(\lambda)}{(n-k)!2^k\sqrt{\pi}\,\Gamma(k+\frac{1}{2}+\lambda)} \int_{0}^{1} (1-x^2)^{k+\lambda-\frac{1}{2}} x^{n-k}\,dx. \end{aligned} \tag{2.4}$$

Let us assume that $n-k \equiv 0 \pmod 2$. Then, by (2.4), we get

$$\begin{aligned} d_k &= \frac{(k+\lambda)n!\Gamma(\lambda)}{(n-k)!2^k\sqrt{\pi}\Gamma(k+\frac{1}{2}+\lambda)} B\left(k+\lambda+\frac{1}{2}, \frac{n-k+1}{2}\right) \\ &= \frac{\Gamma(\frac{n-k+1}{2})\Gamma(k+\lambda+\frac{1}{2})}{\Gamma(\frac{n+k+2\lambda+2}{2})}, \end{aligned} \tag{2.5}$$

where $B(\alpha,\beta)$ is the beta function which is defined by $B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

It is easy to show that

$$\begin{aligned} \Gamma\left(\frac{n-k+1}{2}\right) &= \frac{n-k-1}{2}\Gamma\left(\frac{n-k-1}{2}\right) \\ &= \left(\frac{n-k-1}{2}\right)\left(\frac{n-k-3}{2}\right)\Gamma\left(\frac{n-k-3}{2}\right) = \cdots \\ &= \frac{(\frac{n-k}{2})(\frac{n-k-1}{2})(\frac{n-k-2}{2})\cdots\frac{2}{2}\Gamma(\frac{1}{2})}{(\frac{n-k}{2})(\frac{n-k-2}{2})\cdots(\frac{2}{2})} \\ &= \frac{(n-k)!\sqrt{\pi}}{2^{n-k}(\frac{n-k}{2})!}. \end{aligned} \tag{2.6}$$

Therefore, by (2.5) and (2.6), we obtain the following identity:

$$x^n = \sum_{0\le k\le n, n-k\equiv 0 \pmod 2} \frac{(k+\lambda)n!\Gamma(\lambda)}{2^n(\frac{n-k}{2})!\Gamma(\frac{n+k+2\lambda+2}{2})} C_k^{(\lambda)}(x). \tag{2.7}$$

Let us take $p(x)=B_n(x)\in\mathbb{P}_n$. Then, by (1.10), we get

$$\begin{aligned} d_k &= \frac{(k+\lambda)\Gamma(\lambda)}{(-2)^k\sqrt{\pi}\Gamma(k+\lambda+\frac{1}{2})}\int_{-1}^{1}\left(\frac{d^k}{dx^k}\left(1-x^2\right)^{k+\lambda-\frac{1}{2}}\right)B_n(x)\,dx \\ &= \frac{(k+\lambda)\Gamma(\lambda)(-n)}{(-2)^k\sqrt{\pi}\Gamma(k+\lambda+\frac{1}{2})}\int_{-1}^{1}\left(\frac{d^{k-1}}{dx^{k-1}}\left(1-x^2\right)^{k+\lambda-\frac{1}{2}}\right)B_{n-1}(x)\,dx = \cdots \\ &= \frac{(k+\lambda)\Gamma(\lambda)(-n)(-(n-1))\cdots(-(n-k+1))}{(-2)^k\sqrt{\pi}\Gamma(k+\lambda+\frac{1}{2})}\int_{-1}^{1}\left(1-x^2\right)^{k+\lambda-\frac{1}{2}}B_{n-k}(x)\,dx \\ &= \frac{(k+\lambda)\Gamma(\lambda)}{2^k\sqrt{\pi}\Gamma(k+\lambda+\frac{1}{2})}\times\frac{n!}{(n-k)!}\int_{-1}^{1}\left(1-x^2\right)^{k+\lambda-\frac{1}{2}}B_{n-k}(x)\,dx. \end{aligned} \tag{2.8}$$

