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Convergence theorems for new classes of multivalued hemicontractive-type mappings

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Abstract

Weak and strong convergence theorems are proved in Hilbert spaces for new classes of multivalued demicontractive-type and hemicontractive-type mappings which are related to the class of multivalued pseudocontractive-type mappings studied by Isiogugu (Fixed Point Theory Appl. 2013:61, 2013). Thus our results extend and improve several corresponding results in the contemporary literature.

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1 Introduction

Let E be a normed space. A subset K of E is called proximinal if for each $x \in E$ there exists $k \in K$ such that

$$\|x - k\| = \inf\{\|x - y\| : y \in K\} = d(x, K).$$

It is well known that every closed convex subset of a uniformly convex Banach space is proximinal. For a nonempty set E, we shall denote the family of all nonempty proximinal subsets of E by $P(E)$, the family of all nonempty closed and bounded subsets of E by $CB(E)$, the family of all nonempty closed, convex, and bounded subsets of E by $CVB(E)$, the family of all nonempty closed subsets of E by $C(X)$, the family of all nonempty subsets of E by 2^E, the identity on E by I, the weak topology of E by $\sigma(E, E^*)$, and the norm (or strong) topology of E by $(E, \|\cdot\|)$.

Let H denote the Hausdorff metric induced by the metric d on E, that is, for every $A, B \in 2^E$,

$$H(A, B) = \max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\right\}.$$

If $A, B \in CB(E)$, then

$$H(A, B) = \inf\{\epsilon > 0 : A \subseteq N(\epsilon, B) \text{ and } B \subseteq N(\epsilon, A)\},$$

where $N(\epsilon, C) = \bigcup_{c \in C}\{x \in E : d(x, c) < \epsilon\}$. Let E be a normed space. Let $T : D(T) \subseteq E \to 2^E$ be a multivalued mapping on E. A point $x \in D(T)$ is called a *fixed point* of T if $x \in Tx$. The

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set $F(T) = \{x \in D(T) : x \in Tx\}$ is called the fixed point set of T. A point $x \in D(T)$ is called a *strict fixed point* of T if $Tx = \{x\}$. The set $F_s(T) = \{x \in D(T) : Tx = \{x\}\}$ is called the strict fixed point set of T. A multivalued mapping $T : D(T) \subseteq E \to 2^E$ is called *L-Lipschitzian* if there exists $L \geq 0$ such that for all $x, y \in D(T)$

$$H(Tx, Ty) \leq L\|x - y\|. \tag{1.1}$$

In (1.1) if $L \in [0, 1)$, T is said to be a *contraction*, while T is *nonexpansive* if $L = 1$. T is called *quasi-nonexpansive* if $F(T) = \{x \in D(T) : x \in Tx\} \neq \emptyset$ and for all $p \in F(T)$,

$$H(Tx, Tp) \leq \|x - p\|. \tag{1.2}$$

Clearly every nonexpansive mapping with nonempty fixed point set is quasi-nonexpansive.

Several authors have studied various classes of multivalued mappings. In [1], Shahzad and Zegeye studied certain classes of multivalued nonself mappings in Banach spaces and constructed an appropriate net which converges strongly to a fixed point of the classes of the mappings. Recently, Isiogugu [2] introduced new classes of multivalued mappings as follows.

Definition 1.1 ([2]) Let X be a normed space. A multivalued mapping $T : D(T) \subseteq X \to 2^X$ is said to be *k-strictly pseudocontractive-type* in the sense of Browder and Petryshyn [3] if there exists $k \in [0, 1)$ such that given any $x, y \in D(T)$ and $u \in Tx$, there exists $v \in Ty$ satisfying $\|u - v\| \leq H(Tx, Ty)$ and

$$H^2(Tx, Ty) \leq \|x - y\|^2 + k\|x - u - (y - v)\|^2. \tag{1.3}$$

If $k = 1$ in (1.3) T is said to be a *pseudocontractive-type* mapping. T is called *nonexpansive-type* if $k = 0$. Clearly, every multivalued nonexpansive mapping is nonexpansive-type mapping.

From the definitions, it is clear that every multivalued nonexpansive-type mapping is k-strictly pseudocontractive-type and every k-strictly pseudocontractive-type mapping is pseudocontractive-type. Examples to show that the class of nonexpansive-type mappings is properly contained in the class of k-strictly pseudocontractive-type mappings and that the class of k-strictly pseudocontractive-type mappings is properly contained in the class of pseudocontractive-type mappings were given in [2]. The following theorems were also proved in [2].

