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A new construction on the *q*-Bernoulli polynomials

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Abstract

This paper performs a further investigation on the *q*-Bernoulli polynomials and numbers given by Açikgöz et al. (Adv. Differ. Equ. **2010**, 9, Article ID 951764) some incorrect properties are revised. It is pointed out that the definition concerning the *q*-Bernoulli polynomials and numbers is unreasonable. The purpose of this paper is to redefine the *q*-Bernoulli polynomials and numbers and correct its wrong properties and rebuild its theorems.

1 Introduction/Preliminaries

Many mathematicians have studied the *q*-Bernoulli, *q*-Euler polynomials and related topics (see [1-11]). It is worth that Açikgöz et al. [1] give a new approach to the *q*-Bernoulli polynomials and the *q*-Bernstein polynomials and show some properties. That is, Açikgöz et al. introduced a new generating function related the *q*-Bernoulli polynomials and gave a new construction of these polynomials related to the second kind Stirling numbers and the *q*-Bernstein polynomials in [1]. The purpose of this paper is to redefine a generating function related the *q*-Bernoulli polynomials and numbers and correct its wrong properties and rebuild its theorems.

In this paper, we assume that $q(\in \mathbb{C})$ is indeterminate with $|q| < 1$. The *q*-number is defined by $[x]_q = \frac{q^x-1}{q-1}$ (see [4-9]).

It is known that the Bernoulli polynomials are defined as

$$\frac{t}{e^t - 1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \quad \text{for} \quad |t| < 2\pi \tag{1.1}$$

and that $B_n(0) = B_n$ are called the Bernoulli numbers.

The recurrence formula for the classical Bernoulli numbers B_n is as follows:

$$B_0 = 1 \text{ and } (B+1)^n - B_n = 0 \quad \text{if} \quad n > 0. \tag{1.2}$$

The *q*-extension of the following recurrence formula for the Bernoulli numbers is given by

$$B_{0,q} = 1 \text{ and } q(qB+1)^n - B_{n,q} = \begin{cases} 1 \text{ if } n = 1 \\ 0 \text{ if } n > 1 \end{cases} \tag{1.3}$$

with the usual convention of replacing B_q^n by $B_{n,q}$ (see [2,4]).

2 On the q-Bernoulli polynomials and numbers

In this section, we first recall the q-Bernoulli polynomials and numbers, then indicate the ambiguities on the Açikgöz et al. [1]'s definition for the q-Bernoulli polynomials and redefine it. Counter-examples show that some properties are incorrect. Specially, these examples show that the concept on the generating function of the q-Bernoulli polynomials is unreasonable.

Definition 2.1 (**Açikgöz et al**. [1]) For $q \in \mathbb{C}$ with $|q| < 1$, let us define the q-Bernoulli polynomials as follows,

$$D_q(t,\ x) = -t\sum_{y=0}^{\infty} q^y e^{[x+y]_q t} = \sum_{n=0}^{\infty} B_{n,q}(x)\frac{t^n}{n!}. \tag{2.1}$$

Note that

$$\lim_{q\to 1} D_q(t,\ x) = \frac{t}{e^t - 1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \quad \text{for} \quad |t| < 2\pi, \tag{2.2}$$

where $B_n(x)$ are the classical Bernoulli polynomials.

In the special case $x = 0$, $B_{n,q}(0) = B_{n,q}$ are called the q-Bernoulli number.

That is,

$$D_q(t) = D_q(t,\ 0) = -t\sum_{y=0}^{\infty} q^y e^{[y]_q t} = \sum_{n=0}^{\infty} B_{n,q}\frac{t^n}{n!}. \tag{2.3}$$

Remark 2.2 Definition 2.1 (Açikgöz et al. [1]) is unreasonable, since it is not the generating functions of the q-Bernoulli polynomials and numbers. This can be seen the following counter-examples.

Counter-example 2.3 If we take $t = 0$ in (2.2) of Definition 2.1 (Açikgöz et al. [1]), then we have $\lim_{q\to 1} D_q(0, x) = 0$. But $lim_{t\to 0} \frac{t}{e^t-1}e^{xt} = 1$ does not hold in the sense of Definition 2.1 (Açikgöz et al. [1]).