From (1.10) and (2.7), we can derive the following equation:

$$\begin{aligned} &\int_{-1}^{1}\left(1-x^2\right)^{k+\lambda-\frac{1}{2}}B_{n-k}(x)\,dx \\ &\quad= \sum_{l=0}^{n-k}\binom{n-k}{l}B_{n-k-l}\int_{-1}^{1}\left(1-x^2\right)^{k+\lambda-\frac{1}{2}}x^l\,dx \\ &\quad= \sum_{l=0}^{n-k}\binom{n-k}{l}B_{n-k-l}\left(1+(-1)^l\right)\int_{0}^{1}\left(1-x^2\right)^{k+\lambda-\frac{1}{2}}x^l\,dx. \end{aligned} \tag{2.9}$$

Let us consider that $l \equiv 0 \pmod 2$. Then, by (2.9), we get

$$\begin{aligned}
&\int_{-1}^{1}\left(1-x^2\right)^{k+\lambda-\frac{1}{2}} B_{n-k}(x)\,dx \\
&\quad = 2\sum_{0\le l\le n-k, l\equiv 0 \pmod 2}\binom{n-k}{l}B_{n-k-l}\int_0^1\left(1-x^2\right)^{k+\lambda-\frac{1}{2}}x^l\,dx \\
&\quad = \sum_{0\le l\le n-k, l\equiv 0 \pmod 2}\binom{n-k}{l}B_{n-k-l}\int_0^1(1-y)^{k+\lambda-\frac{1}{2}}y^{\frac{l-1}{2}}\,dy \\
&\quad = \sum_{0\le l\le n-k, l\equiv 0 \pmod 2}\binom{n-k}{l}B_{n-k-l}\frac{\Gamma(k+\lambda+\frac{1}{2})\Gamma(\frac{l+1}{2})}{\Gamma(\frac{2k+2\lambda+l+2}{2})}.
\end{aligned} \tag{2.10}$$

For $l \in \mathbb{Z}_+$ with $l \equiv 0 \pmod 2$, we have

$$\begin{aligned}
\Gamma\left(\frac{l+1}{2}\right) &= \Gamma\left(\frac{l-1}{2}+1\right) = \frac{l-1}{2}\Gamma\left(\frac{l-1}{2}\right) \\
&= \left(\frac{l-1}{2}\right)\left(\frac{l-3}{2}\right)\Gamma\left(\frac{l-3}{2}\right) = \cdots \\
&= \left(\frac{l-1}{2}\right)\left(\frac{l-3}{2}\right)\cdots\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \frac{(\frac{1}{2})^l l!\Gamma(\frac{1}{2})}{(\frac{l}{2})!} = \frac{l!\sqrt{\pi}}{2^l(\frac{l}{2})!}.
\end{aligned} \tag{2.11}$$

By (2.10) and (2.11), we get

$$\begin{aligned}
&\int_{-1}^{1}\left(1-x^2\right)^{k+\lambda-\frac{1}{2}} B_{n-k}(x)\,dx \\
&\quad = \sum_{0\le l\le n-k, l\equiv 0 \pmod 2}\binom{n-k}{l}B_{n-k-l}\frac{\Gamma(k+\lambda+\frac{1}{2})\Gamma(\frac{l+1}{2})}{\Gamma(\frac{2k+2\lambda+l+2}{2})} \\
&\quad = \sum_{0\le l\le n-k, l\equiv 0 \pmod 2}\binom{n-k}{l}B_{n-k-l}\frac{l!\sqrt{\pi}}{2^l(\frac{l}{2})!}\times\frac{\Gamma(k+\lambda+\frac{1}{2})}{\Gamma(\frac{2k+2\lambda+l+2}{2})}.
\end{aligned} \tag{2.12}$$

From (2.8) and (2.12), we have

$$d_k = \frac{n!(k+\lambda)\Gamma(\lambda)}{2^k(n-k)!}\sum_{0\le l\le n-k, l\equiv 0 \pmod 2}\frac{\binom{n-k}{l}B_{n-k-l}l!}{2^l(\frac{l}{2})!\Gamma(\frac{2k+2\lambda+l+2}{2})}. \tag{2.13}$$

Therefore, by (2.13) and Proposition 2.1, we obtain the following theorem.

