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Asymptotically almost periodic solution to a class of Volterra difference equations

Wei Long[*] and Wen-Hai Pan

[*]Correspondence:
hopelw@126.com
College of Mathematics and
Information Science, Jiangxi Normal
University, Nanchang, Jiangxi
330022, People's Republic of China

Abstract

This paper is concerned with an asymptotically almost periodic solution to a class of Volterra-type difference equations. We establish a compactness criterion for the sets of asymptotically almost periodic sequences. Then, by using the compactness criterion and Schauder's fixed point theorem, we present an existence theorem for an asymptotically almost periodic solution to the addressed Volterra-type difference equation. Our existence theorem extends and complements a recent result due to (Ding *et al.* in Electron. J. Qual. Theory Differ. Equ. 6:1-13, 2012).

MSC: 39A24; 34K14

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1 Introduction and preliminaries

In this paper, we consider the following nonlinear Volterra-type difference equation:

$$x(n) = \sum_{i=1}^{\lambda} \left[f_i\big(n, x(n)\big) \cdot \sum_{m \in \mathbb{Z}} a_i(n,m) g_i\big(m, x(m)\big) \right], \quad n \in \mathbb{Z}, \tag{1.1}$$

where λ is a fixed positive integer and $f_i : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$, $a_i : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$, $g_i : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ ($i = 1, 2, \ldots, \lambda$) satisfy some conditions recalled in Section 3.

For the background of discrete Volterra equations, we refer the reader to the well-known monograph [1] by Agarwal. The first motivation for this paper is some recent work on asymptotical periodicity for Volterra-type difference equations in [2–6] by Diblík *et al.* In fact, asymptotical behavior for Volterra-type difference equations, including periodicity, asymptotical periodicity, *etc.*, has been of great interest for many mathematicians. However, to the best of our knowledge, there is seldom literature available about asymptotically almost periodicity for Equation (1.1). Thus, in this paper, we will investigate this problem. In addition, it is needed to note that compared with asymptotically periodic sequences, in general, it is more difficult to obtain the compactness for a set of asymptotically almost periodic sequences.

On the other hand, in a recent work [7], by using the classical Schauder fixed point theorem, Ding *et al.* established an interesting existence theorem for the following functional integral equation:

$$y(t) = e\big(t, y(\alpha(t))\big) + g\big(t, y(\beta(t))\big) \left[h(t) + \int_{\mathbb{R}} k(t,s) f\big(s, y(\gamma(s))\big) ds \right], \quad t \in \mathbb{R}. \tag{1.2}$$

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In fact, the existence of almost periodic type solutions has been an interesting and important topic in the study of qualitative theory of difference equations. We refer the reader to [8–13] and references therein for some recent developments on this topic. Equation (1.1) can be seen as a discrete analogue (but more general) of Equation (1.2). That is another main motivation for this work.

Throughout the rest of this paper, we denote by $\mathbb{Z}$ ($\mathbb{Z}^+$) the set of (nonnegative) integers, by $\mathbb{N}$ the set of positive integers, by $\mathbb{R}$ ($\mathbb{R}^+$) the set of (nonnegative) real numbers, by Ω a subset of $\mathbb{R}$, and by X a Banach space.

First, let us recall some notations and basic results of almost periodic type sequences (for more details, see [11, 14, 15]).

Definition 1.1 [14] A function $f : \mathbb{Z} \to X$ is called almost periodic if $\forall \varepsilon$, $\exists N(\varepsilon) \in \mathbb{N}$ such that among any $N(\varepsilon)$ consecutive integers there exists an integer p with the property that

$$\left\| f(k + p) - f(k) \right\| < \varepsilon, \quad \forall k \in \mathbb{Z}.$$

Denote by $AP(\mathbb{Z}, X)$ the set of all such functions. Moreover, we denote $AP(\mathbb{Z}, \mathbb{R})$ by $AP(\mathbb{Z})$ for convenience.

Lemma 1.2 [14, Theorem 1.26] *A necessary and sufficient condition for the sequence $f : \mathbb{Z} \to \mathbb{R}$ to be almost periodic is that for any integer sequence $\{n_k'\}$, one can extract a subsequence $\{n_k\}$ such that $\{f(n + n_k)\}$ converges uniformly with respect to $n \in \mathbb{Z}$.*

Remark 1.3 Let $f, g \in AP(\mathbb{Z})$. By Lemma 1.2, it is not difficult to show that $\forall \varepsilon$, $\exists N(\varepsilon) \in \mathbb{N}$ such that among any $N(\varepsilon)$ consecutive integers there exists a *common* integer p with the property that

$$\left| f(k + p) - f(k) \right| < \varepsilon \quad \text{and} \quad \left| g(k + p) - g(k) \right| < \varepsilon$$

for all $k \in \mathbb{Z}$.