Counter-example 2.4 From (2.1) of Definition 2.1 (Açikgöz et al. [1]),

$$\begin{aligned} D_q(t,\ x) &= \sum_{n=0}^{\infty} B_{n,q}(x)\frac{t^n}{n!} \\ &= B_{0,q}(x) + \sum_{n=1}^{\infty} B_{n,q}(x)\frac{t^n}{n!}, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} D_q(t,\ x) &= -t\sum_{y=0}^{\infty} q^y e^{[x+y]_q t} \\ &= -t\sum_{y=0}^{\infty} q^y \sum_{n=0}^{\infty} [x+y]_q^n \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(-\tfrac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}\sum_{y=0}^{\infty} q^{(l+1)y}\right)\frac{t^{n+1}}{n!} \\ &= \sum_{n=0}^{\infty}\left(-\tfrac{n}{(1-q)^{n-1}}\sum_{l=0}^{n-1}\binom{n-1}{l}(-1)^l q^{lx}\tfrac{l}{1-q^{l+1}}\right)\frac{t^n}{n!}. \end{aligned} \tag{2.5}$$

Comparing these identities (2.4) and (2.5), we obtain

$$B_{0,q}(x) = 0 \text{ and } B_{n,q}(x) = -\frac{n}{(1-q)^{n-1}}\sum_{l=0}^{n-1}\binom{n-1}{l}(-1)^l q^{lx}\frac{l}{1-q^{l+1}}. \tag{2.6}$$

This cannot satisfy some well-known results related the Bernoulli polynomials and numbers. For example, $B_0 = 1$.

Counter-example 2.5 From Definition 2.1 (Açikgöz et al. [1]), we note that

$$\begin{aligned} qD_q(t,\ 1) - D_q(t) &= -t\sum_{y=0}^{\infty} q^{y+1}e^{[1+y]_q t} - t\sum_{y=0}^{\infty} q^y e^{[y]_q t} \\ &= t, \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} qD_q(t,\ 1) - D_q(t) &= q\sum_{n=0}^{\infty} B_{n,q}(1)\frac{t^n}{n!} - \sum_{n=0}^{\infty} B_{n,q}\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}(qB_{n,q}(1) - B_{n,q})\frac{t^n}{n!}. \end{aligned} \tag{2.8}$$

From (2.7) and (2.8), we can easily derive that

$$B_{n,q} = 0 \text{ and } qB_{n,q}(1) - B_{n,q} = \begin{cases} 1 \text{ if } n = 1 \\ 0 \text{ if } n > 1 \end{cases}. \tag{2.9}$$

From (2.1) of Definition 2.1 (Açikgöz et al. [1]),

$$\begin{aligned} \sum_{n=0}^{\infty} B_{n,q}(x)\frac{t^n}{n!} &= D_q(t,\ x) \\ &= -t\sum_{y=0}^{\infty} q^y e^{[x+y]_q t} \\ &= e^{[x]_q t}\frac{1}{q^x}D_q(tq^x) \\ &= \left(\sum_{l=0}^{\infty}\frac{[x]_q^l t^l}{l!}\right) \times \left(\sum_{m=0}^{\infty} B_{m,q}\frac{q^{(m-1)x}t^m}{m!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}B_{m,q}\,q^{(m-1)x}[x]_q^{n-m}\right)\frac{t^n}{n!}. \end{aligned} \tag{2.10}$$

If we compare the coefficients on the both sides in (2.10),

$$B_{n,q}(x) = \sum_{m=0}^{n}\binom{n}{m}B_{m,q}q^{(m-1)x}[x]_q^{n-m}. \tag{2.11}$$

From (2.9) and (2.11),

$$B_{0,q}(x) = \frac{1}{q^x}B_{0,q} = 0. \tag{2.12}$$

However, these are also incorrect.

Next, we redefine the q-Bernoulli polynomials and numbers.

Definition 2.6 For $q \in \mathbb{C}$ with $|q| < 1$, let us define the q-Bernoulli polynomials $B_{n,q}(x)$ as follows,

$$F_q(t,\ x) = \frac{q-1}{logq} e^{\frac{1}{1-q}t} - t\sum_{m=0}^{\infty} q^{x+m} e^{[x+m]_q t} = \sum_{n=0}^{\infty} B_{n,q}(x)\frac{t^n}{n!}. \tag{2.13}$$

Note that

$$\lim_{q\to 1} F_q(t,\ x) = \frac{t}{e^t - 1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \text{ for } |t| < 2\pi, \tag{2.14}$$

where $B_n(x)$ are the classical Bernoulli polynomials.