Theorem 2.2 *For $n \in \mathbb{Z}_+$, we have*

$$\frac{B_n(x)}{n!} = \Gamma(\lambda)\sum_{k=0}^{n}\left(\frac{(k+\lambda)}{2^k(n-k)!}\sum_{0\le l\le n-k, l\equiv 0 \pmod 2}\frac{\binom{n-k}{l}B_{n-k-l}l!}{2^l(\frac{l}{2})!\Gamma(\frac{2k+2\lambda+l+2}{2})}\right)C_k^{(\lambda)}(x).$$

By the same method, we get

$$\frac{E_n(x)}{n!} = \Gamma(\lambda)\sum_{k=0}^{n}\left(\frac{(k+\lambda)}{2^k(n-k)!}\sum_{0\le l\le n-k, l\equiv 0 \pmod 2}\frac{\binom{n-k}{l}E_{n-k-l}l!}{2^l(\frac{l}{2})!\Gamma(\frac{2k+2\lambda+l+2}{2})}\right)C_k^{(\lambda)}(x). \tag{2.14}$$

From (1.1), we note that

$$\begin{aligned}
&C_{n-k}^{(\lambda)}(x)C_k^{(\lambda)}(x)\\
&\quad=\binom{n-k+2\lambda-1}{n-k}\sum_{l=0}^{n-k}\frac{\binom{n-k}{l}(2\lambda+n-k)_l}{(\lambda+\frac{1}{2})_l}\left(\frac{x-1}{2}\right)^l\binom{k+2\lambda-1}{k}\\
&\qquad\times\sum_{m-0}^{k}\frac{\binom{k}{m}(2\lambda+k)_m}{(\lambda+\frac{1}{2})_m}\left(\frac{x-1}{2}\right)^m\\
&\quad=\binom{n-k+2\lambda-1}{n-k}\binom{k+2\lambda-1}{k}\\
&\qquad\times\sum_{p=0}^{n}\left(\sum_{m=0}^{p}\frac{\binom{n-k}{p-m}\binom{k}{m}(2\lambda+k)_m(2\lambda+n-k)_{p-m}}{(\lambda+\frac{1}{2})_m(\lambda+\frac{1}{2})_{p-m}}\right)\left(\frac{x-1}{2}\right)^p.
\end{aligned}\tag{2.15}$$

Let us take $p(x)=C_k^{(\lambda)}(x)C_{n-k}^{(\lambda)}(x)\in\mathbb{P}_n$. From Proposition 2.1, $p(x)$ can be rewritten as

$$p(x)=C_k^{(\lambda)}(x)C_{n-k}^{(\lambda)}(x)=\sum_{r=0}^{n}d_rC_r^{(\lambda)}(x)\quad(d_r\in\mathbb{R}).\tag{2.16}$$

Then, by Proposition 2.1 and (2.15), we get

$$\begin{aligned}
d_r&=\frac{(r+\lambda)\Gamma(\lambda)}{(-2)^r\sqrt{\pi}\,\Gamma(r+\lambda+\frac{1}{2})}\int_{-1}^{1}\left(\frac{d^r}{dx^r}\left(1-x^2\right)^{r+\lambda-\frac{1}{2}}\right)C_k^{(\lambda)}(x)C_{n-k}^{(\lambda)}(x)\,dx\\
&=\frac{(r+\lambda)\Gamma(\lambda)}{(-2)^r\sqrt{\pi}\,\Gamma(r+\lambda+\frac{1}{2})}\binom{n-k+2\lambda-1}{n-k}\binom{k+2\lambda-1}{k}\\
&\quad\times\sum_{p=0}^{n}\left(\sum_{m=0}^{p}\frac{\binom{n-k}{p-m}\binom{k}{m}(2\lambda+k)_m}{(\lambda+\frac{1}{2})_m(\lambda+\frac{1}{2})_{p-m}}(2\lambda+n-k)_{p-m}\right)\\
&\quad\times\int_{-1}^{1}\left(\frac{d^r}{dx^r}\left(1-x^2\right)^{r+\lambda-\frac{1}{2}}\right)\left(\frac{x-1}{2}\right)^p dx\\
&=\frac{(r+\lambda)\Gamma(\lambda)}{(-2)^r\sqrt{\pi}\,\Gamma(r+\lambda+\frac{1}{2})}\binom{n-k+2\lambda-1}{n-k}\binom{k+2\lambda-1}{k}\\
&\quad\times\sum_{p=r}^{n}\left(\sum_{m=0}^{p}\frac{\binom{n-k}{p-m}\binom{k}{m}(2\lambda+k)_m}{(\lambda+\frac{1}{2})_m(\lambda+\frac{1}{2})_{p-m}}(2\lambda+n-k)_{p-m}\right)\\
&\quad\times\int_{-1}^{1}\left(\frac{d^r}{dx^r}\left(1-x^2\right)^{r+\lambda-1}\right)\left(\frac{x-1}{2}\right)^p dx.
\end{aligned}\tag{2.17}$$