Next, we denote by $C_0(\mathbb{Z}, X)$ the space of all the functions $f : \mathbb{Z} \to X$ such that $\lim_{|n| \to \infty} \| f(n) \| = 0$.

Definition 1.4 A function $f : \mathbb{Z} \to X$ is called asymptotically almost periodic if it admits a decomposition $f = g + h$, where $g \in AP(\mathbb{Z}, X)$ and $h \in C_0(\mathbb{Z}, X)$. Denote by $AAP(\mathbb{Z}, X)$ the set of all such functions. Moreover, we denote $AAP(\mathbb{Z}, \mathbb{R})$ by $AAP(\mathbb{Z})$ for convenience.

Definition 1.5 Let $\Omega \subset \mathbb{R}$ and f be a function from $\mathbb{Z} \times \Omega$ to $\mathbb{R}$ such that $f(n, \cdot)$ is continuous for each $n \in \mathbb{Z}$. Then f is called almost periodic in n $\in \mathbb{Z}$ uniformly for $\omega \in \Omega$ if for every $\varepsilon > 0$ and every compact $\Sigma \subset \Omega$, there corresponds an integer $N_\varepsilon(\Sigma) > 0$ such that among $N_\varepsilon(\Sigma)$ consecutive integers there exists an integer p with the property that

$$\left| f(k + p, \omega) - f(k, \omega) \right| < \varepsilon$$

for all $k \in \mathbb{Z}$ and $\omega \in \Sigma$. Denote by $AP(\mathbb{Z} \times \Omega)$ the set of all such functions.

Similarly, for each subset $\Omega \subset \mathbb{R}$, we denote by $C_0(\mathbb{Z} \times \Omega)$ the space of all the functions $f : \mathbb{Z} \times \Omega \to \mathbb{R}$ such that $f(n, \cdot)$ is continuous for each $n \in \mathbb{Z}$, and $\lim_{|n| \to \infty} f(n, x) = 0$ uniformly for x in any compact subset of Ω.

Definition 1.6 A function $f : \mathbb{Z} \times \Omega \to \mathbb{R}$ is called asymptotically almost periodic in n uniformly for $x \in \Omega$ if it admits a decomposition $f = g + h$, where $g \in AP(\mathbb{Z} \times \Omega)$ and $h \in C_0(\mathbb{Z} \times \Omega)$. Denote by $AAP(\mathbb{Z} \times \Omega)$ the set of all such functions.

Lemma 1.7 *Let $E \in \{AP(\mathbb{Z}, X), AAP(\mathbb{Z}, X)\}$. Then the following hold true*:
  (a) *$f \in E$ implies that f is bounded.*
  (b) *$f, g \in E$ implies that $f + g \in E$. Moreover, $f \cdot g \in E$ if $X = \mathbb{R}$.*
  (c) *E is a Banach space equipped with the supremum norm.*

Proof The proof is similar to that of the continuous case (*cf.* [14, 15]). So, we omit the details. □

2 A compactness criterion
The following theorem is a well-known result for the continuous case (see, *e.g.,* [16, p.24, Theorem 2.5]). Here, we give a discrete version.

Theorem 2.1 *Let f be a function from $\mathbb{Z}$ to $\mathbb{R}$. Then $f \in AAP(\mathbb{Z})$ if and only if $\forall \varepsilon$, $\exists M(\varepsilon), N(\varepsilon) \in \mathbb{N}$ such that among any $N(\varepsilon)$ consecutive integers there exists an integer p with the property that*

$$\left| f(k + p) - f(k) \right| < \varepsilon$$

for all $k \in \mathbb{Z}$ with $|k| \geq M(\varepsilon)$ and $|k + p| \geq M(\varepsilon)$.

Proof We first show the 'only if' part. Let $f \in AAP(\mathbb{Z})$. Then there exist $g \in AP(\mathbb{Z})$ and $h \in C_0(\mathbb{Z}, \mathbb{R})$ such that $f = g + h$. By $g \in AP(\mathbb{Z})$, for each $\varepsilon > 0$, $\exists N(\varepsilon) \in \mathbb{N}$ such that among any $N(\varepsilon)$ consecutive integers there exists an integer p with the property that

$$\left| g(k + p) - g(k) \right| < \frac{\varepsilon}{3}, \quad \forall k \in \mathbb{Z}.$$

In addition, since $h \in C_0(\mathbb{Z}, \mathbb{R})$, for the above $\varepsilon > 0$, there exists $M(\varepsilon) \in \mathbb{N}$ such that $|h(k)| < \frac{\varepsilon}{3}$ for all $k \in \mathbb{Z}$ with $|k| \geq M(\varepsilon)$. Thus, we have

$$\left| f(k + p) - f(k) \right| \leq \left| g(k + p) - g(k) \right| + \left| h(k + p) \right| + \left| h(k) \right| < \varepsilon$$

for all $k \in \mathbb{Z}$ with $|k| \geq M(\varepsilon)$ and $|k + p| \geq M(\varepsilon)$.