In the special case $x = 0$, $B_{n,q}(0) = B_{n,q}$ are called the q-Bernoulli numbers. That is,

$$F_q(t) = F_q(t,\ 0) = \sum_{n=0}^{\infty} B_{n,q}\frac{t^n}{n!}. \tag{2.15}$$

By simple calculations, we get

$$\begin{aligned}\sum_{n=0}^{\infty} B_{n,q}(x)\frac{t^n}{n!} &= F_q(t,\ x)\\ &= e^{[x]_q t}F_q(q^x t)\\ &= \left(\sum_{m=0}^{\infty}\frac{[x]_q^m t^m}{m!}\right)\times\left(\sum_{l=0}^{\infty} B_{l,q}\frac{q^{lx}t^l}{l!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}B_{l,q}q^{lx}[x]_q^{n-l}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.16}$$

Comparing the coefficients on the both sides in (2.16), we obtain

$$B_{n,q}(x) = \sum_{l=0}^{n}\binom{n}{l}B_{l,q}q^{lx}[x]_q^{n-l}. \tag{2.17}$$

From (2.13) and (2.15), we derive the following equation.

$$B_{0,q} = \frac{q-1}{logq} \text{ and } B_{n,q}(1) - B_{n,q} = \begin{cases} 1 & \text{if } n = 1\\ 0 & \text{if } n > 1\end{cases}. \tag{2.18}$$

By (2.17) and (2.18), we can see that

$$B_{0,q} = \frac{q-1}{logq} \text{ and } \sum_{l=0}^{n}\binom{n}{l}B_{l,q}q^{l} - B_{n,q} = \begin{cases} 1 & \text{if } n = 1\\ 0 & \text{if } n > 1\end{cases}. \tag{2.19}$$

Theorem 2.7* For $n \in \mathbb{N}^*$, we have

$$B_{0,q} = \frac{q-1}{logq} \text{ and } (qB_q + 1)^n - B_{n,q} = \begin{cases} 1 & \text{if } n = 1\\ 0 & \text{if } n > 1\end{cases}. \tag{2.20}$$

with the usual convention of replacing B_q^n by $B_{n,q}$.

Remark 2.8 Theorem 2.7* is a revised theorem of Theorem 2.1 in [1].

From (2. 13), we have

$$
\begin{aligned}
\sum_{n=0}^{\infty} B_{n,q}(x)\frac{t^n}{n!} &= F_q(t,\ x) \\
&= \frac{q-1}{logq}e^{\frac{1}{1-q}t} - t\sum_{m=0}^{\infty} q^{x+m}e^{[x+m]_q t} \\
&= \frac{q-1}{logq}\sum_{n=0}^{\infty}\frac{1}{(1-q)^n}\frac{t^n}{n!} - \sum_{m=0}^{\infty} q^{x+m}\sum_{n=0}^{\infty} n[x+m]_q^{n-1}\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\frac{q-1}{logq}\frac{1}{(1-q)^n} - n\sum_{m=0}^{\infty} q^{x+m}[x+m]_q^{n-1}\right)\frac{t^n}{n!} \qquad (2.21) \\
&= \sum_{n=0}^{\infty}\left(-\frac{(1-q)^n}{logq} - \frac{n}{(1-q)^{n-1}}\sum_{m=0}^{\infty} q^{x+m}\sum_{l=0}^{n-1}\binom{n-1}{l}(-1)^l q^{(x+m)l}\right)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\frac{(q-1)^{1-n}}{logq} + \frac{n}{(1-q)^{n-1}}\sum_{l=0}^{n-1}\binom{n-1}{l}(-1)^{l+1} q^{(l+1)x}\frac{1}{1-q^{(l+1)}}\right)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}\frac{l}{[l]_q}\right)\frac{t^n}{n!}.
\end{aligned}
$$

By (2.21), we obtain the following theorem.

Theorem 2.9* For $n \in \mathbb{N}^*$, we have

$$
B_{0,q} = \frac{q-1}{logq} \text{ and } B_{n,q}(x) = \frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}\frac{l}{[l]_q}. \qquad (2.22)
$$

Remark 2.10 Theorem 2.9* is a revised theorem of Theorem 2.3 in [1].

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Authors' contributions
Coresponding author raised the problem and make a sequence to appoach the problem. AB carried out the q-Bernoulli poynomials studies, participated in the making new construction of the q-Bernoulli numbers. EJM carried out the calculation of [1]. JHJ participated in the sequence alignment. SJL performed the correction problem. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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