It is not difficult to show that

$$\begin{aligned}
&\int_{-1}^{1}\left(\frac{d^r}{dx^r}\left(1-x^2\right)^{r+\lambda-\frac{1}{2}}\right)\left(\frac{x-1}{2}\right)^p dx\\
&\quad=\frac{(-1)^rp!}{2^p(p-r)!}\int_{-1}^{1}\left(1-x^2\right)^{r+\lambda-\frac{1}{2}}(1-x)^{p-r}(-1)^{p-r}\,dx\\
&\quad=\frac{(-1)^pp!}{2^p(p-r)!}\int_{-1}^{1}(1-x)^{p+\lambda-\frac{1}{2}}(1+x)^{r+\lambda-\frac{1}{2}}\,dx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(-1)^p p!}{2^p (p-r)!} \int_0^1 (2-2y)^{p+\lambda-\frac{1}{2}} (2y)^{r+\lambda-\frac{1}{2}} 2\, dy \\
&= \frac{(-1)^p 2^{p+\lambda-\frac{1}{2}+r+\lambda-\frac{1}{2}+1}}{2^p} \times \frac{p!}{(p-r)!} \int_0^1 (1-y)^{p+\lambda-\frac{1}{2}} y^{r+\lambda-\frac{1}{2}}\, dy \\
&= (-1)^p 2^{r+2\lambda} \frac{p!}{(p-r)!} \times \frac{\Gamma(p+\lambda+\frac{1}{2})\Gamma(r+\lambda+\frac{1}{2})}{\Gamma(r+p+2\lambda+1)}.
\end{aligned}
\tag{2.18}
$$

From the fundamental theorem of gamma function, we have

$$
\frac{\Gamma(p+\lambda+\frac{1}{2})}{\Gamma(r+p+2\lambda+1)} = \frac{(p+\lambda-\frac{1}{2})\cdots(\lambda+\frac{1}{2})\Gamma(\lambda+\frac{1}{2})}{(r+p+2\lambda)\cdots 2\lambda\Gamma(2\lambda)} = \frac{(\lambda+\frac{1}{2})_p \sqrt{\pi}\, 2^{1-\lambda}}{(2\lambda)_{r+p+1}\Gamma(\lambda)}. \tag{2.19}
$$

By (2.18) and (2.19), we get

$$
\begin{aligned}
&\int_{-1}^1 \left(\frac{d^r}{dx^r}\left(1-x^2\right)^{r+\lambda-\frac{1}{2}}\right)\left(\frac{x-1}{2}\right)^p dx \\
&= (-1)^p 2^{r+2\lambda} \frac{p!}{(p-r)!} \times \frac{\Gamma(p+\lambda+\frac{1}{2})\Gamma(r+\lambda+\frac{1}{2})}{\Gamma(r+p+2\lambda+1)} \\
&= (-1)^p 2^{r+2\lambda} \frac{p!}{(p-r)!} \Gamma\left(r+\lambda+\frac{1}{2}\right) \times \frac{(\lambda+\frac{1}{2})_p \sqrt{\pi}\, 2^{1-\lambda}}{(2\lambda)_{r+p+1}\Gamma(\lambda)} \\
&= (-1)^p 2^{r+\lambda+1} \frac{p!}{(p-r)!} \Gamma\left(r+\lambda+\frac{1}{2}\right) \times \frac{(\lambda+\frac{1}{2})_p \sqrt{\pi}}{(2\lambda)_{r+p+1}\Gamma(\lambda)}.
\end{aligned}
\tag{2.20}
$$