Next, let us prove the 'if' part. First, let us show that f is bounded. Letting $\varepsilon = 1$, there exists $M(1), N(1) \in \mathbb{N}$ such that among any $N(1)$ consecutive integers there exists an integer p with the property that

$$\left| f(k + p) - f(k) \right| < 1$$

for all $k \in \mathbb{Z}$ with $|k| \geq M(1)$ and $|k + p| \geq M(1)$. Then, for each $k \in \mathbb{Z}$ with $|k| \geq M(1)$, there exists $p_k \in [M(1) - k, M(1) + N(1) - k] \cap \mathbb{Z}$ such that

$$\left| f(k + p_k) - f(k) \right| < 1.$$

Noting that $k + p_k \in [M(1), M(1) + N(1)]$, we get

$$\left| f(k) \right| \leq \left| f(k + p_k) \right| + 1 \leq \max_{k \in [M(1), M(1) + N(1)]} \left| f(k) \right| + 1$$

for all $k \in \mathbb{Z}$ with $|k| \geq M(1)$. Thus,

$$\sup_{k \in \mathbb{Z}} \left| f(k) \right| \leq \max_{k \in [-M(1), M(1) + N(1)]} \left| f(k) \right| + 1 < +\infty.$$

Now, let us show that $f \in AAP(\mathbb{Z})$. We divide the remaining proof into three steps.

Step 1. Since f is bounded, we can choose a sequence $\{s_n\} \subset \mathbb{N}$ such that $\lim_{n \to +\infty} s_n = +\infty$ and $\lim_{n \to +\infty} f(k + s_n)$ exists for each $k \in \mathbb{Z}$. Let

$$\overline{g}(k) = \lim_{n \to +\infty} f(k + s_n), \quad k \in \mathbb{Z}.$$

For each $\varepsilon > 0$, among any $N(\varepsilon)$ consecutive integers there exists an integer p with the property that

$$\left| f(k + p) - f(k) \right| < \varepsilon$$

for all $k \in \mathbb{Z}$ with $|k| \geq M(\varepsilon)$ and $|k + p| \geq M(\varepsilon)$. Then, for each fixed $k \in \mathbb{Z}$, we have

$$\left| f(k + s_n + p) - f(k + s_n) \right| < \varepsilon$$

for sufficiently large n, which yields that

$$\left| \overline{g}(k + p) - \overline{g}(k) \right| \leq \varepsilon.$$

Thus, $\overline{g} \in AP(\mathbb{Z})$.

Step 2. Now fix $\varepsilon > 0$. Then, for each $n \in \mathbb{N}$, there exists $t_n \in [s_n - N(\varepsilon), s_n] \cap \mathbb{Z}$ such that

$$\left| f(k + t_n) - f(k) \right| < \varepsilon \tag{2.1}$$

for all $k \in \mathbb{Z}$ with $|k| \geq M(\varepsilon)$ and $|k + t_n| \geq M(\varepsilon)$. Let $r_n = s_n - t_n$. Then $r_n \in \{0, 1, 2, \ldots, N(\varepsilon)\}$, which means that there exist a subsequence $\{r'_n\} \subset \{r_n\}$ and $r(\varepsilon) \in \{0, 1, 2, \ldots, N(\varepsilon)\}$ such that

$$r'_n \equiv r(\varepsilon).$$

Thus, for all $k \in \mathbb{Z}$ with $|k| \geq M(\varepsilon)$, we have

$$\left| f(k) - \overline{g}(k - r(\varepsilon)) \right| = \left| f(k) - f(k + t'_n) \right| + \left| f(k - r(\varepsilon) + s'_n) - \overline{g}(k - r(\varepsilon)) \right|.$$

Combining this with (2.1), $\lim_{n\to+\infty} t'_n = +\infty$, and

$$\overline{g}(k) = \lim_{n\to+\infty} f(k + s_n), \quad \forall k \in \mathbb{Z},$$

we conclude

$$\left| f(k) - \overline{g}\big(k - r(\varepsilon)\big) \right| \leq \varepsilon$$

for all $k \in \mathbb{Z}$ with $|k| \geq M(\varepsilon)$.