Theorem 1.1 *Let K be a nonempty closed and convex subset of a real Hilbert space H. Suppose that $T : K \to P(K)$ is a k-strictly pseudocontractive-type mapping from K into the family of all proximinal subsets of K with $k \in (0, 1)$ such that $F(T) \neq \emptyset$ and $T(p) = \{p\}$ for all $p \in F(T)$. Suppose $(I - T)$ is weakly demiclosed at zero. Then the Mann-type sequence defined by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n y_n$$

converges weakly to $q \in F(T)$, where $y_n \in Tx_n$ with $\|x_n - y_n\| = d(x_n, Tx_n)$ and α_n is a real sequence in $(0, 1)$ satisfying: (i) $\alpha_n \to \alpha < 1 - k$; (ii) $\alpha > 0$; (iii) $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$.

Theorem 1.2 *Let K be a nonempty closed and convex subset of a real Hilbert space X. Suppose that $T : K \to P(K)$ is an L-Lipschitzian pseudocontractive-type mapping from K into the family of all proximal subsets of K such that $F(T) \neq \emptyset$ and $T(p) = \{p\}$ for all $p \in F(T)$. Suppose for any pair $x, y \in K$ and $u \in Tx$ with $\|x - u\| = d(x, Tx)$, there exists $v \in Ty$ with $\|y - v\| = d(y, Ty)$ satisfying the conditions of Definition 1.1. Suppose T satisfies condition (1) (i.e., if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that $d(x, Tx) \geq f(d(x, F(T)))$, $\forall x \in K$. Then the Ishikawa sequence defined by*

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n u_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n w_n \end{cases} \tag{1.4}$$

converges strongly to $p \in F(T)$, where $u_n \in Tx_n$ with $\|x_n - u_n\| = d(x_n, Tx_n)$, $w_n \in Ty_n$ with $\|y_n - w_n\| = d(y_n, Ty_n)$ satisfying the conditions in Definition 1.1 and $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences satisfying: (i) $0 \leq \alpha_n \leq \beta_n < 1$; (ii) $\liminf_{n \to \infty} \alpha_n = \alpha > 0$; (iii) $\sup_{n \geq 1} \beta_n \leq \beta \leq \frac{1}{\sqrt{1+L^2}+1}$.

In [4], Chidume *et al.* also considered a class of multivalued k-strictly pseudocontractive mappings defined as follows.

Let H be a real Hilbert space. A multivalued mapping $T : D(T) \subseteq H \to CB(H)$ is said to be k-strictly pseudocontractive if there exists $k \in (0, 1)$ such that for all $x, y \in D(T)$ one has

$$H^2(Tx, Ty) \leq \|x - y\|^2 + k\|x - u - (y - v)\|^2, \quad \forall u \in Tx, v \in Ty.$$

If $k = 1$, T is said to be pseudocontractive mapping. They constructed a Mann-type iteration scheme which is an approximate fixed point sequence and obtain some strong convergence theorems for the class of k-strictly pseudocontractive mappings.

The following example shows that the class of multivalued pseudocontractive-type mappings considered by Isiogugu [2] is not a subclass of the multivalued pseudocontractive mappings considered by Chidume *et al.* [4].

Example 1.1 Let $X = \mathbb{R}$ (the reals with usual metric). Define $T : [0, \infty) \to CB(\mathbb{R})$ by

$$Tx = \left[-\frac{5x}{2}, -2x \right]. \tag{1.5}$$

It was shown in [2] that T is k-strictly pseudocontractive-type mapping hence pseudocontractive-type. However, for $x = 3$, $y = 2$ if we choose $u = -6 \in Tx$ and $v = -5 \in Ty$ then $H^2(Tx, Ty) = \frac{25}{4}$ and $\|x - y\|^2 + \|x - u - (y - v)\|^2 = 5$. Consequently,

$$H^2(Tx, Ty) > \|x - y\|^2 + \|x - u - (y - v)\|^2,$$

which implies that T is not pseudocontractive and hence not k-strictly pseudocontractive mapping in the sense of Chidume *et al.* [4].

It is our purpose in this work to introduce and study new classes of multivalued demicontractive-type and hemicontractive-type mappings which are more general than the class of multivalued quasi-nonexpansive mappings and are also related to the multivalued k-strictly pseudocontractive-type and pseudocontractive-type mappings of Isiogugu

[2], single-valued mappings of Browder and Petryshyn [3], Hicks and Kubicek [5] and Naimpally and Singh [6]. We also prove weak and strong convergence theorems for approximation of fixed points of our classes of mappings.