From (2.17) and (2.20), we have

$$
\begin{aligned}
d_r &= \frac{(r+\lambda)\Gamma(\lambda)}{(-2)^r \sqrt{\pi}\,\Gamma(r+\lambda+\frac{1}{2})} \binom{n-k+2\lambda-1}{n-k}\binom{k+2\lambda-1}{k} \\
&\quad \times \sum_{p=r}^n \left(\sum_{m=0}^p \frac{\binom{n-k}{p-m}\binom{k}{m}(2\lambda+k)_m}{(\lambda+\frac{1}{2})_m(\lambda+\frac{1}{2})_{p-m}} (2\lambda+n-k)_{p-m} \right) \\
&\quad \times (-1)^p 2^{r+\lambda+1} \frac{p!}{(p-r)!} \Gamma\left(\lambda+\frac{1}{2}+r\right) \times \frac{(\lambda+\frac{1}{2})_p \sqrt{\pi}}{(2\lambda)_{r+p+1}\Gamma(\lambda)} \\
&= (-1)^{r+p} 2^{\lambda+1} (r+\lambda) \binom{n-k+2\lambda-1}{n-k}\binom{k+2\lambda-1}{k} \\
&\quad \times \sum_{p=r}^n \left(\sum_{m=0}^p \frac{\binom{n-k}{p-m}\binom{k}{m}(2\lambda+k)_m}{(\lambda+\frac{1}{2})_m(\lambda+\frac{1}{2})_{p-m}} (2\lambda+n-k)_{p-m} \frac{p!(\lambda+\frac{1}{2})_p}{(p-r)!(2\lambda)_{r+p+1}} \right).
\end{aligned}
\tag{2.21}
$$

Therefore, by (2.21), we obtain the following theorem.

Theorem 2.3 *For $n, k \in \mathbb{Z}_+$ with $n \geq k$, we have*

$$
\begin{aligned}
C_{n-k}^{(\lambda)}(x) C_k^{(\lambda)}(x) &= 2^{\lambda+1} \binom{n-k+2\lambda-1}{n-k}\binom{k+2\lambda-1}{k} \sum_{r=0}^n \sum_{p=r}^n \sum_{m=0}^p \Bigg\{ (r+\lambda)(-1)^{p+r} \\
&\quad \times \frac{\binom{n-k}{p-m}\binom{k}{m}(2\lambda+k)^m (2\lambda+n-k)_{p-m} p! (\lambda+\frac{1}{2})_p}{(\lambda+\frac{1}{2})_m(\lambda+\frac{1}{2})_{p-m}(p-r)!(2\lambda)_{r+p+1}} \Bigg\} C_r^{(\lambda)}(x).
\end{aligned}
$$

Let us take $p(x) = C_n^{(\lambda)}(x) \in \mathbb{P}_n$. Then, from (1.1), we have

$$\begin{aligned}
C_n^{(\lambda)}(x) &= \frac{\Gamma(\lambda+\frac{1}{2})\Gamma(n+2\lambda)}{\Gamma(2\lambda)\Gamma(n+\lambda+\frac{1}{2})} P_n^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(x) \\
&= \frac{(n+2\lambda-1)\cdots(2\lambda)}{(n+\lambda-\frac{1}{2})\cdots(\lambda+\frac{1}{2})} P_n^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(x) = \frac{\binom{n+2\lambda-1}{n}}{\binom{n+\lambda-\frac{1}{2}}{n}} P_n^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(x). \qquad (2.22)
\end{aligned}$$

In the previous paper, we have shown that

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k}\binom{n+\beta}{k}\left(\frac{x-1}{2}\right)^k \left(\frac{x+1}{2}\right)^{n-k} \quad \text{(see [14]).} \qquad (2.23)$$