Step 3. By Step 2, we know that for each $\varepsilon > 0$, there exists $r(\varepsilon) \in \{0, 1, 2, \ldots, N(\varepsilon)\}$ such that

$$\left| f(k) - \overline{g}\big(k - r(\varepsilon)\big) \right| \leq \varepsilon$$

for all $k \in \mathbb{Z}$ with $|k| \geq M(\varepsilon)$. Taking $\varepsilon = 1, 1/2, \ldots$, we get a sequence $\{r(1/m)\}$. On the other hand, it follows from Step 1 that $\overline{g} \in AP(\mathbb{Z})$. Thus, going to a subsequence, if necessary, we may assume that $\overline{g}(\cdot - r(1/m))$ is uniformly convergent on $\mathbb{Z}$. Let

$$g(k) = \lim_{m\to+\infty} \overline{g}\big(k - r(1/m)\big), \quad k \in \mathbb{Z}.$$

Then $g \in AP(\mathbb{Z})$. In addition, noting that

$$\left| f(k) - g(k) \right| \leq \left| f(k) - \overline{g}\big(k - r(1/m)\big) \right| + \left| \overline{g}\big(k - r(1/m)\big) - g(k) \right|$$

$$\leq \frac{1}{m} + \left| \overline{g}\big(k - r(1/m)\big) - g(k) \right|$$

for all $k \in \mathbb{Z}$ with $|k| \geq M(1/m)$, we know that $f - g \in C_0(\mathbb{Z})$. This completes the proof. □

Definition 2.2 $F \subseteq AAP(\mathbb{Z})$ is said to be equi-asymptotically almost periodic if for each $\varepsilon > 0$, there exist $M(\varepsilon), N(\varepsilon) \in \mathbb{N}$ such that among any $N(\varepsilon)$ consecutive integers there exists an integer p with the property that

$$\sup_{f \in F} \left| f(k + p) - f(k) \right| < \varepsilon$$

for all $k \in \mathbb{Z}$ with $|k| \geq M(\varepsilon)$ and $|k + p| \geq M(\varepsilon)$.

Theorem 2.3 *Let $F \subseteq AAP(\mathbb{Z})$. Then F is precompact in $AAP(\mathbb{Z})$ if and only if the following two conditions hold*:

(i) *for each $k \in \mathbb{Z}$, $\{f(k) : f \in F\}$ is bounded*;

(ii) *F is equi-asymptotically almost periodic*.

Proof 'only if' part

Let $F \subset AAP(\mathbb{Z})$ be precompact. Then F is bounded in $AAP(\mathbb{Z})$. So, (i) obviously holds. In addition, $\forall \varepsilon > 0$, there exists $N \in \mathbb{N}$ and $f_1, f_2, \ldots, f_N \in F$ such that

$$F \subset \bigcup_{i=1}^{N} B(f_i, \varepsilon). \tag{2.2}$$

By Remark 1.3, we can get that $\{f_1, f_2, \ldots, f_N\}$ is equi-asymptotically almost periodic. Combing this with (2.2), we can show that F is equi-asymptotically almost periodic, *i.e.*, (ii) holds.

'if part'

Let $\{f_n\} \subset F$. Since $\{f_n(k)\}$ is bounded for each $k \in \mathbb{Z}$, we can assume that (if necessary going to a subsequence) $\{f_n(k)\}$ is convergent for each $k \in \mathbb{Z}$. On the other hand, since F is equi-asymptotically almost periodic, for each $\varepsilon > 0$, there exist $M(\varepsilon), N(\varepsilon) \in \mathbb{N}$ such that among any $N(\varepsilon)$ consecutive integers there exists an integer p with the property that

$$\sup_{n \in \mathbb{Z}} \left| f_n(k + p) - f_n(k) \right| < \varepsilon/3 \tag{2.3}$$

for all $k \in \mathbb{Z}$ with $|k| \geq M(\varepsilon)$ and $|k + p| \geq M(\varepsilon)$. For the above $\varepsilon > 0$, there exists a positive integer K such that for all $n, m > K$, the following hold:

$$\left| f_n(k) - f_m(k) \right| < \varepsilon/3, \quad k \in \left[-M(\varepsilon), M(\varepsilon) + N(\varepsilon) \right] \cap \mathbb{Z}. \tag{2.4}$$

For all $k \in \mathbb{Z}$ with $|k| \geq M(\varepsilon)$, taking $p \in [-k + M(\varepsilon), -k + M(\varepsilon) + N(\varepsilon)] \cap \mathbb{Z}$, by (2.3) and (2.4), we get

$$\left| f_n(k) - f_m(k) \right| \leq \left| f_n(k) - f_n(k + p) \right| + \left| f_n(k + p) - f_m(k + p) \right| + \left| f_m(k + p) - f_m(k) \right|$$
$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon, \quad n, m > K;$$

also, for all $k \in \mathbb{Z}$ with $|k| < M(\varepsilon)$, by (2.4), we have

$$\left| f_n(k) - f_m(k) \right| < \varepsilon/3 < \varepsilon, \quad n, m > K.$$