2 Preliminaries

We shall need the following definitions and lemmas.

Definition 2.1 (see, *e.g.*, [7]) Let E be a Banach space. Let $T : D(T) \subseteq E \to 2^E$ be a multivalued mapping. $I - T$ is said to be *strongly demiclosed at zero* if for any sequence $\{x_n\}_{n=1}^{\infty} \subseteq D(T)$ such that x_n converges strongly to p and a sequence $\{y_n\}$ with $y_n \in Tx_n$ for all $n \in \mathbb{N}$ such that $\{x_n - y_n\}$ converges strongly to *zero*, then $p \in Tp$ (*i.e.*, $0 \in (I - T)p$).

Observe that if T is a multivalued Lipschitzian mapping, then $I - T$ is strongly demiclosed.

Definition 2.2 (see, *e.g.*, [7, 8]) Let E be a Banach space. Let $T : D(T) \subseteq E \to 2^E$ be a multivalued mapping. $I - T$ is said to be *weakly demiclosed at zero* if for any sequence $\{x_n\}_{n=1}^{\infty} \subseteq D(T)$ such that $\{x_n\}$ converges weakly to p and a sequence $\{y_n\}$ with $y_n \in Tx_n$ for all $n \in \mathbb{N}$ such that $\{x_n - y_n\}$ converges strongly to *zero*. Then $p \in Tp$ (*i.e.*, $0 \in (I - T)p$).

Definition 2.3 (see, *e.g.*, [7, 8]) Let E be a Banach space. Let $T : D(T) \subseteq E \to 2^E$ be a multivalued mapping. The graph of $I - T$ is said to be *closed* in $\sigma(E, E^*) \times (E, \| \cdot \|)$ (*i.e.*, $I - T$ is *weakly demiclosed or demiclosed*) if for any sequence $\{x_n\}_{n=1}^{\infty} \subseteq D(T)$ such that $\{x_n\}$ converges weakly to p and a sequence $\{y_n\}$ with $y_n \in Tx_n$ for all $n \in \mathbb{N}$ such that $\{x_n - y_n\}$ converges strongly to y. Then $y \in (I - T)p$ (*i.e.*, $y = p - v$ for some $v \in Tp$).

Definition 2.4 A Banach X is said to satisfy Opial's condition if whenever a sequence $\{x_n\}$ converges weakly to $x \in X$ then it is the case that

$$\liminf \|x_n - x\| < \liminf \|x_n - y\|,$$

for all $y \in X$, $y \neq x$.

Definition 2.5 ([9]) A multivalued mapping $T : K \to P(K)$ is said to satisfy condition (1) (see for example [9]) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$d(x, Tx) \geq f\big(d\big(x, F(T)\big)\big), \quad \forall x \in K.$$

Lemma 2.1 ([10]) *Let $\{a_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be sequences of nonnegative real numbers satisfying the following relation*:

$$a_{n+1} \leq (1 + \beta_n)a_n + \gamma_n, \quad n \geq n_0,$$

where n_0 is a nonnegative integer. If $\sum \beta_n < \infty$, $\sum \gamma_n < \infty$, then $\lim_{n \to \infty} a_n$ exists.

Lemma 2.2 ([11]) *Let K be a normed space. Let $T : K \to P(K)$ be a multivalued mapping and $P_T(x) = \{y \in Tx : \|x - y\| = d(x, Tx)\}$. Then the following are equivalent*:

(1) $x \in Tx$;

(2) $P_T x = \{x\}$;

(3) $x \in F(P_T)$.

Moreover, $F(T) = F(P_T)$.

Lemma 2.3 ([12]) *Let $A, B \in CB(X)$ and $a \in A$. If $\gamma > 0$, then there exists $b \in B$ such that*

$$d(a, b) \leq H(A, B) + \gamma.$$

3 Main results

We now introduce the new classes of multivalued demicontractive-type and hemicontractive-type mappings and prove some convergence theorems for these classes of mappings.

Definition 3.1 Let X be a real normed space. A mapping $T : D(T) \subseteq X \to 2^X$ is said to be *demicontractive* in the terminology of Hicks and Kubicek [5] if $F(T) \neq \emptyset$ and for all $p \in F(T)$, $x \in D(T)$ there exists $k \in [0, 1)$ such that

$$H^2(Tx, Tp) \leq \|x - p\|^2 + kd^2(x, Tx), \tag{3.1}$$

where $H^2(Tx, Tp) = [H(Tx, Tp)]^2$ and $d^2(x, p) = [d(x, p)]^2$.