From (2.22) and (2.23), we have

$$C_n^{(\lambda)}(x) = \frac{\binom{n+2\lambda-1}{n}}{\binom{n+\lambda-\frac{1}{2}}{n}} \sum_{k=0}^{n} \binom{n+\lambda-\frac{1}{2}}{n-k}\binom{n+\lambda-\frac{1}{2}}{k}\left(\frac{x-1}{2}\right)^k \left(\frac{x+1}{2}\right)^{n-k}, \qquad (2.24)$$

and

$$\frac{d^k}{dx^k} C_n^{(\lambda)}(x) = 2^k \lambda^k C_{n-k}^{(\lambda+k)}(x). \qquad (2.25)$$

Let $p(x) = C_n^{(\lambda)}(x) = \sum_{k=0}^{n} d_k C_k^{(\lambda)}(x)$. Then, by Proposition 2.1, we get

$$\begin{aligned}
d_k &= \frac{(k+\lambda)\Gamma(\lambda)}{(-2)^k\sqrt{\pi}\,\Gamma(k+\lambda+\frac{1}{2})} \int_{-1}^{1} \left(\frac{d^k}{dx^k}(1-x^2)^{k+\lambda-\frac{1}{2}}\right) C_n^{(\lambda)}(x)\,dx \\
&= \frac{(k+\lambda)\Gamma(\lambda)}{(-2)^k\sqrt{\pi}\,\Gamma(k+\lambda+\frac{1}{2})}(-1)^k 2^k \lambda^k \int_{-1}^{1} (1-x^2)^{k+\lambda-\frac{1}{2}} C_{n-k}^{(\lambda+k)}(x)\,dx \\
&= \frac{\lambda^k(k+\lambda)\Gamma(\lambda)}{\sqrt{\pi}\,\Gamma(k+\lambda+\frac{1}{2})} \int_{-1}^{1} (1-x^2)^{k+\lambda-\frac{1}{2}} C_{n-k}^{(\lambda+k)}(x)\,dx. \qquad (2.26)
\end{aligned}$$

By (2.24), we get

$$\begin{aligned}
&C_{n-k}^{(\lambda+k)}(x) \\
&\quad= \frac{\binom{n-k+2(\lambda+k)-1}{n-k}}{\binom{n-k+\lambda+k-\frac{1}{2}}{n-k}} \sum_{l=0}^{n-k} \binom{n-k+\lambda+k-\frac{1}{2}}{n-k-l}\binom{n-k+\lambda+k-\frac{1}{2}}{l}\left(\frac{x-1}{2}\right)^l \left(\frac{x+1}{2}\right)^{n-k-l} \\
&\quad= \frac{\binom{n+k+2\lambda-1}{n-k}}{\binom{n+\lambda-\frac{1}{2}}{n-k}} \sum_{l=0}^{n-k} \binom{n+\lambda-\frac{1}{2}}{n-k-l}\binom{n+\lambda-\frac{1}{2}}{l}\left(\frac{x-1}{2}\right)^l \left(\frac{x+1}{2}\right)^{n-k-l}. \qquad (2.27)
\end{aligned}$$

From (2.26) and (2.27), we have

$$\begin{aligned}
d_k &= \frac{\lambda^k(k+\lambda)\Gamma(\lambda)}{\sqrt{\pi}\,\Gamma(k+\lambda+\frac{1}{2})} \times \frac{\binom{n+k+2\lambda-1}{n-k}}{\binom{n+\lambda-\frac{1}{2}}{n-k}} \sum_{l=0}^{n-k} \binom{n+\lambda-\frac{1}{2}}{n-k-l}\binom{n+\lambda-\frac{1}{2}}{l}(-1)^l \left(\frac{1}{2}\right)^{n-k} \\
&\quad \times \int_{-1}^{1} (1-x)^{k+\lambda-\frac{1}{2}+l}(1+x)^{\lambda+n-\frac{1}{2}-l}\,dx. \qquad (2.28)
\end{aligned}$$