Thus, we get

$$\sup_{k \in \mathbb{Z}} \left| f_n(k) - f_m(k) \right| \leq \varepsilon, \quad n, m > K,$$

which means that $\{f_n(k)\}$ is uniformly convergent on $\mathbb{Z}$, *i.e.*, $\{f_n\}$ is convergent in $AAP(\mathbb{Z})$. So, F is precompact in $AAP(\mathbb{Z})$. □

3 Application to Volterra difference equations

In this section, we discuss the existence of an asymptotically almost periodic solution to Volterra difference equation (1.1). Throughout the rest of this paper, $p, q \geq 1$ are two fixed real numbers and

$$\frac{1}{p} + \frac{1}{q} = 1.$$

In addition, we denote by $l^p(\mathbb{Z})$ (resp. $l^q(\mathbb{Z})$) the space of all the functions $f : \mathbb{Z} \to \mathbb{R}$ satisfying

$$\|f\|_p := \left(\sum_{k \in \mathbb{Z}} \left| f(k) \right|^p \right)^{1/p} < +\infty \quad \left(\text{resp. } \|f\|_q := \left(\sum_{k \in \mathbb{Z}} \left| f(k) \right|^q \right)^{1/q} < +\infty \right).$$

For convenience, we first list some assumptions.

(H1) For each $i \in \{1, 2, \ldots, \lambda\}$, $f_i(\cdot, x) \in AAP(\mathbb{Z})$ for any fixed $x \in \mathbb{R}$, and there exists a constant $L_i \geq 0$ such that

$$\left| f_i(k, x) - f_i(k, y) \right| \leq L_i |x - y|, \quad \forall k \in \mathbb{Z}, \forall x, y \in \mathbb{R}.$$

(H2) For each $i \in \{1, 2, \ldots, \lambda\}$, $g_i(k, \cdot)$ is continuous for each $k \in \mathbb{Z}$, and for each $r > 0$, there exists a sequence $\{\mu_i^r\} \subset l^p(\mathbb{Z})$ such that

$$\left| g_i(k, x) \right| \leq \mu_i^r(k), \quad |x| \leq r, k \in \mathbb{Z}.$$

(H3) For each $i \in \{1, 2, \ldots, \lambda\}$, $\widetilde{a}_i \in AAP(\mathbb{Z}, l^q(\mathbb{Z}))$, where $[\widetilde{a}_i(k)](l) = a_i(k, l)$, $\forall k, l \in \mathbb{Z}$.

(H4) There exists a constant $M > 0$ such that

$$\sum_{i=1}^{\lambda} \alpha_i L_i \left\| \mu_i^M \right\|_p < 1,$$

where $\alpha_i = \sup_{n \in \mathbb{Z}} \|\widetilde{a}_i(n)\|_q$; and

$$\sum_{i=1}^{\lambda} \left[\sup_{n \in \mathbb{Z}, |x| \leq K} \left| f_i(n, x) \right| \cdot \alpha_i \cdot \left\| \mu_i^M \right\|_p \right] < K, \quad \forall K > M,$$

Theorem 3.1 *Assume that* (H1)-(H4) *hold. Then Equation* (1.1) *has an asymptotically almost periodic solution.*

Proof We denote

$$(A_i x)(n) = f_i\big(n, x(n)\big), \quad n \in \mathbb{Z}, x \in AAP(\mathbb{Z}), i = 1, 2, \ldots, \lambda;$$

$$(B_i x)(n) = \sum_{m \in \mathbb{Z}} a_i(n, m) g_i\big(m, x(m)\big), \quad n \in \mathbb{Z}, x \in AAP(\mathbb{Z}), i = 1, 2, \ldots, \lambda;$$

and

$$(\mathcal{M} x)(n) = \sum_{i=1}^{\lambda} (A_i x)(n) \cdot (B_i x)(n), \quad n \in \mathbb{Z}, x \in AAP(\mathbb{Z}).$$

It suffices to prove that $\mathcal{M}$ has a fixed point in $AAP(\mathbb{Z})$. We give the proof in three steps.

Step 1. A_i and B_i both map $AAP(\mathbb{Z})$ into $AAP(\mathbb{Z})$, $i = 1, 2, \ldots, \lambda$.

Since f_i is Lipschitz, by Remark 1.3, we can first show that for each compact subset $K \subset \mathbb{R}$ and each $i \in \{1, 2, \ldots, \lambda\}$, $\{f_i(\cdot, x) : x \in K\}$ is equi-asymptotically almost periodic. Then it is easy to show that $A_i x \in AAP(\mathbb{Z})$ for each $x \in AAP(\mathbb{Z})$.