If $k = 1$ in (3.1) then T is called a hemicontractive mapping.

The following are some examples of demicontractive mappings.

Example 3.1 Every multivalued quasi-nonexpansive mapping is demicontractive.

Example 3.2 Let X be a normed space. Suppose that T is a multivalued mapping such that $F(T) \neq \emptyset$ and that P_T is a k-strictly pseudocontractive-type mapping; then P_T is demicontractive.

Example 3.3 Let X be a normed space. Let $T : D(T) \subseteq X \to P(X)$ be a multivalued k-strictly pseudocontractive-type with a nonempty fixed point set. Suppose $Tp = \{p\}$ for all $p \in F(T)$; then for any $x \in D(T)$, $p \in F(T)$ and $u \in Tx$ with $\|u - x\| = d(x, Tx)$ we have

$$H^2(Tx, Tp) \leq \|x - p\|^2 + k\|x - u\|^2 = \|x - p\|^2 + kd^2(x, Tx);$$

therefore, T is demicontractive-type.

Example 3.4 Let $X = \mathbb{R}$ (the reals with usual metric). Define $T : \mathbb{R} \to 2^{\mathbb{R}}$ by

$$Tx = \begin{cases} [-\frac{3x}{2}, -2x], & x \in (-\infty, 0], \\ [-2x, -\frac{3x}{2}], & x \in (0, \infty). \end{cases} \tag{3.2}$$

Then $F(T) = \{0\}$. For each $x \in (-\infty, 0) \cup (0, \infty)$,

$$H^2(Tx, T0) = |-2x - 0|^2 = 4|x - 0|^2 = |x - 0|^2 + 3|x|^2,$$

$$d^2(x, Tx) = \left| x - \left(-\frac{3x}{2}\right) \right|^2 = \left| \frac{5x}{2} \right|^2 = \frac{25}{4}|x|^2.$$

Therefore,

$$H^2(Tx, T0) = |x - 0|^2 + 3|x|^2 = |x - 0|^2 + \frac{12}{25} d^2(x, Tx) \le |x - 0|^2 + kd^2(x, Tx).$$

Consequently, T is demicontractive-type with $k = \frac{12}{25}$. It then follows that T is hemicontractive. Observe that T is not quasi-nonexpansive so that the class of multivalued quasi-nonexpansive mappings is properly contained in the class of multivalued demicontractive-type mappings.

Next is an example of a multivalued mapping T with $F(T) \ne \emptyset$, $Tp = \{p\}$ for all $p \in Tp$ for which P_T is a demicontractive-type but not a k-strictly pseudocontractive-type mapping.

Example 3.5 Let $X = \mathbb{R}$ (the reals with usual metric). Define $T : [-1, 1] \to 2^{[-1,1]}$ by

$$Tx = \begin{cases} [-1, \frac{2}{3}x \sin \frac{1}{x}], & x \in (0, 1], \\ \{0\}, & x = 0, \\ [\frac{2}{3}x \sin \frac{1}{x}, 1], & x \in [-1, 0). \end{cases} \tag{3.3}$$

Then $F(T) = \{0\}$. For each $x \in [-1, 1]$,

$$P_T x = \begin{cases} \{\frac{2}{3}x \sin \frac{1}{x}\}, & x \ne 0, \\ \{0\}, & x = 0, \end{cases} \tag{3.4}$$

which is demicontractive-type but not k-strictly pseudocontractive-type (see for example [5]).

The following example shows that the class of demicontractive mapping is properly contained in the class of hemicontractive mappings.

Example 3.6 Let $X = \mathbb{R}$ (the reals with the usual metric). Define $T : \mathbb{R} \to 2^{\mathbb{R}}$ by

$$Tx = \begin{cases} [-\sqrt{2}x, 0], & x \in [0, \infty), \\ [0, -\sqrt{2}x], & x \in (-\infty, 0). \end{cases} \tag{3.5}$$

Then $F(T) = \{0\}$. For each $x \in (-\infty, 0) \cup (0, \infty)$,

$$H^2(Tx, T0) = |\sqrt{2}x - 0|^2 = 2|x - 0|^2 = |x - 0|^2 + |x - 0|^2,$$
$$d^2(x, Tx) = |x - 0|^2 = |x - 0|^2.$$

Therefore,

$$H^2(Tx, T0) \le |x - 0|^2 + |x - 0|^2 = |x - 0|^2 + d^2(x, Tx) > |x - 0|^2 + kd^2(x, Tx), \tag{3.6}$$

$\forall x \in \mathbb{B}$ and $\forall k \in [0, 1)$. Therefore, T is hemicontractive but not demicontractive.