It is easy to show that

$$\begin{aligned}
&\int_{-1}^{1}(1-x)^{k+\lambda-\frac{1}{2}+l}(1+x)^{\lambda+n-l-\frac{1}{2}}\,dx \\
&\quad = \int_{0}^{1}(2-2y)^{k+\lambda-\frac{1}{2}+l}(2y)^{\lambda+n-l-\frac{1}{2}}2\,dy \\
&\quad = 2^{n+2\lambda+k}\int_{0}^{1}(1-y)^{k+\lambda+l-\frac{1}{2}}y^{\lambda+n-l-\frac{1}{2}}\,dy \\
&\quad = 2^{k+n+2\lambda}\frac{\Gamma(k+\lambda+l+\frac{1}{2})\Gamma(\lambda+n-l+\frac{1}{2})}{\Gamma(k+n+2\lambda)}.
\end{aligned} \tag{2.29}$$

By the fundamental theorem of gamma function, we see that

$$\Gamma\left(k+\lambda+l+\frac{1}{2}\right) = \binom{k+\lambda+l-\frac{1}{2}}{l} l!\,\Gamma\left(k+\lambda+\frac{1}{2}\right), \tag{2.30}$$

$$\Gamma\left(\lambda+n-l+\frac{1}{2}\right) = \binom{\lambda+n-l-\frac{1}{2}}{n-l}(n-l)!\,\Gamma\left(\lambda+\frac{1}{2}\right), \tag{2.31}$$

and

$$\Gamma(k+2\lambda+n) = \binom{k+2\lambda+n-1}{n+k}(n+k)!\,\Gamma(2\lambda). \tag{2.32}$$

As is well known, the duplication formula for the gamma function is given by

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z). \tag{2.33}$$

By (2.29), (2.30), (2.31), and (2.32), we get

$$\int_{-1}^{1}(1-x)^{k+\lambda+l-\frac{1}{2}}(1+x)^{\lambda+n-l-\frac{1}{2}}\,dx = 2^{k+n+1}\frac{\binom{k+\lambda+l-\frac{1}{2}}{l}\binom{\lambda+n-l-\frac{1}{2}}{n-l}\Gamma(k+\lambda+\frac{1}{2})}{\binom{n}{l}\binom{k+2\lambda+n-1}{n+k}\binom{n+k}{k}k!\,\Gamma(\lambda)}\sqrt{\pi}. \tag{2.34}$$

From (2.28) and (2.34), we have

$$\begin{aligned}
d_k &= \lambda^k(k+\lambda)2^{2k+1}\frac{\binom{n+k+2\lambda-1}{n-k}}{\binom{n+\lambda-\frac{1}{2}}{n-k}} \\
&\quad\times\sum_{l=0}^{n-k}\binom{n+\lambda-\frac{1}{2}}{n-k-l}\binom{n+\lambda-\frac{1}{2}}{l}(-1)^l\frac{\binom{k+\lambda+l-\frac{1}{2}}{l}\binom{\lambda+n-l-\frac{1}{2}}{n-l}}{\binom{n}{l}\binom{k+2\lambda+n-1}{n+k}\binom{n+k}{k}k!}.
\end{aligned} \tag{2.35}$$

Therefore, by (2.35), we obtain the following theorem.

Theorem 2.4 *For $n \in \mathbb{Z}_+$, we have*

$$\begin{aligned}
C_n^{(\lambda)}(x) &= \sum_{k=0}^{n}\left\{\frac{\lambda^k(k+\lambda)2^{2k+1}\binom{n+k+2\lambda-1}{n-k}}{\binom{n+\lambda-\frac{1}{2}}{n-k}}\right. \\
&\quad\left.\times\sum_{l=0}^{n-k}\frac{\binom{n+\lambda-\frac{1}{2}}{n-k-l}\binom{n+\lambda-\frac{1}{2}}{l}(-1)^l\binom{k+\lambda+l-\frac{1}{2}}{l}}{\binom{n}{l}\binom{k+2\lambda+n-1}{n+k}\binom{n+k}{k}k!}\binom{\lambda+n-l-\frac{1}{2}}{n-l}\right\}C_k^{(\lambda)}(x).
\end{aligned}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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