Since $\widetilde{a}_i \in AAP(\mathbb{Z}, l^q(\mathbb{Z}))$, there exist $b_i \in AP(\mathbb{Z}, l^q(\mathbb{Z}))$ and $c_i \in C_0(\mathbb{Z}, l^q(\mathbb{Z}))$ such that $\widetilde{a}_i = b_i + c_i$. For each $x \in AAP(\mathbb{Z})$, noting that for $n, p \in \mathbb{Z}$,

$$(B_i x)(n) = \sum_{m \in \mathbb{Z}} a_i(n, m) g_i\big(m, x(m)\big)$$

$$= \sum_{m \in \mathbb{Z}} \big[b_i(n)\big](m) g_i\big(m, x(m)\big) + \sum_{m \in \mathbb{Z}} \big[c_i(n)\big](m) g_i\big(m, x(m)\big),$$

$$\left| \sum_{m \in \mathbb{Z}} [b_i(n+p)](m)g_i\big(m,x(m)\big) - \sum_{m \in \mathbb{Z}} [b_i(n)](m)g_i\big(m,x(m)\big) \right|$$

$$\leq \big\| b_i(n+p) - b_i(n) \big\|_q \cdot \big\| \mu_i^{\|x\|} \big\|_p,$$

and

$$\left| \sum_{m \in \mathbb{Z}} [c_i(n)](m)g_i\big(m,x(m)\big) \right| \leq \big\| c_i(n) \big\|_q \cdot \big\| \mu_i^{\|x\|} \big\|_p,$$

we know that $B_i x \in AAP(\mathbb{Z})$.

Step 2. For each $y \in AAP(\mathbb{Z})$ with $\|y\| \leq M$, there exists a unique $x_y \in AAP(\mathbb{Z})$ such that

$$x_y = \sum_{i=1}^{\lambda} A_i x_y \cdot B_i y.$$

Let

$$(\mathcal{Y}x)(n) = \sum_{i=1}^{\lambda} (A_i x)(n) \cdot (B_i y)(n), \quad n \in \mathbb{Z}, x \in AAP(\mathbb{Z}).$$

Then, by Step 1, $\mathcal{Y}$ maps $AAP(\mathbb{Z})$ into $AAP(\mathbb{Z})$. For all $x_1, x_2 \in AAP(\mathbb{Z})$ and $n \in \mathbb{Z}$, we have

$$\big| (\mathcal{Y}x_1)(n) - (\mathcal{Y}x_2)(n) \big|$$

$$\leq \sum_{i=1}^{\lambda} \big| (A_i x_1)(n) - (A_i x_2)(n) \big| \cdot \big| (B_i y)(n) \big|$$

$$= \sum_{i=1}^{\lambda} \big| f_i\big(n,x_1(n)\big) - f_i\big(n,x_2(n)\big) \big| \cdot \big| (B_i y)(n) \big|$$

$$\leq \sum_{i=1}^{\lambda} L_i \big| x_1(n) - x_2(n) \big| \cdot \left| \sum_{m \in \mathbb{Z}} a_i(n,m)g_i\big(m,y(m)\big) \right|$$

$$\leq \sum_{i=1}^{\lambda} L_i \| x_1 - x_2 \| \cdot \left| \sum_{m \in \mathbb{Z}} a_i(n,m)g_i\big(m,y(m)\big) \right|$$

$$\leq \sum_{i=1}^{\lambda} L_i \| x_1 - x_2 \| \cdot \sum_{m \in \mathbb{Z}} \big| [\widetilde{a}_i(n)](m) \big| \cdot \mu_i^M(m)$$

$$\leq \sum_{i=1}^{\lambda} L_i \| x_1 - x_2 \| \cdot \big\| \widetilde{a}_i(n) \big\|_q \cdot \big\| \mu_i^M \big\|_p$$

$$\leq \left(\sum_{i=1}^{\lambda} \alpha_i L_i \big\| \mu_i^M \big\|_p \right) \| x_1 - x_2 \|,$$

which yields that

$$\| \mathcal{Y}x_1 - \mathcal{Y}x_2 \| \leq \left(\sum_{i=1}^{\lambda} \alpha_i L_i \big\| \mu_i^M \big\|_p \right) \| x_1 - x_2 \|.$$

Noting that $\sum_{i=1}^{\lambda} \alpha_i L_i \|\mu_i^M\|_p < 1$, $\mathcal{Y}$ has a unique fixed point x_y in $AAP(\mathbb{Z})$.

Step 3. $\mathcal{M}$ has a fixed point in $AAP(\mathbb{Z})$.

Let $E = \{y \in AAP(\mathbb{Z}) : \|y\| \leq M\}$ and

$$\mathcal{N}y = x_y, \quad y \in E,$$

where x_y is the unique fixed point of $\mathcal{Y}$ (see Step 2).