Other examples of hemicontractive mappings include the following.

Example 3.7 Let X be a normed space. Suppose T is a multivalued mapping such that $F(T) \neq \emptyset$ and P_T is pseudocontractive-type mapping; then P_T is hemicontractive.

Example 3.8 Let X be a normed space. Let $T : D(T) \subseteq X \to P(X)$ be a multivalued pseudocontractive-type with a nonempty fixed point set. Suppose $Tp = \{p\}$ for all $p \in F(T)$; then for any $x \in D(T)$, $p \in F(T)$ and $u \in Tx$ with $\|u - x\| = d(x, Tx)$ we have

$$H^2(Tx, Tp) \leq \|x - p\|^2 + \|x - u\|^2 = \|x - p\|^2 + d^2(x, Tx).$$

The following lemma shows that Lemma 2.3 is also valid for all $A, B \in P(E)$ and $\gamma = 0$.

Lemma 3.1 *Let E be a metric space. If $A, B \in P(E)$ and $a \in A$, then it is a simple consequence of the Hausdorff metric H that there exists $b \in B$ such that*

$$d(a, b) \leq H(A, B). \tag{3.7}$$

Proof Let E be a metric space and $P(E)$ be the family of all nonempty proximinal subsets of E. Let $A, B \in P(E)$ and $a \in A$. Since B is proximinal, there exists $b_a \in B$ such that

$$d(a, B) = d(a, b_a).$$

Observe that

$$
\begin{aligned}
H(A, B) &= \max\left\{ \sup_{u \in A} d(u, B), \sup_{v \in B} d(v, A) \right\} \\
&\geq \sup_{u \in A} d(u, B) \geq d(a, B) = d(a, b_a).
\end{aligned}
$$

Hence the result follows. $\qquad\square$

Remark 3.1 Lemma 3.1 holds if E is a reflexive real Banach space and $P(E)$ is replaced with $CB(K)$ with B weakly closed (see for example [4]).

We now prove the following theorems.

Theorem 3.1 *Let K be a nonempty closed and convex subset of a real Hilbert space H. Suppose that $T : K \to P(K)$ is a demicontractive mapping from K into the family of all proximinal subsets of K with $k \in (0, 1)$ and $T(p) = \{p\}$ for all $p \in F(T)$. Suppose $(I - T)$ is weakly demiclosed at zero. Then the Mann type sequence defined by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n y_n, \tag{3.8}$$

converges weakly to $q \in F(T)$, where $y_n \in Tx_n$ and α_n is a real sequence in $(0, 1)$ satisfying: (i) $\alpha_n \to \alpha < 1 - k$; (ii) $\alpha > 0$.

Proof Using the well-known identity:

$$\left\| tx + (1 - t)y \right\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2,$$

which holds for all $x, y \in H$ and for all $t \in [0,1]$, we obtain

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &= \left\|(1-\alpha_n)x_n + \alpha_n y_n - p\right\|^2 \\
&= \left\|(1-\alpha_n)(x_n - p) + \alpha_n(y_n - p)\right\|^2 \\
&= (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|y_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - y_n\|^2 \\
&\le (1-\alpha_n)\|x_n - p\|^2 + \alpha_n H^2(Tx_n, Tp) - \alpha_n(1-\alpha_n)\|x_n - y_n\|^2 \\
&\le (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\left[\|x_n - p\|^2 + kd^2(x_n, Tx_n)\right] \\
&\quad - \alpha_n(1-\alpha_n)\|x_n - y_n\|^2 \\
&\le \|x_n - p\|^2 + \alpha_n k\|x_n - y_n\|^2 \\
&\quad - \alpha_n(1-\alpha_n)\|x_n - y_n\|^2 \\
&= \|x_n - p\|^2 - \alpha_n\big(1 - (\alpha_n + k)\big)\|x_n - y_n\|^2.
\end{aligned}
\tag{3.9}
$$

It then follows that $\lim_{n\to\infty}\|x_n - p\|$ exists; hence $\{x_n\}$ is bounded. Also,

$$
\sum_{n=1}^{\infty} \alpha_n\big(1 - (\alpha_n + k)\big)\|x_n - y_n\|^2 \le \|x_0 - p\|^2 < \infty.
$$

Since $\alpha > 0$ from (ii), we have $\lim_{n\to\infty}\|x_n - y_n\| = 0$. Thus $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Also since K is closed and $\{x_n\} \subseteq K$ with $\{x_n\}$ bounded, there exist a subsequence $\{x_{n_t}\} \subseteq \{x_n\}$ such that $\{x_{n_t}\}$ converges weakly to some $q \in K$. Also $\lim_{n\to\infty}\|x_n - y_n\| = 0$ implies that $\lim_{n\to\infty}\|x_{n_t} - y_{n_t}\| = 0$. Since $(I - T)$ is weakly demiclosed at *zero* we have $q \in Tq$. Since H satisfies Opial's condition [13] we find that $\{x_n\}$ converges weakly to $q \in F(T)$. $\qquad\square$

Corollary 3.1 *Let K be a nonempty closed and convex subset of a real Hilbert space H. Suppose that $T : K \to P(K)$ is k-strictly pseudocontractive-type mapping from K into the family of all proximinal subsets of K with $k \in (0,1)$ such that $F(T) \ne \emptyset$ and $T(p) = \{p\}$ for all $p \in F(T)$. Suppose $(I - T)$ is weakly demiclosed at zero. Then the Mann sequence $\{x_n\}$ defined in Theorem* 3.1 *converges weakly to a point of $F(T)$.*

Proof The proof follows easily from Example 3.3 and Theorem 3.1. $\qquad\square$

Corollary 3.2 *Let H be a real Hilbert space and K a nonempty closed and convex subset of H. Let $T : K \to P(K)$ be a multivalued mapping from K into the family of all proximinal subsets of K. Suppose P_T is a demicontractive mapping with $k \in (0,1)$ and $(I - P_T)$ is weakly demiclosed at zero. Then the Mann sequence $\{x_n\}$ defined in Theorem* 3.1 *converges weakly to a point of $F(T)$.*

Proof The proof follows easily from Lemma 2.2 and Theorem 3.1. $\qquad\square$

Remark 3.2 Since the choice of $y_n \in Tx_n$ in the Mann-type iteration scheme is independent of $d(x_n, Tx_n)$, we can also replace $P(K)$ with $CB(K)$ in Theorem 3.1 and its corollaries. Furthermore, since $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, one can impose standard conditions on T or K which guarantee strong convergence.

Theorem 3.2 *Let K be a nonempty closed and convex subset of a real Hilbert space X. Suppose that $T : K \to P(K)$ is an L-Lipschitzian hemicontractive mapping from K into the family of all proximinal subsets of K and $Tp = \{p\}$ for all $p \in F(T)$. Suppose T satisfies condition* (1). *Then the Ishikawa sequence defined by*

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n u_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n w_n \end{cases} \tag{3.10}$$

converges strongly to $p \in F(T)$, where $u_n \in Tx_n$, $w_n \in Ty_n$ satisfying the conditions of Lemma 3.1 *and $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences satisfying:* (i) $0 \le \alpha_n \le \beta_n < 1$; (ii) $\liminf_{n \to \infty} \alpha_n = \alpha > 0$; (iii) $\sup_{n \ge 1} \beta_n \le \beta \le \frac{1}{\sqrt{1+L^2}+1}$.

Proof

$$\|x_{n+1} - p\|^2 = \left\|(1 - \alpha_n)x_n + \alpha_n w_n - p\right\|^2$$
$$= \left\|(1 - \alpha_n)(x_n - p) + \alpha_n(w_n - p)\right\|^2$$
$$= (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n\|w_n - p\|^2$$
$$\quad - \alpha_n(1 - \alpha_n)\|x_n - w_n\|^2$$
$$\le (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n H^2(Ty_n, Tp)$$
$$\quad - \alpha_n(1 - \alpha_n)\|x_n - w_n\|^2$$
$$\le (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n\big[\|y_n - p\|^2$$
$$\quad + d^2(y_n, Ty_n)\big] - \alpha_n(1 - \alpha_n)\|x_n - w_n\|^2$$
$$= (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n\|y_n - p\|^2 + \alpha_n d^2(y_n, Ty_n)$$
$$\quad - \alpha_n(1 - \alpha_n)\|x_n - w_n\|^2, \tag{3.11}$$

$$d^2(y_n, Ty_n) \le \|y_n - w_n\|^2$$
$$= \left\|(1 - \beta_n)x_n + \beta_n u_n - w_n\right\|^2$$
$$= \left\|(1 - \beta_n)(x_n - w_n) + \beta_n(u_n - w_n)\right\|^2$$
$$= (1 - \beta_n)\|x_n - w_n\|^2 + \beta_n\|u_n - w_n\|^2$$
$$\quad - \beta_n(1 - \beta_n)\|x_n - u_n\|^2. \tag{3.12}$$