We claim that $\mathcal{N}(E) \subset E$. In fact, if there exists $y_0 \in E$ such that $\|\mathcal{N}y_0\| > M$, then by (H4), we have

$$
\begin{aligned}
\|\mathcal{N}y_0\| &= \|x_{y_0}\| \\
&= \sup_{n \in \mathbb{Z}} \left| \sum_{i=1}^{\lambda} (A_i x_{y_0})(n) \cdot (B_i y_0)(n) \right| \\
&\leq \sup_{n \in \mathbb{Z}} \left(\sum_{i=1}^{\lambda} \left| f_i\big(n, x_{y_0}(n)\big) \right| \cdot \left| \sum_{m \in \mathbb{Z}} a_i(n,m) g_i\big(m, y_0(m)\big) \right| \right) \\
&\leq \sup_{n \in \mathbb{Z}} \left(\sum_{i=1}^{\lambda} \left| f_i\big(n, x_{y_0}(n)\big) \right| \cdot \alpha_i \cdot \left\| \mu_i^M \right\|_p \right) \\
&\leq \sum_{i=1}^{\lambda} \left[\sup_{n \in \mathbb{Z}, |x| \leq \|\mathcal{N}y_0\|} \left| f_i(n,x) \right| \cdot \alpha_i \cdot \left\| \mu_i^M \right\|_p \right] \\
&< \|\mathcal{N}y_0\|,
\end{aligned}
$$

which is a contradiction.

Next, let us show that $\mathcal{N} : E \to E$ is continuous. For all $y_1, y_2 \in E$, we have

$$
\begin{aligned}
\|\mathcal{N}y_1 - \mathcal{N}y_2\| &= \|x_{y_1} - x_{y_2}\| \\
&= \left\| \sum_{i=1}^{\lambda} A_i x_{y_1} \cdot B_i y_1 - \sum_{i=1}^{\lambda} A_i x_{y_2} \cdot B_i y_2 \right\| \\
&\leq \sum_{i=1}^{\lambda} \|A_i x_{y_1} \cdot B_i y_1 - A_i x_{y_2} \cdot B_i y_1 + A_i x_{y_2} \cdot B_i y_1 - A_i x_{y_2} \cdot B_i y_2 \| \\
&\leq \left(\sum_{i=1}^{\lambda} \alpha_i L_i \left\| \mu_i^M \right\|_p \right) \|x_{y_1} - x_{y_2}\| \\
&\quad + \sum_{i=1}^{\lambda} \left(ML_i + \sup_{n \in \mathbb{Z}} \left| f_i(n,0) \right| \right) \cdot \|B_i y_1 - B_i y_2\|,
\end{aligned}
$$

which gives that

$$\|\mathcal{N}y_1 - \mathcal{N}y_2\| \leq \sum_{i=1}^{\lambda} \beta_i \cdot \|B_i y_1 - B_i y_2\|, \tag{3.1}$$

where

$$\beta_i := \frac{ML_i + \sup_{n \in \mathbb{Z}} |f_i(n,0)|}{1 - (\sum_{i=1}^{\lambda} \alpha_i L_i \|\mu_i^M\|_p)}, \quad i = 1, 2, \ldots, \lambda.$$

Letting $y_k \to y$ in E, by (3.1), we have

$$\|\mathcal{N}y_k - \mathcal{N}y\| \leq \sum_{i=1}^{\lambda} \beta_i \cdot \|B_i y_k - B_i y\|$$

$$\leq \sum_{i=1}^{\lambda} \beta_i \cdot \sup_{n \in \mathbb{Z}} \left(\sum_{m \in \mathbb{Z}} |a_i(n,m)| \cdot |g_i(m, y_k(m)) - g_i(m, y(m))| \right)$$

$$\leq \sum_{i=1}^{\lambda} \beta_i \cdot \sup_{n \in \mathbb{Z}} (\|\tilde{a}_i(n)\|_q \cdot \|g_i(\cdot, y_k(\cdot)) - g_i(\cdot, y(\cdot))\|_p)$$

$$\leq \sum_{i=1}^{\lambda} \alpha_i \beta_i \cdot \|g_i(\cdot, y_k(\cdot)) - g_i(\cdot, y(\cdot))\|_p. \tag{3.2}$$

For each $i = 1, 2, \ldots, \lambda$, noting that

$$|g_i(m, y_k(m)) - g_i(m, y(m))| \leq 2\mu_i^M(m), \quad m \in \mathbb{Z},$$

$g_i(m, \cdot)$ is continuous for each $m \in \mathbb{Z}$, and $y_k(m) \to y(m)$ for each $m \in \mathbb{Z}$, we conclude that

$$\|g_i(\cdot, y_k(\cdot)) - g_i(\cdot, y(\cdot))\|_p \to 0.$$

Combining this with (3.2), we know that $\mathcal{N}y_k \to \mathcal{N}y$. So $\mathcal{N} : E \to E$ is continuous.

Now, let us show that $\mathcal{N}(E)$ is precompact in $AAP(\mathbb{Z})$. In order to show that, we first prove each $B_i(E)$ is precompact in $AAP(\mathbb{Z})$. By a direct calculation, we can get

$$|(B_i y)(n)| \leq \alpha_i \cdot \|\mu_i^M\|_p, \quad i = 1, 2, \ldots, \lambda,$$

for all $y \in E$ and $n \in \mathbb{Z}$. In addition, for all $n_1, n_2 \in \mathbb{Z}$ and $y \in E$, we have

$$|(B_i y)(n_1) - (B_i y)(n_2)| \leq \sum_{m \in \mathbb{Z}} |a_i(n_1, m) - a_i(n_2, m)| \cdot |g_i(m, y(m))|$$

$$\leq \|\tilde{a}_i(n_1) - \tilde{a}_i(n_2)\|_q \cdot \|\mu_i^M\|_p,$$

which yields that each $B_i(E)$ is equi-asymptotically almost periodic since $\tilde{a}_i \in AAP(\mathbb{Z}, l^q(\mathbb{Z}))$. Then, by Theorem 2.3, each $B_i(E)$ is precompact in $AAP(\mathbb{Z})$. Let $\{y_k\} \subset E$. Then $\{B_i y_k\}$, if necessary going to a subsequence, is convergent in $AAP(\mathbb{Z})$ for each $i \in \{1, 2, \ldots, \lambda\}$. By (3.1), we conclude that $\{\mathcal{N}y_k\}$ is convergent in $AAP(\mathbb{Z})$. So, $\mathcal{N}(E)$ is precompact in $AAP(\mathbb{Z})$.

By applying Schauder's fixed point theorem, there exists a fixed point y_* of $\mathcal{N}$ in E. Then we have

$$y_* = \mathcal{N}y_* = x_{y_*} = \sum_{i=1}^{\lambda} A_i x_{y_*} \cdot B_i y_* = \sum_{i=1}^{\lambda} A_i y_* \cdot B_i y_* = \mathcal{M}y_*,$$

which means that y_* is a fixed point of $\mathcal{M}$. This completes the proof. □

Finally, we give a simple example to illustrate our result.

Example 3.2 Let $\lambda = 2$, $p = 1$, $q = \infty$,

$$f_1(n,x) = \frac{x}{10}\left(\sin n + \sin \pi n + \frac{1}{|n|+1}\right), \qquad g_1(n,x) = \frac{\sin(xe^{n^2})}{2(1+n^2)},$$

$$a_1(n,m) = \frac{\cos n + \cos\sqrt{2}n + \frac{1}{n^2+1}}{3(1+m^2)},$$

and

$$f_2(n,x) = \frac{\cos n \sin x}{20}, \qquad g_2(n,x) = \frac{\arctan(nx)}{1+n^2}, \qquad a_2(n,m) = \frac{1}{3}e^{-m^2}\sin n.$$

It is easy to see that (H1) holds with $L_1 = \frac{3}{10}$ and $L_2 = \frac{1}{20}$. Also, (H2) holds with $\mu_1^r(n) \equiv \frac{1}{2(1+n^2)}$ and $\mu_2^r(n) \equiv \frac{\pi}{2}\cdot\frac{1}{1+n^2}$. In addition, (H3) can be easily verified. By a direct calculation, we can get

$$\alpha_1 \le 1, \qquad \alpha_2 \le \frac{1}{3},$$

and

$$\|\mu_1^r\|_1 \le \frac{\pi+1}{2}, \qquad \|\mu_2^r\|_1 \le \frac{\pi^2+\pi}{2}, \quad \forall r > 0.$$

Letting $M = 1$, we have

$$\sum_{i=1}^{2}\alpha_i L_i\|\mu_i^M\|_1 \le \frac{3(\pi+1)}{20} + \frac{\pi^2+\pi}{120} < 1,$$

and

$$\sum_{i=1}^{2}\left[\sup_{n\in\mathbb{Z},|x|\le K}|f_i(n,x)|\cdot\alpha_i\cdot\|\mu_i^M\|_1\right] \le \frac{3(\pi+1)}{20}K + \frac{\pi^2+\pi}{120} < K, \quad \forall K > 1.$$

Thus, (H4) holds with $M = 1$. Then, by using Theorem 3.1, Equation (1.1) has an asymptotically almost periodic solution.

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