Equations (3.11) and (3.12) imply that

$$\|x_{n+1} - p\|^2 \le (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n\|y_n - p\|^2$$
$$\quad + \alpha_n\big[(1 - \beta_n)\|x_n - w_n\|^2 + \beta_n\|u_n - w_n\|^2$$
$$\quad - \beta_n(1 - \beta_n)\|x_n - u_n\|^2\big]$$
$$\quad - \alpha_n(1 - \alpha_n)\|x_n - w_n\|^2, \tag{3.13}$$

$$\|y_n - p\|^2 = \left\|(1 - \beta_n)x_n + \beta_n u_n - p\right\|^2$$
$$= \left\|(1 - \beta_n)(x_n - p) + \beta_n(u_n - p)\right\|^2$$

$$= (1 - \beta_n)\|x_n - p\|^2 + \beta_n\|u_n - p\|^2 - \beta_n(1 - \beta_n)\|x_n - u_n\|^2$$

$$\leq (1 - \beta_n)\|x_n - p\|^2 + \beta_n H^2(Tx_n, Tp) - \beta_n(1 - \beta_n)\|x_n - u_n\|^2$$

$$\leq (1 - \beta_n)\|x_n - p\|^2 + \beta_n\big[\|x_n - p\|^2 + d^2(x_n, Tx_n)\big]$$

$$\quad - \beta_n(1 - \beta_n)\|x_n - u_n\|^2$$

$$\leq (1 - \beta_n)\|x_n - p\|^2 + \beta_n\|x_n - p\|^2 + \beta_n\|x_n - u_n\|^2$$

$$\quad - \beta_n(1 - \beta_n)\|x_n - u_n\|^2$$

$$= \|x_n - p\|^2 + \beta_n^2\|x_n - u_n\|^2. \tag{3.14}$$

Equations (3.13) and (3.14) imply that

$$\|x_{n+1} - p\|^2 \leq (1 - \alpha_n)\|x_n - p\|^2$$

$$\quad + \alpha_n\big[\|x_n - p\|^2 + \beta_n^2\|x_n - u_n\|^2\big]$$

$$\quad + \alpha_n\big[(1 - \beta_n)\|x_n - w_n\|^2 + \beta_n\|u_n - w_n\|^2$$

$$\quad - \beta_n(1 - \beta_n)\|x_n - u_n\|^2\big]$$

$$\quad - \alpha_n(1 - \alpha_n)\|x_n - w_n\|^2$$

$$= (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n\|x_n - p\|^2 + \alpha_n\beta_n^2\|x_n - u_n\|^2$$

$$\quad + \alpha_n(1 - \beta_n)\|x_n - w_n\|^2 + \alpha_n\beta_n\|u_n - w_n\|^2$$

$$\quad - \alpha_n\beta_n(1 - \beta_n)\|x_n - u_n\|^2 - \alpha_n(1 - \alpha_n)\|x_n - w_n\|^2$$

$$\leq \|x_n - p\|^2 + \alpha_n\beta_n^2\|x_n - u_n\|^2 + \alpha_n\beta_n H^2(Tx_n, Ty_n)$$

$$\quad - \alpha_n(\beta_n - \alpha_n)\|x_n - w_n\|^2$$

$$\quad - \alpha_n\beta_n(1 - \beta_n)\|x_n - u_n\|^2$$

$$\leq \|x_n - p\|^2 + \alpha_n\beta_n^2\|x_n - u_n\|^2 + \alpha_n\beta_n^3 L^2\|x_n - u_n\|^2$$

$$\quad - \alpha_n\beta_n(1 - \beta_n)\|x_n - u_n\|^2$$

$$\quad - \alpha_n(\beta_n - \alpha_n)\|x_n - w_n\|^2$$

$$= \|x_n - p\|^2 - \alpha_n\beta_n\big[1 - 2\beta_n - L^2\beta_n^2\big]\|x_n - u_n\|^2$$

$$\quad - \alpha_n(\beta_n - \alpha_n)\|x_n - w_n\|^2$$

$$= \|x_n - p\|^2 - \alpha_n\beta_n\big[1 - 2\beta_n - L^2\beta_n^2\big]\|x_n - u_n\|^2. \tag{3.15}$$

It then follows from Lemma 2.1 that $\lim_{n\to\infty}\|x_n - p\|$ exists. Hence $\{x_n\}$ is bounded so $\{u_n\}$ and $\{w_n\}$ also are. We then have from (3.15), (ii), and (iii)

$$\sum_{n=0}^{\infty}\alpha^2\big[1 - 2\beta - L^2\beta^2\big]\|x_n - u_n\|^2 \leq \sum_{n=0}^{\infty}\alpha_n\beta_n\big[1 - 2\beta_n - L^2\beta_n^2\big]\|x_n - u_n\|^2$$

$$\leq \sum_{n=0}^{\infty}\big[\|x_n - p\|^2 - \|x_{n+1} - p\|^2\big]$$

$$\leq \|x_0 - p\|^2 + D < \infty.$$

It then follows that $\lim_{n\to\infty} \|x_n - u_n\| = 0$. Since $u_n \in Tx_n$ we have $d(x_n, Tx_n) \le \|x_n - u_n\| \to 0$ as $n \to \infty$. Since T satisfies condition (1), $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Thus there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\|x_{n_k} - p_k\| \le \frac{1}{2^k}$ for some $\{p_k\} \subseteq F(T)$. From (3.10)

$$\|x_{n_{k+1}} - p_k\| \le \|x_{n_k} - p_k\|.$$

We now show that $\{p_k\}$ is a Cauchy sequence in $F(T)$. We have

$$\begin{aligned} \|p_{k+1} - p_k\| &\le \|p_{k+1} - x_{n_{k+1}}\| + \|x_{n_{k+1}} - p_k\| \\ &\le \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &= \frac{1}{2^{k-1}}. \end{aligned}$$

Therefore $\{p_k\}$ is a Cauchy sequence and converges to some $q \in K$ because K is closed. Now,

$$\|x_{n_k} - q\| \le \|x_n - p_k\| + \|p_k - q\|.$$

Hence $x_{n_k} \to q$ as $k \to \infty$. We have

$$\begin{aligned} d(q, Tq) &\le \|q - p_k\| + \|p_k - x_{n_k}\| + d(x_{n_k}, Tx_{n_k}) + H(Tx_{n_k}, Tq) \\ &\le \|q - p_k\| + \|p_k - x_{n_k}\| + d(x_{n_k}, Tx_{n_k}) + L\|x_{n_k} - q\|. \end{aligned}$$

Hence, $q \in Tq$ and $\{x_{n_k}\}$ converges strongly to q. Since $\lim \|x_n - q\|$ exists we see that x_n converges strongly to $q \in F(T)$. □

Corollary 3.3 *Let K be a nonempty closed and convex subset of a real Hilbert space X. Suppose that $T : K \to P(K)$ is an L-Lipschitzian pseudocontractive-type mapping from K into the family of all proximinal subsets of K such that $F(T) \ne \emptyset$ and $T(p) = \{p\}$ for all $p \in F(T)$. Suppose T satisfies condition* (1). *Then the Ishikawa sequence $\{x_n\}$ defined in* (3.10) *converges strongly to $p \in F(T)$.*

Proof The proof follows easily from Example 3.8, Lemma 3.1, and Theorem 3.2. □

Corollary 3.4 *Let H be a real Hilbert space and K a nonempty closed and convex subset of H. Let $T : K \to P(K)$ be a multivalued mapping from K into the family of all proximinal subsets of K such that $F(T) \ne \emptyset$. Suppose P_T is an L-Lipschitzian hemicontractive mapping. If T satisfies condition* (1). *Then the Ishikawa sequence $\{x_n\}$ defined in* (3.10) *converges strongly to $p \in F(T)$.*

Proof The proof follows easily from Lemma 2.2 and Theorem 3.2. □

Remark 3.3 In Theorem 3.2 and its corollaries we can replace $P(K)$ with $CB(K)$ with additional condition that T is weakly closed for all $x \in D(T) = K$ in order to ensure that u_n and w_n satisfy Lemma 3.1 as indicated in Remark 3.1. Furthermore, since $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, the additional requirement that $(I - T)$ is weakly demiclosed at zero in Theorem 3.2 yields weak convergence without condition (1).

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